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On Some Weak Monomorphisms and Weak Epimorphisms of Pro-HTop*

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ABSTRACT. Related to Shape Theory, in a previous paper [6] we studied weak monomorphisms and weak epimorphisms in the category of pro-groups. In this note we give some intrinsic characterizations of the weak monomorphisms and the weak epimorphisms in pro-HTop* in the case when one of the two objects of such a morphism is a rudimentary system.

1. INTRODUCTION

If C is a category with zero-objects then a morphism $f: A \to B$ of C is a weak monomorphism if $f \circ u = 0$ implies $u = 0$. A morphism $f: A \to B$ is called a weak epimorphism if $u \circ f = 0$ implies $u = 0$.

Weakened versions of categorical notions of monomorphism and epimorphism have proved to be of some interest in pointed homotopy theory. A study of the comparison between weak monomorphism and monomorphism in homotopy theory was carried by T.Ganea [3] who, in

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particular, obtained examples of weak monomorphisms which are not monomorphisms. Examples of homotopy weak monomorphisms which are not homotopy epimorphisms have been given by J.Roitberg [7]. Certainly, the study of shape monomorphisms and epimorphisms and their weakened versions can be interesting (see, for the homotopy case, the recently papers of E.Dyer & J.Roitberg [2] and J.Dydak [1]). In [6] we characterized weak monomorphisms and weak epimorphisms in the category of pro-groups and we defined the notion of weakly exact sequence and we studied this notion in the category of pro-groups.

In this note we consider the pro-category of HTop*, the homotopy category of pointed topologal spaces, and we give some intrinsic characterizations of weak monomorphisms and weak epimorphisms $\underline{f}: \underline{X} \to \underline{Y}$ in pro-HTop*, when $\underline{X}$ or $\underline{Y}$ is a rudimentary system. These results can be interesting (and maybe sufficient) so a shape morphism $F: X \to Y$ between topological spaces X and Y can be given by means of such morphisms $\underline{f}: \underline{X} \to \underline{Y}$ in pro-HTop* (approaching morphisms). The study of an arbitrary morphism $f: \underline{X} \to \underline{Y}$ of pro-HTop* is more complicated.

The notions and properties of pro-categories which are used in this paper are those of the book of S.Mardešić and J.Segal [4].

2. WEAK MONOMORPHISMS IN THE CATEGORY PRO-HTOP*

The category pro-HTop* is a category with zero objects. A zero-object is a single point rudimentary system.

If $(X,*)$ is a rudimentary system in pro-HTop* and if $\underline{Y} = ((Y_\lambda,*), q_{\lambda\lambda'}, \Lambda)$ is an arbitrary object in pro-HTop*, then the morphisms $\underline{f} = (f_\lambda): (X,*) \to \underline{Y}$ coincide with the morphisms in inv-HTop*, the category of inverse systems in HTop* [4, p.20]. This means that for each $\lambda \in \Lambda$ is given a morphism $f_\lambda: (X,*) \to (Y_\lambda,*)$ in HTop* and for each pair $\lambda \leq \lambda'$ we have $q_{\lambda\lambda'} f_{\lambda'} = f_\lambda$.

Lemma 1. *For a morphism $\underline{f}: (X,*) \to \underline{Y} = ((Y_\lambda,*), q_{\lambda\lambda'}, \Lambda)$ in pro-HTop*, there exist an object $\underline{P} = ((P_\lambda,*), r_{\lambda\lambda'}, \Lambda)$ and two morphisms $\underline{p} = (p_\lambda, 1_\Lambda): \underline{P} \to \underline{Y}$, $\underline{h} = (h_\lambda): (X,*) \to \underline{P}$ such that for each $\lambda \in \Lambda$:*

(i) $h_\lambda: (X,*) \to (P_\lambda,*)$ is a pointed homotopy equivalence,

(ii) $p_\lambda : (P,*) \to (Y_\lambda,*)$ *is a pointed fiber map,*

(iii) $f_\lambda = p_\lambda \circ h_\lambda$.

Proof. The existence for each $\lambda \in \Lambda$ of a factorization (iii), satisfying (i) and (ii), is well known [5, p.249]. For a pair $\lambda \leq \lambda'$ in Λ we define $r_{\lambda\lambda'} = h_\lambda \circ h_{\lambda'}^{-1}$, for which is immediate that $\underline{P} = ((P_\lambda,*), r_{\lambda\lambda'}, \Lambda)$ is an inverse system in HTop* and that $\underline{h} = (h_\lambda) : (X,*) \to \underline{P}$ is a morphism in pro-HTop*. Also, from the relations $q_{\lambda\lambda'} \circ f_{\lambda'} \approx f_\lambda, f_\lambda = p_\lambda h_\lambda, f_{\lambda'} = p_{\lambda'} \circ h_{\lambda'}$, we deduce that $q_{\lambda\lambda'} p_\lambda \approx p_{\lambda'} \circ r_{\lambda\lambda'}$, which shows that $\underline{p} = (p_\lambda, 1_\Lambda) : \underline{P} \to \underline{Y}$ a morphism of pro-HTop*.

Remark 1. It is obvious from Lemma 1 that we can write the equality $\underline{f} = \underline{p} \circ \underline{h}$, in the category pro-HTop*, where $\underline{h}$ is an isomorphism. Then it is clear that $\underline{f}$ is a weak monomorphism if and only if $\underline{p}$ is a weak monomorphism. We will refer to the morphism $\underline{p} : \underline{P} \to \underline{Y}$ as the *fibred factor* of the morphism $\underline{f}$.

Remark 2. If $\underline{p} = (p_\lambda, 1_\Lambda) : \underline{P} = ((P_\lambda,*), r_{\lambda\lambda'}, \Lambda) \to \underline{Y} = ((Y_\lambda,*), p_{\lambda\lambda'}, \Lambda)$ is a fibred factor in pro-HTop*, we can consider the *fiber* of this morphism by the object $\underline{F} = ((F_\lambda,*), r'_{\lambda\lambda'}, \Lambda)$, where $F_\lambda = p_\lambda^{-1}(*)$ and $r'_{\lambda\lambda'} = r_{\lambda\lambda'}/F_{\lambda'}$, for $\lambda \leq \lambda'$. Then we can define a morphism $\underline{i} = (i_\lambda, 1_\Lambda) : \underline{F} \to \underline{P}$, where i_λ is the inclusion of $(F_\lambda,*)$ in $(P_\lambda,*)$.

Definition 1. *We will say that the fiber* $\underline{F} = ((F_\lambda,*), r'_{\lambda\lambda'}, \Lambda)$ *of the fibered factor* $\underline{p} = (p_\lambda, 1_\Lambda) : \underline{p} = ((P_\lambda,*), r_{\lambda\lambda'}, \Lambda) \to \underline{Y} = ((Y_\lambda,*), p_{\lambda\lambda'}, \Lambda)$ *is contractible in* $\underline{P}$ *if for each* $\lambda \in \Lambda$ *there is a* $\lambda' \geq \lambda$ *such that* $i_\lambda \circ r'_{\lambda\lambda'} \approx *$.

Theorem 1. *A morphism* $\underline{f} = (f_\lambda) : (X,*) \to \underline{Y}$ *is a weak monomorphism in the category pro-HTop* if and only if the fibre* $\underline{F}$ *of every fibred factor* $\underline{p} = \underline{P} \to \underline{Y}$ *is contractible in* $\underline{P}$.

Proof. By Remark 1 it is sufficient to prove that $\underline{p}$ is a weak monomorphism if and only if $\underline{F}$ is contractible in $\underline{P}$.

Suppose that $\underline{p} : \underline{P} \to \underline{Y}$ is a weak monomorphism in the category pro-HTop*. For the morphism $\underline{i} : \underline{F} \to \underline{P}$ from Remark 2 we have $\underline{p} \circ \underline{i} = \underline{*}$ and by hypothesis it follows $\underline{i} = \underline{*}$. If $\underline{*} = (*, \Phi)$ then we have an equivalence $(i_\lambda, 1_\Lambda) \sim (*, \Phi)$ [4, p.6] which implies that for each

$\lambda \in \Lambda$ there is $\lambda' \geq \lambda$ (and $\lambda' \geq \Phi(\lambda)$) such that the following diagram in HTop* commutes

$$\begin{array}{ccc} F_{\Phi(\lambda)} & \xleftarrow{\;r'_{\Phi(\lambda)\lambda'}\;} & F_{\lambda'} \\ *\downarrow & & \downarrow r_{\lambda\lambda'} \\ P_\lambda & \xleftarrow{\;i_\lambda\;} & F_\lambda \end{array}$$

This implies that $i_\lambda o r'_{\lambda\lambda'}$ is pointed null-homotopic, i.e. $i_\lambda o r'_{\lambda\lambda'} \approx *$. Thus, $\underline{F}$ is contractible in $\underline{P}$.

Conversely, suppose that the libre $\underline{F}$ is contractible in $\underline{P}$ and let $\underline{u} = (u_\lambda, \Phi) : \underline{Z} = ((Z_\mu, *), s_{\mu\mu'}, M) \to \underline{P} = ((P_\lambda, *), r_{\lambda\lambda'}, \Lambda)$ be a morphism, such that $\underline{p} o \underline{u} = \underline{*}$. But $\underline{p} o \underline{u} = (p_\lambda o u_\lambda, \Phi)$, with the function $\Phi : \Lambda \to M$, and $p_\lambda \circ u_\lambda : Z_{\Phi(\lambda)} \to P_\lambda \to Y_\lambda$. This relation implies that each $\lambda \in \Lambda$ admits $\mu \in M$, $\mu \geq \Phi(\lambda)$ such that $p_\lambda \circ u_\lambda \circ s_{\Phi(\lambda)\mu} \approx *$, by a pointed homotopy $H_\lambda : Z_\mu \times [0,1] \to Y_\lambda$. Then, by the homotopy covering property of p_λ, there exists a pointed homotopy $K_\lambda : Z_\mu \times [0,1] \to P_\lambda$ such that $K_\lambda(\cdot, 0) = u_\lambda \circ s_{\phi(\lambda)\mu}$ and $p_\lambda \circ K_\lambda = H_\lambda$. Thus we have $u_\lambda \circ s_{\Phi(\lambda)\mu} \approx K_\lambda(\cdot, 1)$ and $Im\ K_\lambda \subseteq F_\lambda$. By the proof of Lemma 1 and since the index sets are directed, for each $\lambda \in \Lambda$ we can choose the indices $\lambda' \in \Lambda$ and $\mu, \mu' \in M$ such that $i_\lambda \circ r_{\lambda\lambda'} \approx *$ and the following diagram commutes

$$\begin{array}{ccccc} Z_{\mu'} & \xleftarrow{\;s_{\mu'\mu}\;} & Z_\mu & \xrightarrow{\;s_{\Phi(\lambda)\mu}\;} & Z_{\Phi(\lambda)} \\ K_{\lambda'}(\cdot,1)\downarrow & & \downarrow K_\lambda(\cdot,1) & & \downarrow u_\lambda \\ F_{\lambda'} & \xrightarrow{\;r_{\lambda\lambda'}\;} & F_\lambda & \xrightarrow{\;i_\lambda\;} & P_\lambda \end{array}$$

This means that $u_\lambda s_{\phi(\lambda)\mu} \approx i_\lambda r_{\lambda\lambda'} K_{\lambda'}(\cdot, 1) s_{\mu'\mu} \approx *$, i.e. $(u_\lambda, \Phi) \sim (*, \Phi')$ for satisfactory function $\Phi' : \Lambda \to M$. Thus we obtained $\underline{u} = \underline{*}$, what finishes the proof of the theorem.

Remark 3. If $f : (X, *) \to (Y, *)$ is a pointed continuous map then f is a weak monomorphism in HTop* if and only if it is a weak monomorphism in pro-HTop*. Theorem 1 generalizes the usual result for pointed continuous map [7, Prop. 2.2, (ii)].

3. WEAK EPIMORPHISMS IN THE CATEGORY PRO-HTOP*

In this section we consider only morphisms of the form $\underline{f} : \underline{X} \to (Y,*)$, where $\underline{X}$ is an arbitrary inverse system in HTop*. In fact the morphism $\underline{f}$ can be represented by a continuous map $f_\lambda : (X_\lambda,*) \to (Y,*)$, if $\underline{X} = ((X_\lambda,*), p_{\lambda\lambda'}, \Lambda)$, and two such maps $f_{\lambda_1}, f_{\lambda_2}$ define the same morphism $\underline{f}$ if there is $\lambda \geq \lambda_1, \lambda_2$ such that $f_\lambda p_{\lambda_1\lambda} = f_{\lambda_2} p_{\lambda_2\lambda}$ in HTop*.

Lemma 2. *For a morphism $\underline{f} : \underline{X} = ((X_\lambda,*), p_{\lambda\lambda'}, \Lambda) \to (Y,*)$ there exist an object $\underline{M} = ((M_\lambda,*), r_{\lambda\lambda'}, \Lambda), \Phi : \Lambda' \hookrightarrow \Lambda$ and two morphisms $\underline{j} = (j_\lambda, \Phi) : \underline{X} \to \underline{M}$, $\underline{h} : \underline{M} \to (Y,*)$, such that for each $\lambda \in \Lambda'$:*

(i) $h_\lambda : (M_\lambda,*) \to (Y,*)$ is a pointed homotopy equivalence,

(ii) $j_\lambda : (X_\lambda,*) \to (M_\lambda,*)$ is a pointed cofiber inclusion map,

(iii) $f_\lambda = h_\lambda \circ j_\lambda$.

Proof. Denote by Λ' the subset of Λ such that an index λ is in Λ' if and only if there is a map $f_\lambda : (X_\lambda,*) \to (Y,*)$ defining $\underline{f}$.

The existence for each $\lambda \in \Lambda'$ of a factorization (iii) satisfying (i) and (ii) is well known [5, p.246]. For a pair $\lambda \leq \lambda'$ in Λ' define $r_{\lambda\lambda'} = h_\lambda^{-1} \circ h_{\lambda'}$, from which is immediate that $\underline{M} = (M_\lambda,*), r_{\lambda\lambda'}, \Lambda')$ is an inverse system in HTop* and that all maps $h_\lambda, \lambda \in \Lambda'$ define the same morphism $\underline{h} : \underline{M} \to (Y,*)$. Finally, if $\Phi : \Lambda' \hookrightarrow \Lambda$ is the inclusion function, then $\underline{j} = (j_\lambda, \Phi) : \underline{X} \to \underline{M}$ is a morphism of pro-HTop*.

Remark 4. It is obvious from Lemma 2 that we can write $\underline{f} = \underline{h} \circ \underline{j}$, in the category pro-HTop*, where $\underline{h}$ is an isomorphism. Then it is clear that $\underline{f}$ is a weak epimorphism if and only if $\underline{j}$ is a weak epimorphism. We will refer to the morphism $\underline{j} : \underline{X} \to \underline{M}$ as the *cofibred factor* of the morphism $\underline{f}$.

Remark 5. Let $\underline{j} = (j_\lambda, \Phi) : \underline{X} = ((X_\lambda,*), p_{\lambda\lambda'}, \Lambda) \to \underline{M} = ((M_\lambda,*), r_{\lambda\lambda'}, \Lambda')$ be a cofibred factor in pro-HTop*. Then for each $\lambda \in \Lambda'$ we can consider the pointed quotient space M_λ / X_λ with the pointed identification map $\pi_\lambda : M_\lambda \to M_\lambda / X_\lambda$. We can consider the inverse system $\underline{M}/\underline{X} = ((M_\lambda/X_\lambda, *), \bar{r}_{\lambda\lambda'}, \Lambda')$ and the morphism $\underline{\pi} = (\pi_\lambda, 1_{\Lambda'}) : \underline{M} \to \underline{M}/\underline{X}$. For the morphism $\underline{f} : \underline{X} \to (Y, *)$ we will say that $(Y, *)$ *is contractible in the cofibred factor of f* if each $\lambda \in \Lambda'$ admits a $\lambda' \geq \lambda$ such that $\pi_\lambda \circ r_{\lambda\lambda'} \approx *$.

Theorem 2. *A morphism $\underline{f} : \underline{X} \to (Y, *)$ of pro-HTop* is a weak epimorphism if and only if $(Y, *)$ is contractible in every cofibred factor.*

Proof. Suppose that $\underline{f}$ is a weak epimorphism, what is equivalent to the fact that the morphism $\underline{j} : \underline{X} \to \underline{M}$ is a weak epimorphism. Since $\underline{\pi} \circ \underline{j} = \underline{*}$, the hypothesis implies $\underline{\pi} = \underline{*}$ in pro HTop*. This means $(\pi_\lambda, 1_{\Lambda'}) \sim (*, \Phi')$, i.e. each $\lambda \in \Lambda'$ admits a $\lambda' \geq \lambda$ such that $\pi_\lambda \circ r_{\lambda\lambda'} = *$ in HTop*. Thus, $(Y, *)$ is contractible in the cofibred factor $\underline{j} : \underline{X} \to \underline{M}$ of $\underline{f}$.

Conversely, suppose that $(Y, *)$ is contractible in the cofibred factor of $\underline{f}$. It is sufficient to prove that $\underline{f}$ is a weak epimorphism. For this, suppose that for a morphism $\underline{u} = (u_\nu, \Psi) : \underline{M} \to \underline{Z} = ((Z_\nu, *), s_{\nu\nu'}, N)$ we have $\underline{u} \circ \underline{j} = \underline{*}$. This implies that for each $\nu \in N$ there is a pointed homotopy $H_\nu : u_\nu \circ j_{\Psi(\nu)} \approx *$. Then, by the pointed homotopy extension property of the pair $(M_{\Psi(\nu)}, X_{\Psi(\nu)})$ there exists a pointed homotopy $K_\nu : M_{\Psi(\nu)} \times [0,1] \to Z_\nu$, such that $K_\nu(\cdot, 0) = u_\nu$ and $K_\nu / K_{\Psi(\nu)} \times [0,1] = H_\nu$. Now, if we consider the pointed map $\varphi_\nu : M_{\Psi(\nu)} \to Z_\nu, \varphi_\nu = K_\nu(\cdot, 1)$, then we have $\varphi_\nu / X_{\Psi(\nu)} = K_\nu / X_{\Psi(\nu)} \times \{1\} = H_\nu(\cdot, 1) = *$. Therefore, we can define the pointed map $\tilde{\varphi}_\nu : M_{\Psi(\nu)}/X_{\Psi(\nu)} \to Z_\nu$, such that $\tilde{\varphi}_\nu \circ \pi_{\Psi(\nu)} = \varphi_\nu$ and the pointed homotopy $\tilde{\varphi}_\nu \circ F_{\Psi(\nu)} : M_{\lambda'} \times [0,1] \to M_{\Psi(\nu)}/X_{\Psi(\nu)} \to Z_\nu$, where $F_{\Psi(\nu)} : \pi_{\Psi(\nu)} \circ r_{\Psi(\nu)\lambda'} \approx *$, for a convenient $\lambda' \geq \Psi(\nu)$. For this we have $\tilde{\varphi}_{\Psi(\nu)} \circ F_{\Psi(\nu)} : \varphi_\nu \circ r_{\Psi(\nu)\lambda} \approx *$ in Top*. On the other hand K_ν is a pointed homotopy, $K_\nu : u_\nu \circ r_{\Psi(\nu)\lambda} \approx \varphi_\nu \circ r_{\Psi(\nu)\lambda}$, and therefore $\tilde{\varphi}_\nu \circ F_{\Psi(\nu)} \circ K_\nu : u_\nu r_{\Psi(\nu)\lambda} \approx *$. This proves the equivalence $(u_\nu, \Psi) \sim (*, \Psi)$ for every $\nu \in N$ and therefore $\underline{u} = \underline{*}$, what finishes the proof of the theorem.

Remark 6. If $f : (X, *) \to (Y, *)$ is a pointed continuous map then f is a weak epimorphism in HTop* if and only if it is a weak epimorphism in pro-HTop*. Particularly, Theorem 2 generalizes the usual intrinsic characterization of a weak epimorphism in HTop* [7, Prop. 2.2 (i)].

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