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# Approximation of Almost Periodic Functions by Convolution Type Operators

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**ABSTRACT.** For $S^p$- and $S^*$-almost periodic functions $f$ the convolution type operators $L_\mu f$ are considered. The rates of convergence of $L_\mu f(x)$ to $f(x)$ at the Lebesgue or Lebesgue-Denjoy points $x$ of $f$ are estimated.

## 1. PRELIMINARIES

Let $L^p_{loc}$ $(1 \le p < \infty)$ be the class of all measurable complex-valued functions Lebesgue-integrable with $p$-th power on each finite interval and let $D^*_{loc}$ be the set of all complex-valued functions integrable in the Denjoy-Perron sense on each finite interval. Denote by $S^p$ and by $S^*$ the spaces of all functions $f \in L^p_{loc}$ and $f \in D^*_{loc}$ which are $S^p$-almost periodic and $S^*$-almost periodic, respectively, with the norms

$$||f||_{S^p} := \sup_{-\infty<v<\infty} \left( \int_v^{v+1} |f(t)|^p dt \right)^{1/p}$$

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and

$$\|f\|_{S^*} := \sup_{-\infty<v<\infty} \left( \sup_{0\le u\le 1} \left| \int_v^{v+u} f(t)dt \right| \right).$$

Write $S = S^1$ and use the symbol $B$ for the space of all complex-valued functions $f$ almost periodic in the Bohr sense, i.e. uniformly almost periodic, with the norm

$$\|f\|_B := \sup_{-\infty<v<\infty} |f(v)|.$$

The theory of Bohr's and $S^p$-almost periodic functions is given in [6]. Some properties of $S^*$-almost periodic functions can be found e.g. in [7], [8].

Let $E$ be a set of positive numbers, having the accumulation point at infinity. Introduce the convolution type operators $L_\mu$ ($\mu \in E$), defined for functions $f \in S$ or $f \in S^*$ by the improper Denjoy-Perron integral

$$L_\mu f(x) := (f * \psi_\mu)(x) \equiv \int_{\to -\infty}^{\to \infty} f(x-t)\psi_\mu(t)dt \ (x \in R := (-\infty, \infty)), \quad (1)$$

where $\psi_\mu$ are measurable (complex-valued) functions satisfying some additional assumptions. In particular, if $f \in S^p$ with some $p \ge 1$ and if $\psi_\mu$ is Lebesgue-integrable on $R$ (in symbols $\psi_\mu \in L$), then $L_\mu f$ is of class $S^p$. If $f \in S^p$ ($p > 1$), $\psi_\mu \in L^q_{loc}$ (where $\frac{1}{q} = 1 - \frac{1}{p}$) and

$$\|\psi_\mu\|_q := \sum_{k=-\infty}^{\infty} \left( \int_k^{k+1} |\psi_\mu(t)|^q dt \right)^{1/q} < \infty,$$

then $L_\mu f$ is uniformly almost periodic; the same is also true if $f \in S$, $\psi_\mu \in L^\infty_{loc}$ (i.e. $\psi_\mu$ is measurable and essentially bounded on each finite interval) and if

$$\|\psi_\mu\|_\infty := \sum_{k=-\infty}^{\infty} \operatorname*{ess\,sup}_{k\le t\le k+1} |\psi_\mu(t)| < \infty.$$

In the case when $f \in S^*$, the assumptions

$$\|\psi_\mu\|_\infty < \infty \text{ and } \operatorname{var}_{-\infty<t<\infty}\psi_\mu(t) < \infty$$

imply the uniform almost periodicity of $L_\mu f$, too (see [8], [9]).

In this paper, letting $\mu \to \infty$, we present some estimates for the rate of convergence of $L_\mu f$ at the Lebesgue or Lebesgue-Denjoy points $x$ of $f$. As a measure of deviation of $L_\mu f(x)$ from $f(x)$ we take the quantities

$$w_x(h;f) := \frac{1}{h}\int_0^h |\varphi_x(t)|dt \quad \text{if } f \in S,$$

$$w_x^*(h;f) := \sup_{0<v\leq h} \frac{1}{v}\left|\int_0^v \varphi_x(t)dt\right| \quad \text{if } f \in S^*,$$

where $h > 0$ and $\varphi_x(t) := f(x+t) + f(x-t) - 2f(x)$. For $f \in S$ we also use the quantity

$$\bar{w}_x(h;f) := \sup_{0<v\leq h} w_x(v;f).$$

Clearly, $w_x(h;f) < \infty$ for all $x$ and $h > 0$. In view of the well-known Lebesgue theorem and the fundamental properties of the Denjoy-Perron integral [5], for almost every $x$,

$$\lim_{h\to 0+} w_x(h;f) = \lim_{h\to 0+} \bar{w}_x(h;f) = 0 \text{ and } \lim_{h\to 0+} w_x^*(h;f) = 0$$

(we call these $x$ the Lebesgue and the Lebesgue-Denjoy points of $f$, respectively). Further, $\bar{w}_x(h;f)$ and $w_x^*(h;f)$ are non-decreasing functions of $h$ on $(0,\infty)$, provided they are finite at $x$. The so-called local integral modulus $\bar{w}_x(h;f)$ (in a slightly different form) was first used in [1] to obtain the quantitative version of the known Fejér-Lebesgue theorem.

For $f \in S^p$ $(p > 1)$ we introduce also the quantities

$$w_x(h;f)_p := \left(\frac{1}{h}\int_0^h |\varphi_x(t)|^p dt\right)^{1/p} \quad (h > 0),$$

which have the properties similar to that of $w_x(h;f)$.

Throughout, the integral part of a real number $a$ is denoted by $[a]$. The symbol $\zeta(s)$, $s > 1$, means the well-known Riemann zeta function.

## 2. MAIN RESULTS

Consider operators $L_\mu$ defined by (1), in which $\psi_\mu$ are even measurable functions such that $||\psi_\mu||_\infty < \infty$ or $||\psi_\mu||_q < \infty$ with some $q > 1$ (clearly, this implies that $\psi_\mu$ are Lebesgue-integrable on $R$).

**Theorem 1.** *Suppose that* $||\psi_\mu||_\infty < \infty$,

$$\int_{-\infty}^{\infty} \psi_\mu(t)dt = 1 \text{ for all } \mu \in E \tag{2}$$

*and that there exist positive numbers* $\sigma, \alpha_\mu$ *such that*

$$|\psi_\mu(t)| \leq \alpha_\mu t^{-\sigma} \text{ for a.e. } t \in (0,1] \text{ and all } \mu \in E. \tag{3}$$

*If* $f \in S$, *then for every real* $x$,

$$\begin{aligned} |L_\mu f(x) - f(x)| \leq &2(||f||_S + |f(x)|)(\alpha_\mu + \gamma_\mu) \\ &+ \beta_\mu \delta_\mu w_x(\delta_\mu; f) + \sigma\alpha_\mu \int_{\delta_\mu}^1 t^{-\sigma} w_x(t;f)dt, \end{aligned} \tag{4}$$

*where* $\delta_\mu$ *are arbitrary positive numbers not greater than 1 and*

$$\beta_\mu := \operatorname*{ess\,sup}_{0<t\leq\delta_\mu} |\psi_\mu(t)|, \quad \gamma_\mu := \sum_{k=1}^{\infty} \operatorname*{ess\,sup}_{k\leq t\leq k+1} |\psi_\mu(t)|.$$

**Proof.** In view of our assumptions, the convolution (1) exists for all $x$ as the ordinary Lebesgue integral and

$$|L_\mu f(x) - f(x)| \le \left( \int_0^{\delta_\mu} + \int_{\delta_\mu}^{1} + \int_1^{\infty} \right) |\varphi_x(t)\psi_\mu(t)| dt = I_1 + I_2 + I_3, \text{ say.}$$

Clearly,

$$I_1 \le \beta_\mu \int_0^{\delta_\mu} |\varphi_x(t)| dt = \beta_\mu \delta_\mu w_x(\delta_\mu; f),$$

$$I_3 \le \sum_{k=1}^{\infty} \operatorname*{ess\,sup}_{k \le t \le k+1} |\psi_\mu(t)| \int_k^{k+1} |\varphi_x(t)| dt \le 2\Big( ||f||_S + |f(x)| \Big) \gamma_\mu .$$

Further, by (3) and partial integration,

$$I_2 \le \alpha_\mu \int_{\delta_\mu}^{1} |\varphi_x(t)| t^{-\sigma} dt = \alpha_\mu \int_{\delta_\mu}^{1} \left( \int_0^t |\varphi_x(u)| du \right)' t^{-\sigma} dt$$

$$\le \alpha_\mu \left\{ w_x(1; f) + \sigma \int_{\delta_\mu}^{1} t^{-\sigma} w_x(t; f) dt \right\}.$$

Collecting the results and observing that $w_x(1; f) \le 2(||f||_S + |f(x)|)$ we get (4), immediately.

**Remark 1.** Assuming that $\bar{w}_x(1; f) < \infty$, one can easily verify that

$$\int_{\delta_\mu}^{1} t^{-\sigma}\bar{w}_x(t;f)dt \le \tau(\sigma)\sum_{k=1}^{m} k^{\sigma-2}\bar{w}_x\Big(\frac{1}{k};f\Big),$$

where $m := [1/\delta_\mu]$, $\tau(\sigma) := \max\{1, 2^{\sigma-2}\}$. Also, if $\sigma \ge 1$,

$$\bar{w}_x(\delta_\mu;f) \le \bar{w}_x\Big(\frac{1}{m};f\Big) \le \frac{\sigma+1}{m^{\sigma+1}}\sum_{k=1}^{m} k^{\sigma}\bar{w}_x\Big(\frac{1}{m};f\Big)$$

$$\le \frac{\sigma+1}{(m+1)^{\sigma-1}}2^{\sigma-1}\sum_{k=1}^{m} k^{\sigma-2}\bar{w}_x\Big(\frac{1}{k};f\Big)$$

$$\le 2^{\sigma-1}(\sigma+1)\delta_\mu^{\sigma-1}\sum_{k=1}^{m} k^{\sigma-2}\bar{w}_x\Big(\frac{1}{k};f\Big).$$

Consequently, under assumptions of Theorem 1 (with $\sigma \ge 1$) we have

$$|L_\mu f(x) - f(x)| \le 2(||f||_S + |f(x)|)(\alpha_\mu + \gamma_\mu) + c_\mu(\sigma)\sum_{k=1}^{m} k^{\sigma-2}\bar{w}_x\Big(\frac{1}{k};f\Big),$$

where $c_\mu(\sigma) = 2^{\sigma-1}(\sigma+1)\beta_\mu\delta_\mu^{\sigma} + \sigma\tau(\sigma)\alpha_\mu$. In the case when $\sigma = 1, 2$ or 3, a direct calculation shows that the term $2^{\sigma-1}$ in $c_\mu(\sigma)$ may be omitted.

Let us note that Theorem 1 remains valid for functions $f$ of class $S^p$ with $p > 1$, because $S^p \subset S$. Nevertheless, in this case, the argumentation similar to that of the proof of Theorem 1 leads to

**Theorem 2.** *Let $f \in S^p$ $(p > 1)$ and let $||\psi_\mu||_q < \infty$ for all $\mu \in E$, where $q = p/(p-1)$. Suppose, moreover, that conditions (2) and (3) are satisfied. Then, for every $x \in R$,*

$$|L_\mu f(x) - f(x)| \le 2(\|f\|_{S^p} + |f(x)|)(\alpha_\mu + \gamma_{\mu,q})$$

$$+ \beta_{\mu,q}\delta_\mu^{1/p} w_x(\delta_\mu; f)_p + \sigma\alpha_\mu \int_{\delta_\mu}^{1} t^{-\sigma} w_x(t; f)_p dt,$$

where $0 < \delta_\mu \le 1$,

$$\beta_{\mu,q} := \Big(\int_0^{\delta_\mu} |\psi_\mu(t)|^q dt\Big)^{1/q}, \quad \gamma_{\mu,q} := \sum_{k=1}^{\infty} \Big(\int_k^{k+1} |\psi_\mu(t)|^q dt\Big)^{1/q}.$$

The corresponding result for almost periodic functions integrable in the Denjoy-Perron sense can be stated as follows.

**Theorem 3.** *Let* $\|\psi_\mu\|_\infty < \infty$, $\mathrm{var}_{-\infty<t<\infty}\psi_\mu(t) < \infty$ *for all* $\mu \in E$ *and let condition (2) be satisfied. Assume, moreover, that* $\psi_\mu$ *are absolutely continuous on* $(0,1]$ *and that*

$$|\psi_\mu'(t)| \le \alpha_\mu^* t^{-\rho} \text{ for a.e. } t \in (0,1] \text{ and all } \mu \in E, \tag{5}$$

$\rho, \alpha_\mu^*$ *being some positive numbers. If* $f \in S^*$ *and if* $w_x^*(1; f) < \infty$ *then*

$$|L_\mu f(x) - f(x)| \le 2(\|f\|_{S^*} + |f(x)|)\gamma_\mu^* + \frac{1}{2}\beta_\mu^*\delta_\mu^2 w_x^*(\delta_\mu; f)$$

$$+ \alpha_\mu^* \int_{\delta_\mu}^{1} t^{-\rho+1} w_x^*(t; f) dt,$$

*where*

$$\beta^*_\mu := \operatorname*{ess\,sup}_{0<t\le\delta_\mu} |\psi'_\mu(t)|, \quad \gamma^*_\mu := 2\gamma_\mu + \mathrm{var}_{1\le t<\infty}\psi_\mu(t),$$

*$\delta_\mu$ and $\gamma_\mu$ have the same meaning as in Theorem 1.*

**Proof.** In view of (1) and (2),

$$L_\mu f(x) - f(x) = \Big(\int_0^1 + \int_1^{\to\infty}\Big)\varphi_x(t)\psi_\mu(t)dt = J_1 + J_2, \text{ say.}$$

Applying the known inequalities for the Denjoy-Perron integral ([5] p. 45, or [8] p. 187) we obtain

$$|J_2| = \Big|\sum_{k=1}^{\infty}\int_k^{k+1}\varphi_x(t)\psi_\mu(t)dt\Big|$$

$$\le \sum_{k=1}^{\infty}\Big(\sup_{k\le t\le k+1}|\psi_\mu(t)| + \mathrm{var}_{k\le t\le k+1}\psi_\mu(t)\Big)\max_{k\le\xi\le k+1}\Big|\int_k^{\xi}\varphi_x(t)dt\Big|$$

$$\le 2\Big(\gamma_\mu + \mathrm{var}_{1\le t<\infty}\psi_\mu(t)\Big)(\|f\|_{S^*} + |f(x)|).$$

Further, putting

$$\Phi_x(t) := \int_0^t \varphi_x(u)du$$

and integrating by parts ([5] p. 42) we get

$$|J_1| = \left|\Phi_x(1)\psi_\mu(1) - \int_0^1 \Phi_x(t)\psi'_\mu(t)dt\right|$$

$$\leq |\Phi_x(1)||\psi_\mu(1)| + \left(\int_0^{\delta_\mu} + \int_{\delta_\mu}^1\right) tw_x^*(t;f)|\psi'_\mu(t)|dt.$$

Hence, assumption (5) and the obvious inequalities

$$|\psi_\mu(1)| \leq \gamma_\mu, \; |\Phi_x(1)| \leq 2(||f||_{S^*} + |f(x)|)$$

give

$$|J_1| \leq 2\gamma_\mu(||f||_{S^*} + |f(x)|) + \frac{1}{2}\beta_\mu^*\delta_\mu^2 w_x^*(\delta_\mu;f) + \alpha_\mu^* \int_{\delta_\mu}^1 t^{-\rho+1} w_x^*(t;f)dt.$$

Collecting the results we get the desired assertion.

**Remark 2.** In the same way as in Remark 1, the estimate given in Theorem 3 can be stated in the form

$$|L_\mu f(x) - f(x)| \leq 2(||f||_{S^*} + |f(x)|)\gamma_\mu^* + c_\mu^*(\rho)\sum_{k=1}^m k^{\rho-3} w_x^*(\frac{1}{k};f),$$

where $m = [1/\delta_\mu]$, $c_\mu^*(\rho) = 2^{\rho-3}(\rho+1)\beta_\mu^*\delta_\mu^\rho + \alpha_\mu^* \max\{1, 2^{\rho-3}\}$, provided that $\rho \geq 2$.

Now, denoting by $Y$ the space $B$, $S^p$ $(p \geq 1)$ or $S^*$, let us define the modulus of smoothness of $f \in Y$ with respect to the norm of $Y$ by

$$\omega_2(h;f)_Y := \sup_{0\le t\le h} ||f(\cdot+t)+f(\cdot-t)-2f(\cdot)||_Y \qquad (h\ge 0).$$

Clearly, if $f\in B$ then, for all $x\in R$ and $h>0$,

$$w_x(h;f)\le \omega_2(h;f)_B.$$

In case $f\in S^p$ we have

$$\sup_{-\infty<v<\infty}\Big(\int_v^{v+1}(w_x(h;f))^p dx\Big)^{1/p}\le \omega_2(h;f)_{S^p} \qquad (h>0),$$

by the generalized Minkowski inequality. These estimates and Theorem 1 together with Remark 1 lead to the following

**Corollary.** Let $f\in Y$, where $Y=B$ or $S^p$ ($p\ge 1$), and let conditions (2), (3) with $\sigma\ge 1$ be satisfied. Then, for all $\mu\in E$,

$$||L_\mu f-f||_Y\le 4(\alpha_\mu+\gamma_\mu)||f||_Y + c_\mu(\sigma)\sum_{k=1}^m k^{\sigma-2}\omega_2\big(\frac{1}{k};f\big)_Y\ ,$$

where $m,\alpha_\mu,\gamma_\mu,c_\mu(\sigma)$ have the same meaning as in Theorem 1 and Remark 1.

For almost periodic functions integrable in the Denjoy-Perron sense a direct calculation gives

**Theorem 4.** *Suppose that $f\in S^*$ and that conditions (2) and (5) with $\rho\ge 2$ are satisfied. Then, for all $\mu\in E$,*

$$||L_\mu f-f||_{S^*}\le 4\gamma_\mu^*||f||_{S^*}+c_\mu^*(\rho)\sum_{k=1}^m k^{\rho-3}\omega_2\big(\frac{1}{k};f\big)_{S^*},$$

*where $m, \gamma_\mu^*$ and $c_\mu^*(\rho)$ have the same meaning as in Theorem 3 and Remark 2.*

## 3. EXAMPLES

I. Let $0 \le \lambda \equiv \lambda(\mu) < \mu$ for $\mu \in E = (0, \infty)$ and let $\Psi_{\lambda,\mu}$ be the continuous functions defined for $t \ne 0$ by the formula

$$\Psi_{\lambda,\mu}(t) := \frac{\left(4 \sin \frac{1}{4}(\mu - \lambda)t\right)^2 \sin \frac{1}{2}(\mu + \lambda)t}{\pi(\mu - \lambda)^2 t^3}.$$

Denote by $L_{\lambda,\mu}$ the operators (1) with $\psi_\mu = \Psi_{\lambda,\mu}$. As in known ([3] p. 256), condition (2) is satisfied. Introducing the auxiliary function $g_z(t) := (\sin zt)/t$ for $t \ne 0$, $g_z(0) = z$, with a positive parameter $z$, we can write

$$\Psi_{\lambda,\mu}(t) = \frac{1}{\pi} a^{-2} g_a^2(t) g_b(t) \text{ with } a = \frac{\mu - \lambda}{4},\ b = \frac{\mu + \lambda}{2}.$$

Since

$$|g_z(t)| \le \frac{1}{t},\ |g_z'(t)| = \left|\frac{zt \cos zt - \sin zt}{t^2}\right| \le \frac{2z}{t} \text{ for } t > 0$$

and

$$|g_z(t)| \le z,\ |g_z'(t)| \le \frac{2}{3} z^3 t \text{ for } t \ge 0,$$

we have

$$|\Psi_{\lambda,\mu}(t)| \le \frac{16}{\pi(\mu - \lambda)^2 t^3},\ |\Psi_{\lambda,\mu}'(t)| \le \frac{32\mu}{\pi(\mu - \lambda)^2 t^3} \text{ for } t > 0$$

and

$$|\Psi_{\lambda,\mu}(t)| \le \frac{\mu + \lambda}{2},\ |\Psi_{\lambda,\mu}'(t)| \le \frac{(\mu + \lambda)^3 t}{8\pi} \text{ for } t \ge 0.$$

These inequalities ensure that for every $f \in S^p$ $(p \geq 1)$ or $f \in S^*$ the functions $L_{\lambda,\mu}f$ are uniformly almost periodic. Moreover, under the assumption $\mu - \lambda \geq 1$, Theorems 1, 3, 4 apply with $\sigma = 3$, $\rho = 3$,

$$\delta_\mu = \frac{1}{\mu - \lambda},\ \alpha_\mu = \frac{16}{\pi(\mu-\lambda)^2},\ \beta_\mu \leq \frac{\mu+\lambda}{2},\ \gamma_\mu \leq \frac{16\zeta(3)}{\pi(\mu-\lambda)^2},$$

$$\alpha^*_\mu = \frac{32\mu}{\pi(\mu-\lambda)^2},\ \beta^*_\mu \leq \frac{(\mu+\lambda)^3}{8\pi(\mu-\lambda)},\ \gamma^*_\mu \leq \frac{32(1+\mu)\zeta(3)}{\pi(\mu-\lambda)^2}.$$

Assuming additionally that $\frac{\lambda}{\mu} \leq \theta < 1$ for all $\mu > 0$, we easily verify that the right-hand sides of the estimates given in Theorems 1 - 3 and Remarks 1, 2 converge to zero as $\mu \to \infty$, for almost every $x$. In particular, setting $\lambda(\mu) = \frac{\mu}{2}$ we get for $f \in S$ the result of [3] (Th. 5). Moreover, from Corollary it follows the estimate of $||L_{\lambda,\mu}f - f||_{S^p}$ in terms of the modulus of smoothness of $f \in S^p$. Namely,

$$||L_{\lambda,\mu}f - f||_{S^p} \leq \frac{21(1+\zeta(3))}{(\mu-\lambda)^2}||f||_{S^p} +$$

$$+ 2\left(\frac{1+\theta}{1-\theta} + 8\right)\frac{1}{(\mu-\lambda)^2}\sum_{k=1}^{m} k\omega_2\left(\frac{1}{k}; f\right)_{S^p},$$

where $m = [\mu - \lambda]$ (clearly, the right-hand side of this inequality converges to zero as $\mu \to \infty$). Taking into account the integral modulus of continuity

$$\omega_1(h; f)_{S^p} := \sup_{0 \leq t \leq h} ||f(\cdot + t) - f(\cdot)||_{S^p}$$

and applying its basic properties, we easily verify that for $f \in S^p$ with $\omega_1(1; f)_{S^p} \neq 0$ there holds the relation

$$||L_{\lambda,\mu}f - f||_{S^p} = \mathcal{O}\left(\omega_1\left(\frac{1}{\mu-\lambda}; f\right)_{S^p}\right),$$

which is equivalent to Theorem 1 of [3]. Note, that the corresponding estimates for $f \in S^*$ follow from Theorem 4.

II. The Bernstein integral operators $Q_\mu \equiv L_\mu$ are defined by (1), in which $\mu \in E = (0, \infty)$, $\psi_\mu = G_\mu$ are continuous functions on $R$ with values

$$G_\mu(t) := \frac{c(r)}{\mu^{2r-1}} \left( \frac{1}{t} \sin \frac{\mu t}{2r} \right)^{2r}, \; c(r) := (2r)^{2r-1} / \int_{-\infty}^{\infty} \left( \frac{\sin v}{v} \right)^{2r} dv,$$

for $t \neq 0$, and $r$ is a fixed positive integer (see [4]). It is easy to verify that Theorems 1, 3, 4 are true for $\mu \geq 1$ with

$$\delta_\mu = 1/\mu, \; \sigma = \rho = 2r,$$

$$\alpha_\mu = \frac{c(r)}{\mu^{2r-1}}, \; \beta_\mu \leq \frac{c(r)}{(2r)^{2r}}, \; \gamma_\mu \leq \frac{c(r)\zeta(2r)}{\mu^{2r-1}},$$

$$\alpha_\mu^* = \frac{2c(r)}{\mu^{2r-2}}, \; \beta_\mu^* \leq \frac{4c(r)\mu^2}{3(2r)^{2r+1}}, \; \gamma_\mu^* \leq \frac{4c(r)\zeta(2r)}{\mu^{2r-2}}.$$

For almost every $x$, the right-hand side of the estimate corresponding to Theorem 1 converges to zero as $\mu \to \infty$, provided that $r \geq 1$. The same relation for the estimate following from Theorem 3 needs the assumption $r \geq 2$.

Note, that for some classes of functions the above results cannot be essentially improved. To see this, let us fix a point $x$ and let us consider the class $\Omega_x$ of all functions $f \in S$ such that $w_x(h; f) \leq h$ for $0 < h \leq 1$. In view of Theorem 1, for every $f \in \Omega_x$ and every $\mu \geq 1$,

$$|Q_\mu f(x) - f(x)|$$

$$\leq c(r) \left\{ 2\left(1 + \zeta(2r)\right)(\|f\|_S + |f(x)|) + (2r)^{-2r} + \frac{2r}{2r-2} \right\} \frac{1}{\mu}$$

whenever $r \geq 2$. On the other hand, the function $\eta_x$ of period 2, defined by $\eta_x(t) := |t-x|$ if $|t-x| \leq 1$, belongs to $\Omega_x$ and, for $\mu \geq \pi r$,

$$|Q_\mu \eta_x(x) - \eta_x(x)| = \int_0^\infty (\eta_x(x+t) + \eta_x(x-t))G_\mu(t)dt \geq 2\int_0^1 tG_\mu(t)dt$$

$$\geq \frac{2c(r)}{\mu^{2r-1}} \int_{1/\mu}^{\pi r/\mu} t^{-2r+1}\left(\sin\frac{\mu t}{2r}\right)^{2r} dt \geq \frac{2c(r)}{\mu^{2r-1}} \int_{1/\mu}^{\pi r/\mu} t^{-2r+1}\left(\frac{\mu t}{\pi r}\right)^{2r} dt$$

$$= \frac{c(r)}{(\pi r)^{2r}}(\pi^2 r^2 - 1)\frac{1}{\mu}.$$

III. Let us suppose that the Fourier series of a function $f \in S$ is of the form

$$f(x) \sim \sum_{k=-\infty}^{\infty} A_k e^{i\lambda_k x} \text{ with } A_k := \lim_{T\to\infty} \frac{1}{T}\int_0^T f(t)e^{-i\lambda_k t}dt,$$

$$0 < \lambda_k < \lambda_{k+1} \text{ if } k \in N := \{1, 2, \ldots\},\ \lim_{k\to\infty} \lambda_k = \infty,\ \lambda_{-k} = -\lambda_k,$$

$|A_k| + |A_{-k}| > 0$, and let us consider its partial sums

$$S_n f(x) := \sum_{|\lambda_k| \leq \lambda_n} A_k e^{i\lambda_k x} \qquad (n \in N).$$

As is known ([6] p. 83 and [2] Lemma 2), $S_n f$ can be represented in the form (1), in which $\mu = n \in N$ and $\psi_\mu = D_n$, where

$$D_n(t) := \frac{2}{\pi(\lambda_{n+1} - \lambda_n)} t^{-2} \sin\frac{1}{2}(\lambda_{n+1} - \lambda_n)t \sin\frac{1}{2}(\lambda_{n+1} + \lambda_n)t$$

for $t \neq 0$. If $\lambda_{n+1} - \lambda_n \geq d > 0$, where $d$ is independent of $n$, then Theorem 1 gives the estimate

$$|S_n f(x) - f(x)| \leq 2\left(\frac{\pi}{3} + \frac{2}{\pi}\right)(\|f\|_S + |f(x)|)\delta_n$$
$$+ \frac{1}{2\pi}(\lambda_{n+1} + \lambda_n)\delta_n w_x(\delta_n; f) + \frac{4}{\pi}\delta_n \int_{\delta_n}^{1} t^{-2} w_x(t; f)dt \tag{6}$$

with $\delta_n = d(\lambda_{n+1} - \lambda_n)^{-1}$.

Assume that the Fourier series of $f \in S$ is a lacunary series, i.e. there exists a positive number $\theta < 1$ such that

$$\frac{\lambda_n}{\lambda_{n+1}} \leq \theta \quad \text{for all } n \in N.$$

Then inequality (6) holds with $d = \lambda_1(1 - \theta)$. Letting in this inequality $n \to \infty$ and observing that $\delta_n \to 0$ we easily state that $S_n f(x) \to f(x)$ at every Lebesgue point $x$ of the function $f$. Thus, from (6) it follows Theorem 2(1°) of [2], in a sharpened form.

If the Fourier exponents of $f \in S$ satisfy the conditions

$$\lambda_{n+1} - \lambda_n \to \infty \text{ and } \frac{\lambda_n}{\lambda_{n+1}} \to 1 \text{ as } n \to \infty,$$

then estimate (6) ensures that $S_n f(x) \to f(x)$ at a Lebesgue point $x$ of $f$, provided that the additional assumption

$$\lim_{n\to\infty} w_x\left(\frac{1}{\lambda_{n+1} - \lambda_n}; f\right)\left(1 - \frac{\lambda_n}{\lambda_{n+1}}\right)^{-1} = 0$$

is satisfied (cf. Th. 2(2°) in [2]).

Finally, let us note that at the point $x$ of continuity of $f$,

$$w_x(h; f) \leq 2\omega(x; h; f), \text{ where } \omega(x; h; f) := \sup_{0\leq t\leq h} |f(x + t) - f(x)|.$$

In this case it is convenient to estimate the term $I_1$ in the proof of Theorem 1 as follows:

$$I_1 \leq 2\omega(x;\delta_\mu;f)\vartheta_\mu, \text{ where } \vartheta_\mu := \int_0^\infty |\psi_\mu(t)|dt.$$

Hence, inequalities (4) and (6) remain valid with $w_x(h;f)$ replaced by $\omega(x;h;f)$; the term $\beta_\mu\delta_\mu$ in (4) and the corresponding term $\frac{1}{2\pi}(\lambda_{n+1}+\lambda_n)\delta_n$ in (6) may be replaced by $2\vartheta_\mu$ and by

$$2\int_0^\infty |D_n(t)|dt \leq \frac{4}{\pi} + \frac{2}{\pi}\log\frac{\lambda_{n+1}+\lambda_n}{\lambda_{n+1}-\lambda_n},$$

respectively. So, inequality (6) in this form contains also Theorem 2(2°) of [2].

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