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On Multiple Solutions for Nonhomogeneous System of Elliptic Equations

J. CHABROWSKI

ABSTRACT. We establish the existence of at least two solutions of system (1) under some restrictions on $\lambda, \alpha, \beta, f$ and g. Solutions, depending on the case, are obtained by applying the mountain pass theorem, local, global and constrained minimization.

1. INTRODUCTION

The purpose of this paper is to investigate the existence of solutions $(u, v) \in \overset{\circ}{W}{}^{1,p}(Q) \times \overset{\circ}{W}{}^{1,q}(Q)$ of the system of equations

$$\begin{cases} -\Delta_p u = \lambda |u|^{\alpha-1} u |v|^{\beta+1} + f \\ -\Delta_q v = \lambda |u|^{\alpha+1} |v|^{\beta-1} v + g, \end{cases} \tag{1}$$

where $Q \subset \mathbb{R}_N$ is a bounded domain, $\lambda \in \mathbb{R}$, with $\lambda \neq 0$, is a parameter, $(f, g) \in L^{p'}(Q) \times L^{q'}(Q)$, with $\frac{1}{p} + \frac{1}{p'} = 1$ and $\frac{1}{q} + \frac{1}{q'} = 1$. Δ_p is the p-Laplacian defined by

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$$-\Delta_p u = D_i(|\nabla u|^{p-2} D_i u).$$

We assume that $1 < p, q < N$, $-1 < \alpha$ and $-1 < \beta$. We distinguish the following cases: (i) $\frac{\alpha+1}{p^*} + \frac{\beta+1}{q^*} = 1$, (ii) $\frac{\alpha+1}{p} + \frac{\beta+1}{q} = 1$, (iii) $\frac{\alpha+1}{p^*} + \frac{\beta+1}{q^*} < 1$ and $\frac{\alpha+1}{p} + \frac{\beta+1}{q} > 1$, (iv) $\frac{\alpha+1}{p} + \frac{\beta+1}{q} < 1$. Here p^* and q^* denote critical Sobolev exponents: $p^* = \frac{Np}{N-p}$ and $q^* = \frac{Nq}{N-q}$.

The case of the system (1), where $f \equiv 0$ and $g \equiv 0$ on Q, has been studied by several authors and we refer to papers [7], [9] and [10] where additional bibliographical references can be found.

In case (iii), inspired by paper [11], we obtain one solution by applying the mountain pass theorem and a second solution by a local minimization. In cases (ii) and (iv) a solution will be obtained by a global minimization. However, a second solution will be obtained if $f \not\equiv 0$ and $g \equiv 0$ on Q.

Case (i) seems to be more difficult. It is known (see [7]) in this case, that the homogeneous system of equations (1) is not solvable on star-like domains. By constrast, if $(f, g) \not\equiv (0, 0)$, then system (1) has always a solution provided the norms of f and g are not too large.

In this case we develop a method that can be used to find norm-estimates of f and g guaranteeing the solvability of system (1). This method can also be used to show the existence of a solution of one nonhomogeneous equation involving a critical Sobolev exponent and we shall return to this question in a final section (Section 6) of this paper. The result presented in Section 6, recovers in the case $p = 2$ a recent result of paper [6] (see Theorem 1 there).

In this paper we use standard terminology and notations. Let $X = \overset{\circ}{W}{}^{1,p}(Q) \times \overset{\circ}{W}{}^{1,q}(Q)$ be equipped with norm $\|(u,v)\| = \|\nabla u\|_p + \|\nabla v\|_q$. We define a functional $J: X \to \mathbb{R}$ by

$$J(u,v) = \frac{\alpha+1}{p} \int_Q |\nabla u|^p dx + \frac{\beta+1}{q} \int_Q |\nabla v|^q dx - \lambda \int_Q |u|^{\alpha+1} |v|^{\beta+1} dx$$

$$- (\alpha+1) \int_Q f u dx - (\beta+1) \int_Q g v dx.$$

Any critical point $(u, v) \in X$ of the functional J is a solution of (1).

In this work we always denote in a given Banach space Y weak convergence by "$\rightharpoonup$" and strong convergence by "$\to$".

To prove that a minimizing sequence of the functional J is convergent we need the Palais-Smale condition . We say that J satisfies the Palais-Smale condition if every sequence $\{(u_m, v_m)\} \subset X$ such that $J(u_m, v_m)$ is bounded and $J'(u_m, v_m) \to 0$ in X^* as $m \to \infty$ is relatively compact in X.

In each case we find conditions under which J satisfies the Palais-Smale condition.

If $f \not\equiv 0$ and $g \equiv 0$ on Q, then it is easy to find a solution if (1), namely, if $u_0 \in \overset{\circ}{W}{}^{1,p}(Q)$ satisfies

$$-\Delta_p u = f \text{ in } Q, \tag{2}$$

then $(u_0, 0) \in X$ is a solution of system (1). This observation will be frequently used in this paper.

2. CASE $\frac{\alpha+1}{p^*} + \frac{\beta+1}{q^*} = 1$

A solution to problem (1) will be obtained by a minimization of J subject to an artificial constraint. We present here a result for the case $p = q$. Moreover we assume that $\alpha > 0$ and $\beta > 0$. We put

$$M = \{(u, v) \in X - (0,0); \langle J'(u, v), (u, v)\rangle = 0\}.$$

Since $\alpha + \beta + 2 = p^*$ we see that

$$J_{|M}(u, v) = \frac{\alpha+1}{N} \int_Q |\nabla u|^p dx + \frac{\beta+1}{N} \int_Q |\nabla v|^p dx$$

$$- \frac{(\alpha+1)(p^*-1)}{p^*} \int_Q fu dx - \frac{(\beta+1)(p^*-1)}{p^*} \int_Q gv dx.$$

We define a constant $k = k(N, p, q, \alpha, \beta, \lambda), \lambda > 0$, by

$$k = \frac{p^{1+\frac{p}{p'(p^*-p)}}(p^*-p)(p-1)^{\frac{p}{p'(p^*-p)}}}{rR^{\frac{p}{p'(p^*-p)}}\lambda^{\frac{p}{p'(p^*-p)}}(p^*)^{\frac{p}{p'(p^*-p)}}(p^*-p)^{1+\frac{p}{p'(p^*-p)}}},$$

where

$$R = \frac{(\alpha+1)^{\frac{p-p^*}{p}}p^{\frac{p^*}{p}}}{p^*S^{\frac{p^*}{p}}} + \frac{(\beta+1)^{\frac{p-p^*}{p}}p^{\frac{p^*}{p}}}{p^*S^{\frac{p^*}{p}}} \text{ and}$$

$$r = p^{\frac{1}{p}}\left[(\alpha+1)^{\frac{1}{p'}} + (\beta+1)^{\frac{1}{p'}}\right]$$

and S denotes the best Sobolev constant, that is,

$$S = \inf\left\{ \int_Q |\nabla u|^p dx; \int_Q |u|^{p^*} dx = 1,\ u \in \overset{\circ}{W}{}^{1,p}(Q)\right\}. \tag{3}$$

It will be convenient to discuss the solvability of system (1) under more general assumption: $(f,g) \in W^{-1,p'}(Q) \times W^{-1,p'}(Q)$.

Theorem 1. *(i) Suppose that $\lambda < 0$ and $(f,g) \in W^{-1,p'}(Q) \times W^{-1,p'}(Q)$, with $(f,g) \not\equiv (0,0)$. Then system (1) has at least one solution in X.*

(ii) Suppose that $\lambda > 0$ and that $(f,g) \in W^{-1,p'}(Q) \times W^{-1,p'}(Q)$, with $(f,g) \not\equiv (0,0)$ and

$$||f||_{W^{-1,p'}}, ||g||_{W^{-1,p'}} < k. \tag{4}$$

Then system (1) has at least one solution in X.

Proof. Without loss of generality we may assume that $f \not\equiv 0$. It is easy to check, using the Young inequality, that $J_{|_M}$ is bounded from below. Let $u_0 \in \overset{\circ}{W}{}^{1,p}(Q)$ be a solution of equation (2). Since $\int_Q |\nabla u_0|^p dx = \int_Q fu_0 dx > 0$, we see that $(u_0, 0) \in M$ and

$$J(u_0,0) = -\frac{\alpha+1}{p'}\int_q |\nabla u_0|^p dx < 0$$

and hence

$$m_J = \inf_{(u,v)\in M} J(u,v) < 0.$$

Let

$$I(u,v) = \langle J'(u,v),(u,v)\rangle \text{ for } (u,v) \in X.$$

First we show that

$$I'(u,v) \neq 0 \text{ for } (u,v) \in M.$$

Assuming that $I'(u,v) = 0$ for some $(u,v) \in M$ and setting

$$A = \frac{\alpha+1}{p}\int_Q |\nabla u|^p dx + \frac{\beta+1}{p}\int_Q |\nabla v|^p dx,$$

$$B = \int_Q |u|^{\alpha+1}|v|^{\beta+1} dx$$

and

$$C = (\alpha+1)\int_Q fu dx + (\beta+1)\int_Q gv dx,$$

we see that quantities A, B and C satisfy the following system of equations

$$\begin{cases} A - \lambda B - C & = J(u,v) \equiv m \\ pA - \lambda p^* B - C & = 0 \\ p^2 A - \lambda (p^*)^2 B - C & = 0. \end{cases} \tag{5}$$

A unique solution of this system is given by

$$A = -m\frac{p^*}{(p^*-p)(p-1)}, \lambda B = -m\frac{p}{(p^*-1)(p^*-p)},$$

$$C = -m\frac{p^* p}{(p^*-1)(p-1)}.$$

Since $A > 0$ we see that $m < 0$. Hence if $\lambda < 0$, then $B < 0$ which is impossible. Therefore it remains to consider the case $\lambda > 0$. Letting $k_1 = \max(||f||_{W^{-1,p'}}, ||g||_{W^{-1,p'}})$ we get by a straightforward estimation

$$C \leq (\alpha+1)||f||_{W^{-1,p'}}||\nabla u||_p + (\beta+1)||g||_{W^{-1,p'}}||\nabla v||_p$$

$$\leq (\alpha+1)k_1\Big(\int_Q |\nabla u|^p dx\Big)^{\frac{1}{p}} + (\beta+1)k_1\Big(\int_Q |\nabla v|^p dx\Big)^{\frac{1}{p}}$$

$$\leq k_1(\alpha+1)^{\frac{1}{p'}}p^{\frac{1}{p}}\Big(\frac{\alpha+1}{p}\int_Q |\nabla u|^p dx\Big)^{\frac{1}{p}}$$

$$+ k_1(\beta+1)^{\frac{1}{p'}}p^{\frac{1}{p}}\Big(\frac{\beta+1}{p}\int_Q |\nabla v|^p dx\Big)^{\frac{1}{p}}$$

$$\leq k_1\big[((\alpha+1)^{\frac{1}{p'}} + (\beta+1)^{\frac{1}{p'}}\big]p^{\frac{1}{p}}$$

$$\Big(\frac{\alpha+1}{p}\int_Q |\nabla u|^p dx + \frac{\beta+1}{p}\int_Q |\nabla v|^p dx\Big)^{\frac{1}{p}}$$

$$\leq r k_1 A^{\frac{1}{p}}.$$

Using the formulae for A and C we derive from the last inequality that

$$\frac{(-m)^{\frac{1}{p'}}(p^*)^{\frac{1}{p'}}p(p^*-p)^{\frac{1}{p}}}{r(p^*-1)(p-1)^{\frac{1}{p'}}} \leq k_1. \tag{6}$$

Similarly we derive from the Sobolev inequality that

$$\lambda B \leq \lambda\frac{\alpha+1}{p^*}\int_Q |\nabla u|^p dx + \lambda\frac{\beta+1}{p^*}\int_Q |\nabla v|^p dx$$

$$\leq \lambda\frac{\alpha+1}{p^*}S^{-\frac{p^*}{p}}\left(\int_Q |\nabla u|^p dx\right)^{\frac{p^*}{p}} + \lambda\frac{\beta+1}{p^*}S^{-\frac{p^*}{p}}\left(\int_Q |\nabla v|^p dx\right)^{\frac{p^*}{p}}$$

$$= \frac{\lambda(\alpha+1)^{\frac{p-p^*}{p}} p^{\frac{p^*}{p}}}{p^* S^{\frac{p^*}{p}}}\left(\frac{\alpha+1}{p}\int_Q |\nabla u|^p dx\right)^{\frac{p^*}{p}}$$

$$+ \frac{\lambda(\beta+1)^{\frac{p-p^*}{p}} p^{\frac{p^*}{p}}}{p^* S^{\frac{p^*}{p}}}\left(\frac{\beta+1}{p}\int_Q |\nabla v|^p dx\right)^{\frac{p^*}{p}} \leq \lambda R A^{\frac{p^*}{p}}.$$

Taking into account formulae for B and C, we derive from this inequality the lower estimate for $(-m)^{\frac{1}{p'}}$

$$\frac{p^{\frac{p}{p'(p^*-p)}}(p^*-p)^{\frac{1}{p'}}(p-1)^{\frac{p^*}{p'(p^*-p)}}}{(\lambda R)^{\frac{p}{p'(p^*-p)}}(p^*-1)^{\frac{p}{p'(p^*-p)}}(p^*)^{\frac{p}{p'(p^*-p)}}} \leq (-m)^{\frac{1}{p'}}.$$

This combined with (6) leads to the following estimate from below for k_1:

$$\frac{p^{1+\frac{p}{p'(p^*-p)}}(p^*-p)(p-1)^{\frac{p}{p'(p^*-p)}}}{r(\lambda R)^{\frac{p}{p'(p^*-p)}}(p^*)^{\frac{p}{p'(p^*-p)}}(p^*-1)^{1+\frac{p}{p'(p^*-p)}}} \leq k_1,$$

which contradicts assumption (4). Using the Ekeland variational principle [2] we can choose a sequence $\{(u_m, v_m)\} \subset M$ such that

$$J(u_m, v_m) \to m_J \text{ and } J'|_M(u_m, v_m) \to 0 \text{ in } X^* \tag{7}$$

as $m \to \infty$. We now show that $J'(u_m, v_m) \to 0$ in X^* as $m \to \infty$. Since $I'(u, v) \neq 0$ on M, we have

$$J'(u_m, v_m) = J'_{|_M}(u_m, v_m) - \lambda_m I'(u_m, v_m) \tag{8}$$

for some $\lambda_m \in \mathbb{R}$. Since $\{(u_m, v_m)\} \subset M$ we have

$$\langle J'_{|_M}(u_m, v_m), (u_m, v_m)\rangle - \lambda_m \langle I'(u_m, v_m), (u_m, v_m)\rangle$$

$$= \langle J'(u_m, v_m), (u_m, v_m)\rangle = 0.$$

It follows from (7) that $\{(u_m, v_m)\}$ is a bounded sequence in X. We now show that there exists a constant $\delta > 0$ such that

$$|\langle I'(u_m, v_m), (u_m, v_m)\rangle| \geq \delta \tag{9}$$

for all m. In the contrary case we can assume that up to a subsequence

$$\lim_{m\to\infty} \langle I'(u_m, v_m), (u_m, v_m)\rangle = 0. \tag{10}$$

Since sequences $\{\int_Q |u_m|^{\alpha+1}|v_m|^{\beta+1}dx\}$ and $\{(\alpha+1)\int_Q f u_m dx + (\beta + 1)\int_Q g v_m dx\}$ are bounded, we can also assume that the following limits exist

$$A = \lim_{m\to\infty} \left(\frac{\alpha+1}{p}\int_Q |\nabla u_m|^p dx + \frac{\beta+1}{p}\int_Q |\nabla v_m|^p dx\right)$$

$$B = \lim_{m\to\infty} \int_Q |u_m|^{\alpha+1}|v_m|^{\beta+1}dx$$

and

$$C = \lim_{m\to\infty} \left((\alpha+1)\int_Q f u_m dx + (\beta+1)\int_Q g v_m dx\right).$$

It follows from (7) and (10) and the fact that $\{(u_m, v_m)\} \subset M$ that A, B and C satisfy of equations (5) with $m = m_J$. A unique solution of system (5) is given by

$$A = -m_J \frac{p^*}{(p^*-p)(p-1)}, \lambda B = -m_J \frac{p}{(p^*-1)(p^*-p)},$$

$$C = -m_J \frac{p^* p}{(p^*-1)(p-1)}.$$

If $\lambda < 0$, then $B < 0$, which is impossible. Therefore it remains to consider the case $\lambda > 0$. Let us set

$$C_m = (\alpha+1)\int_Q f u_m dx + (\beta+1)\int_Q g v_m dx$$

and

$$A_m = \frac{\alpha+1}{p}\int_Q |\nabla u_m|^p dx + \frac{\beta+1}{p}\int_Q |\nabla v_m|^p dx.$$

Letting $k_1 = \max(\|f\|_{W^{-1,p'}}, \|g\|_{W^{-1,p'}})$ we get as in the previous part of the proof that

$$C_m \le r k_1 A_m^{\frac{1}{p}}.$$

Letting $m \to \infty$ we get

$$C \le r k_1 A^{\frac{1}{p}}. \tag{11}$$

Similarly, we show that

$$\lambda B \le \lambda R A^{\frac{p^*}{p}}. \tag{12}$$

However, the previous part of the proof shows that (11) and (12) lead to a contradiction with (4). Consequently (9) holds and by (8) $\lambda_m \to 0$ as $m \to \infty$. This in conjunction with (7) implies that $J'(u_m, v_m) \to 0$ in X^* as $m \to \infty$. Since $\{(u_m, v_m)\}$ is bounded in X we may assume that $(u_m, v_m) \to (u, v)$ in X and $(u_m, v_m) \to (u, v)$ in $L^q(Q) \times L^q(Q)$ for all $p \le q < p^*$ and a.e. on Q. Since $J'(u_m, v_m) \to 0$ in X^* we see that $J'_u(u_m, v_m) \to 0$ and $J'_v(u_m, v_m) \to 0$ in $W^{-1,p'}(Q)$. Consequently,

$$-\Delta_p u_m = \lambda |u_m|^{\alpha-1} u_m |v_m|^{\beta+1} + f + f_m$$

with $f_m \to 0$ in $W^{-1,p'}(Q)$. Since $g_m = \lambda |u_m|^{\alpha-1} u_m |v_m|^{\beta+1}$ belongs to $W^{-1,p'}(Q)$ and is bounded in $W^{-1,p'}(Q)$ and in $L^1(Q)$ we can apply Theorem 2.1 from [2] (see also Remark 2.7 there). By virtue of this result $Du_m \to Du$, up to subsequence, in $\left(L^r(Q)\right)^N$ for every $r < p$. Similarly, we show that up to a subsequence $Dv_m \to Dv$ in $\left(L^s(Q)\right)^N$ for every $s < p$. In particular, for every $(\varphi, \psi) \in X$ we have

$$
\begin{aligned}
&\langle J'(u_m,v_m),(\varphi,\psi)\rangle \\
&=(\alpha+1)\int_Q |\nabla u_m|^{p-2} Du_m D\varphi dx \\
&+(\beta+1)\int_Q |\nabla v_m|^{p-2} Dv_m D\psi dx \\
&-\lambda(\alpha+1)\int_Q |u_m|^{\alpha-1}u_m\varphi|v_m|^{\beta+1}dx \\
&-\lambda(\beta+1)\int_Q |u_m|^{\alpha+1}|v_m|^{\beta-1}v_m\psi dx \\
&-(\alpha+1)\int_Q f\varphi dx-(\beta+1)\int_Q g\psi dx
\end{aligned}
$$

and letting $m \to \infty$ we get

$$\langle J'(u,v),(\varphi,\psi)\rangle = 0.$$

This means that (u,v) is a solution of (1) and hence $(u,v) \in M$. Since $J_{|_M}$ is weakly lower semicontinuous we get

$$
\begin{aligned}
m_J \leq & J(u,v) = \frac{\alpha+1}{N}\int_Q |\nabla u|^p dx + \frac{\beta+1}{N}\int_Q |\nabla v|^p dx \\
& - \frac{(\alpha+1)(\alpha+\beta+1)}{p^*}\int_Q fudx - \frac{(\beta+1)(\alpha+\beta+1)}{p^*}\int_Q gvdx \\
\leq & \lim_{m\to\infty} J(u_m,v_m) = m_J.
\end{aligned}
$$

Thus $m_J = J(u,v)$ and

$$\lim_{m\to\infty}\left[\frac{\alpha+1}{N}\int_Q|\nabla u_m|^p dx+\frac{\beta+1}{N}\int_Q|\nabla v_m|^p dx\right]$$

$$=\frac{\alpha+1}{N}\int_Q|\nabla u|^p dx+\frac{\beta+1}{N}\int_Q|\nabla v|^p dx$$

which implies that $(u_m, v_m) \to (u, v)$ in X.

The method used in this proof breaks down when $p \neq q$ and we were unable to find a correct argument in this case.

3. CASE $\frac{\alpha+1}{p} + \frac{\beta+1}{q} = 1$

In this case the homogeneous system (1) is in fact an eigenvalue problem. It is known (see [7]) that the minimization problem

$$\lambda_1 = \inf\left\{\frac{\alpha+1}{p}\int_Q|\nabla u|^p dx+\frac{\beta+1}{q}\int_Q|\nabla v|^q dx;\right.$$

$$\left.(u,v)\in X, \int_Q |u|^{\alpha+1}|v|^{\beta+1}dx = 1\right\}$$

has a solution (u_0, v_0) and λ_1 is the smallest eigenvalue with an eigenfunction (u_0, v_0) of the eigenvalue problem

$$\begin{cases} -\Delta_p u = \lambda|u|^{\alpha-1}u|v|^{\beta+1} \\ -\Delta_q v = \lambda|u|^{\alpha+1}|v|^{\beta-1}v \end{cases} \tag{13}$$

in Q.

We commence by investigating the Palais-Smale condition for J.

We shall show that the Palais-Smale condition holds for every $\lambda < \lambda_1$. First, we observe that

$$\lambda_1 \int_Q |u|^{\alpha+1}|v|^{\beta+1}dx \le \frac{\alpha+1}{p}\int_Q |\nabla u|^p dx + \frac{\beta+1}{q}\int_Q |\nabla v|^p dx. \quad (14)$$

Indeed, if $(u,v) \in X$, with $u \not\equiv 0$ and $v \not\equiv 0$ we put

$$\bar{u} = \frac{u}{\left(\int_Q |u|^{\alpha+1}|v|^{\beta+1}dx\right)^{\frac{1}{p}}} \text{ and } \bar{v} = \frac{v}{\left(\int_Q |u|^{\alpha+1}|v|^{\beta+1}dx\right)^{\frac{1}{q}}}$$

and we see that

$$\int_Q |\bar{u}|^{\alpha+1}|\bar{v}|^{\beta+1}dx = 1.$$

It follows from the definition of λ_1 that

$$\lambda_1 \le$$

$$\frac{1}{2}\left(\int_Q |u|^{\alpha+1}|v|^{\beta+1}dx\right)^{-1}\left(\frac{\alpha+1}{p}\int_Q |\nabla u|^p dx + \frac{\beta+1}{q}\int_Q |\nabla v|^q dx\right),$$

which implies (14). Obviously, the estimate of this nature can be obtained by applying the Hölder inequality and the Sobolev inequality, stated below, to the product $|u|^{\alpha+1}|v|^{\beta+1}$. However, inequality (14) involves the optimal constant λ_1 wich is the smallest eigenvalue of problem (13).

In the sequel we shall refer to the following estimate: for every $u \in \overset{\circ}{W}{}^{1,p}(Q)$ we have (see [3], p.45)'

$$||u||_s \le c|Q|^{\frac{1}{s}-\frac{1}{p^*}}||\nabla u||_p \quad (15)$$

for $1 \le s \le p^*$, where $c > 0$ is a constant depending on N and p, and $|Q|$ denotes the Lebesgue measure of Q.

Proposition 1. *Suppose that $\lambda < \lambda_1$. Then the functional J satisfies the Palais-Smale condition.*

Proof. Let $\{(u_m, v_m)\} \subset X$ be a sequence such that $J(u_m, v_m)$ is bounded and $J'(u_m, v_m) \to 0$ in X^* as $m \to \infty$. First we show that the sequence $\{(u_m, v_m)\}$ is bounded in X. This is obvious if $\lambda < 0$, so we only consider the case $0 < \lambda < \lambda_1$. It follows from the Hölder inequality and (15) that

$$\left| \int_Q f u_m dx \right| \leq ||f||_{p'} ||u_m||_p \leq c||f||_{p'} |Q|^{\frac{1}{N}} ||\nabla u_m||_p.$$

Hence by the Young inequality we obtain

$$\left| \int_Q f u_m dx \right| \leq \frac{c^{p'}}{p' \varepsilon^{p'}} ||f||_{p'}^{p'} |Q|^{\frac{p'}{N}} + \frac{\varepsilon^p}{p} \int_Q |\nabla u_m|^p dx. \tag{16}$$

Similarly, we have

$$\left| \int_Q g v_m dx \right| \leq \frac{c^{q'}}{q' \varepsilon^{q'}} ||g||_{q'}^{q'} |Q|^{\frac{q'}{N}} + \frac{\varepsilon^q}{q} \int_Q |\nabla v_m|^q dx. \tag{17}$$

Consequently, using (14), (16) and (17) we obtain the following estimate

$$\begin{aligned} J(u_m, v_m) \geq & \left(\frac{\alpha + 1}{p} - \frac{\varepsilon^p}{p} \right) \int_Q |\nabla u_m|^p dx \\ & + \left(\frac{\beta + 1}{q} - \frac{\varepsilon^p}{q} \right) \int_Q |\nabla v_m|^q dx \\ & - \frac{\lambda}{\lambda_1} \int_Q \left(\frac{(\alpha+1)}{p} |\nabla u_m|^p + \frac{(\beta+1)}{q} |\nabla v_m|^q \right) dx \\ & - \frac{c^{p'} |Q|^{\frac{p'}{N}}}{p' \varepsilon^{p'}} ||f||_{p'}^{p'} - \frac{c^{p'} |Q|^{\frac{q'}{N}}}{q' \varepsilon^{q'}} ||g||_{q'}^{q'}. \end{aligned}$$

Since $\lambda < \lambda_1$ we can choose $\varepsilon > 0$ so that

$$\alpha+1-\varepsilon^p-\frac{\lambda}{\lambda_1}(\alpha+1)>0$$

and

$$\beta+1-\varepsilon^q-\frac{\lambda}{\lambda_1}(\beta+1)>0.$$

This implies that the sequence $\{(u_m,v_m)\}$ is bounded in X. We may assume that $(u_m,v_m)\rightharpoonup(u,v)$ in X, $(u_m,v_m)\to(u,v)$ in $L^p(Q)\times L^q(Q)$ and a.e. on Q. It is obvious that

$$\lim_{m\to\infty}\int_Q|u_m|^{\alpha+1}|v_m|^{\beta+1}dx=\int_Q|u|^{\alpha+1}|v|^{\beta+1}dx.$$

To show that the sequence $\{(u_m,v_m)\}$ is relatively compact in X we use the following algebraic inequality [5]

$$|\nabla u_m-\nabla u_l|^p\leq C\big\{\big[|\nabla u_m|^{p-2}D_iu_m-|\nabla u_l|^{p-2}D_iu_l\big](D_iu_m-D_iu_l)\big\}^{\frac{s}{2}}\times (|\nabla u_m|^p+|\nabla u_l|^p)^{(1-\frac{s}{2})} \tag{18}$$

with

$$s=\begin{cases}p & \text{for } 1<p\leq 2,\\ 2 & \text{for } 2<p,\end{cases}$$

where $C>0$ is a constant independent of m and l. We now observe that

$$(\alpha+1)\int_Q\big(|\nabla u_m|^{p-2}D_iu_m-|\nabla u_l|^{p-2}D_iu_l\big)(D_iu_m-D_iu_l)dx$$

$$=\langle J'(u_m,v_m)-J'(u_l,v_l),(u_m-u_l,0)\rangle$$

$$+ (\alpha+1)\int_Q \big(|u_m|^{\alpha-1}u_m|v_m|^{\beta+1}$$

$$- |u_l|^{\alpha-1}u_l|v_l|^{\beta+1}\big)(u_m-u_l)dx + (\alpha+1)\int_Q f(u_m-u_l)dx = A_{ml}.$$

Since $u_m \to u$ in $L^p(Q)$ and $\langle J'(u_m,v_m)-J'(u_l,v_l),(u_m-u_l,0)\rangle \to 0$ as $m,l\to\infty$, using the Hölder inequality we check that $A_{ml}\to 0$ as $m,l\to\infty$. Similarly, we have

$$B_{ml} = \langle J'(u_m,v_m)-J'(u_l,v_l),(0,v_m-v_l)\rangle$$

$$+ (\beta+1)\int_Q \big(|u_m|^{\alpha+1}|v_m|^{\beta-1}v_m$$

$$- |u_l|^{\alpha+1}|v_l|^{\beta-1}v_l\big)(v_m-v_l)dx + (\beta+1)\int_Q g(v_m-v_l)dx \to 0$$

as $m,l\to\infty$. In then follows from (18) that

$$||\nabla u_m - \nabla u_l||_p^p \le C|A_{ml}|^{\frac{s}{2}}(||\nabla u_m||_p^p + ||\nabla u_l||_p^p)^{1-\frac{s}{2}}$$

and

$$||\nabla v_m - \nabla v_l||_q^q \le C|B_{ml}|^{\frac{s}{2}}(||\nabla v_m||_q^q + ||\nabla v_l||_q^q)^{1-\frac{s}{2}}$$

and this completes the proof.

The existence result for system (1) is obtained by a global minimization of the functional J.

Theorem 2. *Suppose that $\lambda < \lambda_1$. Then for each $(f,g)\in L^{p'}(Q)\times L^{q'}(Q)$, with $(f,g)\not\equiv(0,0)$, system (1) has at least one solution in X.*

Proof. If $0<\lambda<\lambda_1$, then by virtue of (14) we have

$$J(u,v) \geq \left(1 - \frac{\lambda}{\lambda_1}\right) \int_Q \left(\frac{(\alpha+1)}{p}|\nabla u|^p + \frac{(\beta+1)}{q}|\nabla v|^q\right)dx$$

$$- (\alpha+1)\int_Q fudx - (\beta+1)\int_Q gvdx,$$

while for $\lambda < 0$ we have

$$J(u,v) \geq \int_Q \left(\frac{(\alpha+1)}{p}|\nabla u|^p + \frac{(\beta+1)}{q}|\nabla v|^q\right)dx$$

$$- (\alpha+1)\int_Q fudx - (\beta+1)\int_Q gvdx.$$

In both cases J is bounded from below on X. Without loss of generality we may assume that $f \not\equiv 0$. It is obvious that there exists $(u_1, v_1) \in X$ such that

$$\int_Q fu_1 dx > 0 \text{ and } \int_Q gv_1 dx \geq 0.$$

Hence

$$J(tu_1, tv_1) = t\left[\frac{t^{p-1}(\alpha+1)}{p}\int_Q |\nabla u_1|^p dx + \frac{t^{q-1}(\beta+1)}{q}\int_Q |\nabla v_1|^q dx\right.$$

$$- \lambda t^{\alpha+\beta+1}\int_Q |u_1|^{\alpha+1}|v_1|^{\beta+1}dx$$

$$\left. -(\alpha+1)\int_Q fu_1 dx - (\beta+1)\int_Q gv_1 dx\right] < 0$$

for $t > 0$ sufficiently small and consequently

$$M = \inf_{(u,v)\in X} J(u,v) < 0.$$

According to the Ekeland variational principle [3] there exists a sequence $\{(u_m, v_m)\} \subset X$ such that $J(u_m, v_m) \to M$ and $J'(u_m, v_m) \to 0$ in X^* as $m \to \infty$. Since by Proposition 1, J satisfies the Palais-Smale condition , $\{(u_m, v_m)\}$ must be relatively compact in X. Therefore $(u_m, v_m) \to (u, v)$ in X up to a subsequence and (u, v) is a solution of (1) with $J(u, v) < 0$.

If $f \not\equiv 0$ and $g \equiv 0$ on Q, then by a remark made at the end of Section 1, $(u_0, 0)$ is a solution of system (1), where u_0 is a unique solution of equation (2).

In Proposition 2 below we relate this observation to a global minimization of J.

Proposition 2. *Suppose that $f \in L^{p'}(Q)$, with $f \not\equiv 0$ on Q, and $g \equiv 0$ on Q.*

(i) If $0 < \lambda < \lambda_1$, then there exist two distinct solutions in X of system (1).
(ii) If $\lambda < 0$, then $J(u_0, 0) = \inf\limits_{(u,v)\in X} J(u, v)$.

Proof. (i) If $u_0 \in \overset{\circ}{W}{}^{1,p}(Q)$ is a solution of equation (2), then $\int_Q |\nabla u_0|^p dx = \int_Q f u_0 dx$ and hence for each $v \in \overset{\circ}{W}{}^{1,q}(Q)$ we have

$$J(u_0, v) =$$

$$-\frac{\alpha+1}{p'} \int_Q |\nabla u_0|^p dx - \lambda \int_Q |u_0|^{\alpha+1} |v|^{\beta+1} dx + \frac{\beta+1}{q} \int_Q |\nabla v|^q dx.$$

If $v \not\equiv 0$ and $t > 0$ is sufficiently small, then

$$J(u_0, tv) = -\frac{\alpha+1}{p'} \int_Q |\nabla u_0|^p dx$$

$$+ t^{\beta+1} \Big[-\lambda \int_Q |u_0|^{\alpha+1} |v|^{\beta+1} dx$$

$$+\frac{\beta+1}{q}t^{q-\beta-1}\int_Q |\nabla v|^q dx\Big] < J(u_0,0).$$

This imples that

$$\inf_{(u,v)\in X} J(u,v) < J(u_0,0)$$

and by Theorem 2 there exists a minimizer $(\bar{u},\bar{v}) \in X$ such that $J(\bar{u},\bar{v}) = \inf_{(u,v)\in X} J(u,v)$ and $(\bar{u},\bar{v}) \neq (u_0,0)$.

(ii) If $\lambda < 0$, then for each $(u,v) \in X$ we have

$$J(u_0,0) = (\alpha+1) \inf_{w\in \mathring{W}^{1,p}(Q)} \left(\frac{1}{p}\int_Q |\nabla w|^p dx - \int_Q fw dx\right) \leq J(u,v)$$

and the assertion (ii) readily follows.

4. CASE $\frac{\alpha+1}{p^*} + \frac{\beta+1}{q^*} < 1$ and $\frac{\alpha+1}{p} + \frac{\beta+1}{q} > 1$

Repeating the argument of Theorem 2 in [7] we can easily show that the functional J satisfies the Palais-Smale condition.

Proposition 3. *Let $\lambda \in \mathbb{R}$ and let $(f,g) \in L^{p'}(Q) \times L^{q'}(Q)$. Then the functional J satisfies the Palais-Smale condition.*

If $\lambda > 0$, then system (1) has a mountain pass solution.

Theorem 3. *Let $0 < \lambda < \infty$ and let $(f,g) \in L^{p'}(Q) \times L^{q'}(Q)$. Then there exists a constant $m = m(p,q,\alpha,\beta,\lambda) > 0$ such that if $||f||_{p'} + ||g||_{q'} \leq m$, then system (1) has a solution $(\bar{u},\bar{v}) \in X$ with $J(\bar{u},\bar{v}) > 0$.*

Proof. Applying (15) and the Young and Hölder inequalities we get

$$J(u,v) \geq \frac{\alpha+1}{p}\int_Q |\nabla u|^q dx + \frac{\beta+1}{q}\int_Q |\nabla v|^q dx$$

$$-\lambda\Big(\frac{1}{r}\int_Q |u|^{(\alpha+1)r}dx + \frac{1}{r'}\int_Q |v|^{(\beta+1)r'}dx\Big)$$

$$-(\alpha+1)||f||_{p'}||u||_p - (\beta+1)||g||_{g'}||v||_q$$

$$\geq \frac{\alpha+1}{p}\int_Q |\nabla u|^p dx + \frac{\beta+1}{q}\int_Q |\nabla v|^q dx$$

$$-\Big(\frac{c^{(\alpha+1)r}}{r}|Q|^{\frac{1}{(\alpha+1)r}-\frac{1}{p^*}}||\nabla u||_p^{(\alpha+1)r} +$$

$$+\frac{c^{(\beta+1)r}}{r'}|Q|^{\frac{1}{(\beta+1)r'}-\frac{1}{q^*}}||\nabla v||_q^{(\beta+1)r'}\Big)$$

$$-c|Q|^{\frac{1}{N}}\big((\alpha+1)||f||_{p'}||\nabla u||_p + (\beta+1)||g||_{q'}||\nabla v||_q\big).$$

where $r(\alpha+1) > p, r'(\beta+1) > q$ and $\frac{1}{r}+\frac{1}{r'} = 1$. Letting $||\nabla u||_p = s_1$ and $||\nabla v||_q = s_2$ we write

$$J(u,v) \geq \frac{\alpha+1}{p}s_1^p + \frac{\beta+1}{q}s_2^q - \lambda\big(as_1^{r(\alpha+1)} + bs_2^{r'(\beta+1)}\big)$$

$$-c|Q|^{\frac{1}{N}}\big((\alpha+1)||f||_{p'}s_1 + (\beta+1)||g||_{q'}s_2\big),$$

where $a = \frac{c^{(\alpha+1)r}}{r}|Q|^{\frac{1}{(\alpha+1)r}-\frac{1}{p^*}}$ and $b = \frac{c^{(\beta+1)r'}}{r'}|Q|^{\frac{1}{(\beta+1)r'}-\frac{1}{q^*}}$. We next define a function

$$h(s_1,s_2) = \frac{1}{s_1+s_2}\Big[\frac{\alpha+1}{p}s_1^p + \frac{\beta+1}{q}s_2^q - \lambda\big(as_1^{r(\alpha+1)} + bs_2^{r'(\beta+1)}\big)\Big]$$

for $s_1 > 0$ and $s_2 > 0$ and write the last estimate in the form

$$J(u,v) \geq (s_1+s_2)\big[h(s_1,s_2) - c|Q|^{\frac{1}{N}}((\alpha+1)||f||_{p'} + (\beta+1)||g||_{q'})\big]$$

Since $r(\alpha+1) > p$ and $r'(\beta+1) > q$, for a given $\kappa > 0$ sufficiently small, there corresponds a constant $\rho = \rho(p,q,\alpha,\beta,\lambda,|Q|,\kappa) > 0$ such that $h(s_1,s_2) \geq \rho$ for $s_1 + s_2 = \kappa$ with $s_1 \geq 0$ and $s_2 \geq 0$. Taking $m = \frac{\rho}{2c|Q|^{\frac{1}{N}} \max\left((\alpha+1),(\beta+1)\right)}$, we see that

$$J(u,v) \geq \frac{\rho\kappa}{2} \text{ for } ||\nabla u||_p + ||\nabla v||^q = \kappa \text{ and } ||f||_{p'} + ||g||_{q'} \leq m.$$

Let $(u_1,v_1) \in X$ with $u_1 \not\equiv 0$ and $v_1 \not\equiv 0$, then

$$J(t^{\frac{1}{p}}u_1, t^{\frac{1}{q}}v_1)) = t\left(\frac{\alpha+1}{p}\int_Q |\nabla u|^p dx + \frac{\beta+1}{q}\int_Q |\nabla v_1|^q dx\right)$$

$$- \lambda t^{\frac{\alpha+1}{p}+\frac{\beta+1}{q}} \int_Q |u|^{\alpha+1}|v|^{\beta+1} dx$$

$$- t^{\frac{1}{p}}(\alpha+1)\int_Q f u_1 dx - t^{\frac{1}{q}}(\beta+1)\int_Q g v_1 dx < 0$$

for $t > 0$ sufficiently large. Hence we can choose $t_0 > 0$ so that $u_0 = t_0^{\frac{1}{p}} u_1, v_0 = t_0^{\frac{1}{q}} v_q$ satisfy: $J(u_0,v_0) < 0$ and $(u_0,v_0) \notin B(0,\kappa)$. Since J satisfies the Palais-Smale condition, we deduce from the mountain pass theorem [1] the existence of a critical point $(\bar{u},\bar{v}) \in X$ of J such that $J(\bar{u},\bar{v}) \geq \frac{\rho\kappa}{2}$ and this completes the proof.

A second solution of (1) will be obtained by a local minimization of J.

Theorem 4. *(i) Suppose that $\lambda < 0$ and $(f,g) \in L^{p'}(Q) \times L^{q'}(Q)$ with $(f,g) \not\equiv (0,0)$. Then system (1) has a solution $(u^*,v^*) \in X$ such that $J(u^*,v^*) < 0$.*

(ii) Suppose that $0 < \lambda < \infty$ *and* $(f,g) \in L^{p'}(Q) \times L^{q'}(Q)$ *with* $(f,g) \not\equiv (0,0)$ *and* $\|f\|_{p'} + \|g\|_{q'} \le m$*, where* m *is a constant from Theorem 3. Then system (1) has a solution* $(u^*,v^*) \in X$ *such that* $J(u^*,v^*) < 0$.

Proof. (i) We may assume that there exists $(\varphi,\psi) \in X$ such that $\int_Q f\varphi dx > 0$ and $\int_Q g\psi dx \ge 0$. Then $J(t\varphi,t\psi) < 0$ for $t > 0$ sufficiently small. Since $\lambda < 0$, we have for each $(u,v) \in X$ the estimate

$$J(u,v) \ge \frac{\alpha+1}{p}\int_Q |\nabla u|^p dx + \frac{\beta+1}{q}\int_Q |\nabla v|^q dx$$

$$- (\alpha+1)\int_Q fudx - (\beta+1)\int_Q gvdx,$$

which implies that J is bounded from below on X. With the aid of Proposition 3 and the Ekeland variational principle, we show, as in part (i) of Theorem 2, that there exists $(u^*,v^*) \in X$ such that $J(u^*,v^*) = \inf_{(u,v)\in X} J(u,v) < 0$.

(ii) Let $(\varphi,\psi) \in X$ be as in part (i). Since $J(t_\varphi,t_\psi) < 0$ for $t > 0$ sufficiently small, we must have

$$\inf_{\|\nabla u\|_p + \|\nabla v\|_q \le \kappa} J(u,v) < 0,$$

where $\kappa > 0$ is a constant from the proof of Theorem 3. Inspection of the proof of Theorem 3 shows that for a given $\varepsilon > 0$ there exists $\delta > 0$ such that $J(u,v) \ge -\varepsilon$ for every $(u,v) \in X$ satisfying $\kappa - \delta \le \|\nabla u\|_p + \|\nabla v\|_q \le \kappa$. This observation implies that

$$J(u,v) \ge \frac{1}{2} \inf_{\|\nabla u\|_p + \|\nabla v\|_q \le \kappa} J(u,v) \tag{19}$$

for all $(u,v) \in X$ satisfying $\kappa_1 \le \|\nabla u\|_p + \|\nabla v\|_q \le \kappa$ for some $\kappa_1 < \kappa$. Let $\{(u_j,v_j)\} \subset X$ be a minimizing sequence for $\inf_{\|\nabla u\|_p + \|\nabla v\|_q \le \kappa} J(u,v)$. By virtue of (19) we may assume that $\{(u_j,v_j)\} \subset B(0,\kappa)$. Let a closed ball $\overline{B(0,\kappa)}$ in X be equipped with a metric dist $((u,v),(u_1,v_1)) = \|\nabla(u-u_1)\|_p + \|\nabla(v-v_1)\|_q$ for $(u,v),(u_1,v_1) \in \overline{B(0,\kappa)}$. It is clear that $\overline{B(0,\kappa)}$ with this distance is a complete metric space. According to the Ekeland variational principle we may assume that every (u_j, v_j) is a minimizer for

$$\inf\{J(u,v) + \delta_j(\|\nabla(u_j - u)\|_p + \|\nabla(v_j - v)\|_q); (u,v) \in \overline{B(0,\kappa)}\}$$

for some $\delta_j > 0$ with $\delta_j \to 0$ as $j \to \infty$. This implies that $J'(u_j, v_j) \to 0$ in X^* as $j \to \infty$. It then follows from Proposition 3 that there exists $(u^*, v^*) \in X$ such that $J(u^*, v^*) = \inf_{(u,v)\in\overline{B(0,\kappa)}} J(u,v)$.

In case $0 < \lambda < \infty$ we deduce from Theorems 3 and 4 the following multiplicity result.

Corollary 1. *Let* $0 < \lambda < \infty$ $(f,g) \in L^{p'}(Q) \times L^{q'}(Q)$ *with* $(f,g) \not\equiv (0,0)$. *Then there exists a constant* $m > 0$ *such that for* $\|f\|_{p'} + \|g\|_{q'} \leq m$, *system (1) has at least two distinct solutions.*

5. CASE $\frac{\alpha+1}{p} + \frac{\beta+1}{q} < 1$

A solution of system (1) will be obtained by a global minimization of J on X.

Proposition 4. *Let* $\lambda \in \mathbb{R}$. *Then* J *satisfies the Palais-Smale condition.*

Proof. Let $\{(u_m, v_m)\} \subset X$ be a sequence such that $J(u_m, v_m)$ is bounded and $J'(u_m, v_m) \to 0$ in X^*. We have

$$J(u_m, v_m) - \langle J'(u_m, v_m), \left(\frac{u_m}{p}, \frac{v_m}{q}\right)\rangle$$

$$= \lambda\left(1 - \frac{\alpha+1}{p} - \frac{\beta+1}{q}\right)\int_Q |u_m|^{\alpha+1}|v_m|^{\beta+1}dx$$

$$- \frac{\alpha+1}{p'}\int_Q f u_m dx - \frac{\beta+1}{q'}\int_Q g v_m dx,$$

wich implies that

$$-M - \varepsilon_m \|(u_m, v_m)\|_X$$

$$\leq \lambda\left(1 - \frac{\alpha+1}{p} - \frac{\beta+1}{q}\right) \int_Q |u_m|^{\alpha+1} |v_m|^{\beta+1} dx$$

$$- \frac{\alpha+1}{p'} \int_Q f u_m dx - \frac{\beta+1}{q'} \int_Q g v_m dx \leq M + \varepsilon_m \|(u_m, v_m)\|_X$$

for all m, where $M > 0$ and $\varepsilon_m \to 0$. This, combined with the fact that J is bounded, implies that the sequence $\{(u_m, v_m)\}$ is bounded in X. Consequently we may assume that $(u_m, v_m) \rightharpoonup (u, v)$ in X, $(u_m, v_m) \to (u, v)$ in $L^{p'}(Q) \times L^{q'}(Q)$. Since $\frac{\alpha+1}{p} + \frac{\beta+1}{q} < 1$, it is easy to show that $\int_Q |u_m|^{\alpha+1} |v_m|^{\beta+1} dx \to \int_Q |u|^{\alpha+1} |v|^{\beta+1} dx$ and we can complete the proof as in Proposition 1.

As in Theorem 1 we find a solution of system 1 by a global minimization of J on X.

Theorem 5. *Let $\lambda \in \mathbb{R}$ and let $((f, g) \in L^{p'}(Q) \times L^{q'}(Q)$ with $(f, g) \not\equiv (0, 0)$. Then system (1) has a solution in X.*

If one of functions f or g is identically zero we have a multiplicity result:

Proposition 5. *Suppose that $f \in L^{p'}(Q)$ with $f \not\equiv 0$ and $g \equiv 0$ on Q.*

(i) If $0 < \lambda < \infty$, then there exist two distinct solutions in X of system (1).

(ii) If $\lambda < 0$, then $J(u_0, 0) = \inf_{(u,v) \in X} J(u, v)$, where $u_0 \in \overset{\circ}{W}{}^{1,p}(Q)$ is a solution of equation (2).

6. NONHOMOGENEOUS p-LAPLACIAN INVOLVING A CRITICAL SOBOLEV EXPONENT

The aim of this section is to establish the existence of a solution in $\overset{\circ}{W}{}^{1,p}(Q)$ of the equation

$$-\Delta_p u = |u|^{p^*-2}u + f \text{ in } Q, \tag{20}$$

where $f \in W^{-1,p'}(Q)$ and $f \not\equiv 0$. A solution will be obtained by a constrained minimization of a variational functional $F : \overset{\circ}{W}{}^{1,p}(Q) \to \mathbb{R}$ given by

$$F(u) = \frac{1}{p}\int_Q |\nabla u|^p dx - \frac{1}{p^*}\int_Q |u|^{p^*} dx - \int_Q fu dx.$$

Let

$$M = \{u \in \overset{\circ}{W}{}^{1,p}(Q) - \{0\};\ \langle F'(u), u\rangle = 0\},$$

then

$$F_{|M}(u) = \frac{1}{N}\int_Q |\nabla u|^p dx - \left(1 - \frac{1}{p^*}\right)\int_Q fu dx.$$

It is easy to see that F is bounded on M and that

$$m_F = \inf\{F(u); u \in M\} < 0.$$

We put

$$\kappa = \frac{S^{\frac{p^*}{p'(p^*-p)}}(p-1)^{\frac{p}{p'(p^*-p)}}}{(p^*-1)^{1+\frac{p}{p'(p^*-p)}}}.$$

Theorem 6. *If $||f||_{W^{-1,p'}} < \kappa$ and $f \not\equiv 0$, the there exists at least one solution of equation (20).*

Proof. It follows from the Ekeland variational principle that there exists a sequence $\{u_m\} \subset M$ such that

$$F(u_m) \to m_F \text{ and } F'|_M(u_m) \to 0 \text{ in } W^{-1,p'}(Q)$$

as $m \to \infty$. As in the proof of Theorem 1 we show that

$$F'(u_m) \to 0 \text{ in } W^{-1,p'}(Q)$$

as $m \to \infty$. Towards this end we put

$$G(u) = \langle F'(u), u \rangle \text{ for } u \in \overset{\circ}{W}{}^{1,p}(Q)$$

and we show that $G'(u) \neq 0$ for all $u \in M$ and that

$$|\langle G'(u_m), u_m \rangle| \geq \delta \tag{21}$$

for some $\delta > 0$ and for all m. Since the proofs of both claims are similar, we only show (21). In the contrary case, $\lim_{m\to\infty} \langle G'(u_m), u_m \rangle = 0$ up to a subsequence. Since $\{u_m\}$ is a bounded sequence in $\overset{\circ}{W}{}^{1,p}(Q)$ we may assume that

$$\lim_{m\to\infty} \int_Q |\nabla u_m|^p dx = A, \lim_{m\to\infty} \int_Q |u_m|^{p^*} dx = B \text{ and}$$

$$\lim_{m\to\infty} \int_Q f u_m dx = C.$$

Constants A, B and C satisfy the following system of equations

$$\begin{cases} \frac{1}{p}A - \frac{1}{p^*}B - C &= m_F \\ A - B - C &= 0 \\ pA - p^*B - C &= 0, \end{cases}$$

whose solution is given by

$$A = -m_F \frac{2p^*}{p^* - 2p - 2}, \quad B = -m_F \frac{2(p-1)p^*}{(p^*-1)(p^*-2p+2)}$$

and

$$C = -m_F \frac{2p^*(p^*-p)}{(p^*-1)(p^*-2p+2)}.$$

By the Sobolev inequlity we have

$$S\left(\int_Q |u_m|^{p^*} dx\right)^{\frac{p}{p^*}} \leq \int_Q |\nabla u_m|^p dx$$

and letting $m \to \infty$ we get

$$SB^{\frac{p}{p^*}} \leq A.$$

Using the formulae for A and B we derive from this inequality

$$S^{\frac{p^*}{p'(p^*-p)}}(2p^*)^{-\frac{1}{p'}}(p^*-2p+2)^{\frac{1}{p'}}(p-1)^{\frac{p}{p'(p^*-p)}}(p^*-1)^{-\frac{p}{p'(p^*-p)}} \leq (-m_F)^{\frac{1}{p'}}. \tag{22}$$

On the other hand, since

$$\int_Q f u_m dx \leq ||f||_{W^{-1,p'}} ||\nabla u_m||_p$$

we get, letting $m \to \infty$, that

$$C \leq ||f||_{W^{-1,p'}} A^{\frac{1}{p}}$$

This inequality implies that

$$(-m_F)^{\frac{1}{p'}}(2p^*)^{\frac{1}{p'}}(p^*-p)(p^*-1)(p^*-2p+2)^{-\frac{1}{p'}} \leq ||f||_{W^{-1,p'}}. \tag{23}$$

Combining (22) and (23) we obtain

$$S^{\frac{p^*}{p'(p^*-p)}} \frac{(p^*-p)(p-1)^{\frac{p}{p'(p^*-p)}}}{(p^*-1)^{1+\frac{p}{p'(p^*-p)}}} \leq ||f||_{W^{-1,p'}},$$

which is impossible. Therefore (21) holds. Since $G'(u_m) \neq 0$ on M, for each u_m there exists $\lambda_m \in \mathbb{R}$ such that

$$F'(u_m) = F|_M(u_m) - \lambda_m G'(u_m). \tag{24}$$

We now observe that

$$0 = \langle F'(u_m), u_m \rangle = \langle F'_{|M}(u_m), u_m \rangle - \lambda_m \langle G'(u_m), u_m \rangle,$$

and this implies that $\lambda_m \to 0$. This, combined with (24), yields that $F'(u_m) \to 0$ in $W^{-1,p'}(Q)$ as $m \to \infty$. The rest of the proof is similar to that of Theorem 1 and therefore is omitted.

Finally, we note that if $p = 2$, then

$$\kappa = S^{\frac{N}{4}} \frac{4}{N-2} \left(\frac{N-2}{N+2} \right)^{\frac{N+2}{4}}$$

and we recover a result from paper [5] (see Theorem 1 there).

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Department of Mathematics,
The University of Queensland,
St. Lucia 4072, Qld, Australia

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