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Bases of the Homology Spaces of the Hilbert Scheme of Points in an Algebraic Surface

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ABSTRACT. We find two basis of the spaces of rational homology of the Hilbert scheme of points in an algebraic surface, by exhibiting two candidates having as cardinalities the known Betti numbers of this scheme and showing that both intersect in a matrix of nonzero determinant.

Let S be a complex algebraic surface, proper, smooth and connected. Gotsche and Soergel ([G], [GS]) have found the rational homology $H_n(\mathrm{Hilb}^d S)_Q$ of the Hilbert scheme of subschemes of S of length d. We find in this article two bases for these spaces, one of them described by nonreduced subschemes, and another one described by reduced schemes i.e. by sets of distinct points (thus more interesting for potential applications in numerative geometry). In fact, our work only uses the value of the Betti numbers, thus providing an alternative construction of these homology spaces. The technique consists in showing

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that the elements of the two candidates intersect with a triangular matrix of nonzero diagonal entries, as in Mallavibarrena's work [M] on a base of $H_n(\mathrm{Hilb}^4 P^2)$. In fact our candidates are generalizations of types O' and 2 in work [MS] of Mallavibarrena and the second author, although the proof in that article was different, not based in intersection theory. The role of the vertical lines of P^2 (i.e. passing by (0,0,1)) is now played by a pencil of very ample divisors, which we call "verticals".

Fantechi [F] has arrived independently to essentially the same results. We are gratefull by generously sharing her manuscript. In particular it helped us to provide to our candidates the natural structure of oriented cycles in an easier way than we previously intended.

0. PRELIMINARIES AND STATEMENT

We choose a linear pencil V of very ample divisors, with no fixed components, and call "vertical" to such divisors (to help intuition). If $P \in S$ is not a base point of the pencil, we denote $V(P)$ the vertical divisor passing by it, and say a subscheme of S is vertical if it is contained in one vertical divisor.

For each $i = 0, \dots, 4$, we consider classes of oriented cycles

$$c_{i1}, \dots, c_{ib_i} \in H_i(S); \ \tilde{c}_{i1}, \dots, c_{ib_i} \in H_{4-i}(S)$$

(where $b_i = \dim H_i(S)_Q = \dim H_{4-i}(S)_Q)$ such that $c_{ij} \cdot c_{ij'} = 0$ if $j \neq j'$. It is a consecuence of Poincaré duality theorem $H_i(S)_Q^* \cong H_{4-i}(S)_Q$ for the compact oriented manifold S, that we can find such classes. (We could have simplified using rather classes $c_i \in H_i(S)_Q$ and $\tilde{c}_i \in H_{4-i}(S)_Q$ so that furthermore $c_i \cdot \tilde{c}_i = 1$, but this introduces unnecessary restrictions in potential applications. This is the case if, for instance, we want to work only with one base: $\tilde{c}_{ij} = c_{ij}$ for all i, j which amounts to diagonalize the symmetric bilinear form of intersection in $H_2(S)_Q$; or, in cases other than $q(S) = p_g(S) = 0$ where all homological classes are realized by algebraic cycles, if we want to use the result to bound the dimension of the space of such classes).

We can represent classes $c_{ij}, \tilde{c}_{ij}$ by oriented, piecewise smooth cycles $C_{ij}, \tilde{C}_{ij}$ mutually intersecting in the proper dimension, and being the intersections $C_{ij} \cap \tilde{C}_{ij}$ transverse, thus in a finite number of points which are smooth points of C_{ij} and of $\tilde{C}_{ij}$ (with their oriented tangent spaces giving the tangent space to S as direct sum, with orientation direct or reversed depending on whether the intersection at that point is +1 or -1 (See [GH] pp. 49-53, for instance)). We can furthermore assume that the intersection of C_{ij} and $\tilde{C}_{ij'}$, is geometric i.e. at exactly $|c_{ij} \cdot \tilde{c}_{ij'}|$ points, thus with positive sign or negative at all of them, so in particular $C_{ij} \cap \tilde{C}_{ij'} = \emptyset$ if $j \neq j'$ (This is also standard: roughly, you can assume each C_{ij}, $\tilde{C}_{ij'}$, is furthermore connected, by eventually deforming it inside its homology class, then move two intersection points of different sign along an arch connecting them, until both cancel). In fact we will need for technical reasons, several representants $\tilde{C}^0_{ij}, \ldots, \tilde{C}^k_{ij}, \ldots$ of $\tilde{c}_{ij}$ (d representants will be, in any case, enough). We can assume each of them satisfies the above generality conditions, and that all intersections in the finite set of all C_{ij} and $\tilde{C}^{k'}_{i'j'}$ happen in the proper dimension. Let E^k_{ij} be the set of points $C_{ij} \cap \tilde{C}^k_{ij}$, and $E = \sqcup E^k_{ij}$. We can assume each point of E is neither a base point nor a singular point of a vertical divisor, and the vertical divisor passing by the point intersects both C_{ij} and $\tilde{C}^{k'}_{i'j'}$, transversally at that point. We can also assume not two points of E lie in the same vertical, and that each $C_{ij}, \tilde{C}^k_{ij}$ ($i \neq 0, 4$) intersects the general element of the pencil transversally almost everywhere.

Let $\underline{A}$ be the set of sequences

$$\underline{a} = (\underline{a}_{ij}) = (\underline{a}_4 = \underline{a}_{41}; \underline{a}_{3b_3}, \ldots, \underline{a}_{31}; \underline{a}_{2b_2}, \ldots, \underline{a}_{21}; \underline{a}_{1b_1}, \ldots, \underline{a}_{11}; \underline{a}_0)$$

where each $\underline{a}_{ij}$ is a monotone sequence $a^0_{ij} \geq \ldots \geq a^k_{ij} \geq \ldots \geq a^{r_{ij}}_{ij}$, strict monotone if i is odd, and such that

$$\sum a^k_{ij} = d$$

Let the subset $\underline{A}_n$ consist of those $\underline{a} \in \underline{A}$ with

$$n = \sum_{i,j} (ir_{ij}) + 2(d - \sum_{i,j} r_{ij})$$

or equivalently

$$4d - n = \sum_{i,j} (4-i) r_{ij} + 2(d - \sum_{i,j} r_{ij})$$

Attach to each $\underline{a} \in \underline{A}_n$ the subset $\mathcal{Z}^{\underline{a}} \subset \mathrm{Hilb}^d S$ parametrizing subschemes

$$Z = \cup\{Z_{ij} | i = 0, \ldots, 4; j = 1, \ldots, b_i\}$$

of length d obtained as disjoint union of schemes Z_{ij} of length $a_{ij} = \Sigma_k a_{ij}^k$ supported in C_{ij}, whose irreducible components are r_{ij} punctual schemes of lengths $a_{ij}^0, \ldots, a_{ij}^{r_{ij}}$ if $i > 0$, and $Z_0 = \sqcup Z_0^k$ with Z_0^k of lenght a_0^k supported in the point C_0^k.

Attach also the subset $\tilde{\mathcal{Z}}^{\underline{a}} \subset \mathrm{Hilb}^d S$ parametrizing subschemes $\tilde{Z} = \sqcup \tilde{Z}_{ij}^k \subset S$ of length d obtained as disjoint union of schemes $\tilde{Z}_{ij}^k$ of length a_{ij}^k lying in different vertical divisors $\tilde{V}_{ij}^k$ and, if $i > 0$, intersecting $\tilde{C}_{ij}^k$ in one point $\tilde{z}_{ij}^k$.

The closures $\overline{\mathcal{Z}}^{\underline{a}}$ and $\overline{\tilde{\mathcal{Z}}}^{\underline{a}}$ in $\mathrm{Hilb}^d S$ have a natural structure of oriented cycle. We borrow from the analogous in [F] a very simple way to present this structure. Consider the algebraic variety $W = \prod_{ijk} S_{ij}^k \times \prod_{ijk} \mathrm{Hilb}^{a_{ij}^k}(S) \times \mathrm{Hilb}^d(S)$ with obvious projections p and q to $\sqcap S_{ij}^k$ and $\mathrm{Hilb}^d S$, where all $S_{ij}^k = S$. It is smooth and compact. Consider in W the subvariety Inc (incidence) consisting of triples (z_{ij}^k, Z_{ij}^k, Z) such that $z_{ij}^k \in Z_{ij}^k$ and $Z = \sqcup Z_{ij}^k$. Define also Punct or Vert, subvarieties of W, by imposing Z_{ij}^k to be punctual (i.e. supported in a point) or vertical (supported in a vertical). Clearly $p^{-1}(\sqcap C_{ij}^k)$ (taking $C_{ij}^k = C_{ij}$ for all k) and $p^{-1}(\sqcap \tilde{C}_{ij}^k)$ are also oriented cycles, since p is just the projection of a cartesian product with a proper algebraic manifold, and so are the intersections $Punct \cap \overline{Inc} \cap p^{-1}(\sqcap C_{ij}^k)$ and $Vert \cap \overline{Inc} \cap p^{-1}(\sqcap \tilde{C}_{ij}^k)$ since they are, by our generality asumptions, transversal almost everywhere (cf. GH, p. 52). These two cycles of W apply with degree 1 onto their images, which are $\overline{\mathcal{Z}}^{\underline{a}}$ and $\overline{\tilde{\mathcal{Z}}}^{\underline{a}}$ so these are oriented cycles. In fact we see from this construction that if we replace C_{ij}, $\tilde{C}_{ij}^k$ by homologous cycles $(C_{ij})'$, $(\tilde{C}_{ij}^k)'$ we obtain $(\overline{\mathcal{Z}}^{\underline{a}})'$ and $(\overline{\tilde{\mathcal{Z}}}^{\underline{a}})'$ homologous to $\overline{\mathcal{Z}}^{\underline{a}}$ and $\overline{\tilde{\mathcal{Z}}}^{\underline{a}}$.

Theorem. *The homology classes of the closures* $[\overline{\mathcal{Z}}^{\underline{a}}$ *and* $[\overline{\tilde{\mathcal{Z}}}^{\underline{a}}]$ *are bases of* $H_n(\mathrm{Hilb}^d S)_Q$ *and* $H_{4d-n}(\mathrm{Hilb}^d S)_Q$

We will prove this theorem by showing:

T1) The intersection matrix of both sets $[\overline{\mathcal{Z}^{\underline{a}}}]\cdot[\overline{\tilde{\mathcal{Z}}^{\underline{\tilde{a}}}}]$ *is triangular*

T2) The diagonal entries of this matrix are nonzero.

T3) The cardinality of both sets is the known Betti number.

1. PROOF OF T1

Observe that the same closure $\overline{\mathcal{Z}^{\underline{a}}}$ is obtained if we redefine $\mathcal{Z}^{\underline{a}}$ adding the following technical condition: if $Z \in \mathcal{Z}^{\underline{a}}$, then each point $z_{ij}^k \in C_{ij}$ lies in fact in $\overset{0}{C}_{ij} = C_{ij} \setminus \cup\{C_{i'j'} | (i',j') < (i,j)$ lexicographically$\}$ (Observe that $S = \sqcup \overset{0}{C}_{ij}$).

Assume that $X \in \overline{\mathcal{Z}^{\underline{a}}} \cap \overline{\tilde{\mathcal{Z}}^{\underline{\tilde{a}}}}$ and lexicographically,

$$(r_4; r_{3b_3}, \ldots, r_{31}; r_{2b_2}, \ldots, r_{21}; r_{1b_1}, \ldots, r_{11}; r_0) \leq$$

$$\leq (\tilde{r}_4; \tilde{r}_{3b_3}, \ldots, \tilde{r}_{31}; \tilde{r}_{2b_2}, \ldots, \tilde{r}_{21}; \tilde{r}_{1b_1}, \ldots, \tilde{r}_{11}; \tilde{r}_0)$$

and decompose X as $\sqcup X_{ij}$, with supports $x = \sqcup x_{ij}$ so that $x_{ij} \subseteq \overset{0}{C}_{ij}$. We clearly get proved T1 (and get in good position to prove T2 and T3) if we show

T11) $\underline{a} = \underline{\tilde{a}}$

T12) Decomposing X as $\sqcup X_{ij}$ with supports $x = \sqcup x_{ij}$ so that $x_{ij} \subseteq \overset{0}{C}_{ij}$, it is $x_{ij} = \{x_{ij}^k | k = 0, \ldots, r_{ij}\}$ with the point $x_{ij}^k = x_{ij} \cap C_{ij}^k$

T13) Each X_{ij}^k is the a_{ij}^k-th neighborhood of x_{ij}^k in the vertical divisor passing by it.

Let $Z(t)$, $t \in (-\epsilon, \epsilon)$, be a differentiable curve in $\overline{\mathcal{Z}^{\underline{a}}}$ so that $Z(t) = \sqcup Z_{ij}(t) \in \mathcal{Z}^{\underline{a}}$ for $t \neq 0$, and $Z(0) = X$. Analogously, the support point $z_{ij}(t)$, for $t \neq 0$, define as limit a set $z_{ij}(0) \in X$, with $\# z_{ij}(0) \leq \# z_{ij}(t) = r_{ij} + 1$. Since X is also the limit $\tilde{Z}(0)$ of a curve $\tilde{Z}(t) = \sqcup \tilde{Z}_{ij}^k(t)$, $t \neq 0$, as $t \to 0$, we can define analogously $\tilde{Z}_{ij}^k(0) \subseteq \tilde{Z}(0) = X$ of length $\tilde{a}_{ij}^k(t)$ and the points $\tilde{z}_{ij}^k(t)$define a limit point $\tilde{z}_{ij}^k(0) \in X$.

First we prove, by descending induction on i, assert A_i : For $j = 1, \ldots, b_i$ it is
$A_i 1)$ $r_{ij} = \tilde{r}_{ij}$

$A_i2)$ $x_{ij} = \{x_{ij}^k\}$ with $x_{ij}^k = x_{ij} \cap E_{ij}^k$
$A_i3)$ $x_{ij} = z_{ij}(0)$
(this range for index j will always be understood, as well as the range $k \in \{0, \ldots, \tilde{r}_{ij}\}$.

Start with $i = 4$. For all $t \neq 0$, the set of points $\tilde{C}_4 = \{\tilde{C}_4^k\}$ is contained in $\tilde{z}(t) = \sqcup \tilde{z}_{ij}^k(t)$, thus $\tilde{C}_4 \subseteq \tilde{z}(0) \subseteq x$. On the other hand, by dimensionality and our generality assumptions, $\tilde{C}_4$ is disjoint with $\cup_{i \leq 3} C_i$, thus with $\cup_{i \leq 3} x_i$ (Here $C_i = \cup_j C_{ij}$. In general, whenever we omit an index in a letter denoting a subset of S we will understand the union running that index). Therefore $(\tilde{C}_4 \leq x_4$, thus $\tilde{r}_4 + 1 = \#\tilde{C}_4 \leq \#x_4$. Furthermore, $x_4 \subseteq \overset{0}{C}_4$ is disjoint with the closed set $\cup_{i \leq 3} C_i$, which contains $\sqcup_{i \leq 3} z_i(t)$ for all $t \neq 0$. Therefore it is disjoint with the set $\cup_{i \leq 3} z_i(0)$, thus $x_4 \leq z_4(0)$ and $1 + \tilde{r}_4 \leq \#\tilde{C}_4 \leq \#x_4 \leq \#z_4(0) \leq z_4(t) = 1 + r_4$.

As a consequence $\tilde{r}_4 = r_4$ and $x_4 = z_4(0) = \tilde{C}_4 = E_4$.

Now let $0 < i < 4$ and assme $A_{i'}$ for all $i' > i$. We prove assert A_i. Let $j \in \{1, \ldots, b_i\}$ and $k \in \{0, \ldots, \tilde{r}_{ij}\}$. Observe first that $x \cap \tilde{C}_{ij}^k \neq 0$, since $\tilde{z}(t) \cap \tilde{C}_{ij}^k$ for all $t \neq 0$. We know $x_{i'} \cap \tilde{C}_{ij}^k = \emptyset$, if $i' > i$ since by induction hypothesis $x_{i'}$ is contained in the **finite set $E_{i'}$, which by generality assumption is disjoint to $\tilde{C}_{ij}^k$ as it has dimension** $4 - i < 4$. We also know that, for $i' < i$, all $x_{i'} \cap \tilde{C}_{ij}^k \leq C_{i'} \cap \tilde{C}_{ij}^k$. Therefore $x \cap \tilde{C}_i \neq \emptyset$. Now $x_{ij'} \cap \tilde{C}_{ij}^k \leq C_{ij'} \cap \tilde{C}_{ij}^k = \emptyset$ for all $j' \neq j$ as $c_{ij'} \cdot c_{ij} = 0$, thus $x_{ij} \cap \tilde{C}_{ij}^k \neq \emptyset$. This holds for each $k = 0, \ldots, \tilde{a}_{ij}^k$. Furthermore, for two distinct $k, k' \in \{0, \ldots, \tilde{a}_{ij}\}$ it is $x_{ij} \cap (\tilde{C}_{ij}^{k'} \cap \tilde{C}_{ij}^k) \subseteq C_{ij} \cap (\tilde{C}_{ij}^{k'} \cap \tilde{C}_{ij}^k) = \emptyset$ by their dimensionality and our generality assumptions. Thus $\#x_{ij} \geq 1 + \tilde{r}_{ij}$ and in case of equality, each $x_{ij} \cap \tilde{C}_{ij}^k$ consists of exactly one point, say x_{ij}^k, which must be in $E_{ij}^k = C_{ij} \cap \tilde{C}_{ij}^k$.

On the other hand, $x_{ij} = x \cap \overset{0}{C}_{ij} \subseteq x \cap C_{ij} = z(0) \cap C_{ij}$, thus, for $t \neq 0$,

$$1 + \tilde{r}_{ij} \leq \#x_{ij} \leq \#z(0) \cap C_{ij} \leq \#z(t) \cap C_{ij} = 1 + r_{ij}$$

Since $\tilde{r}_{ij} \geq r_{ij}$, these are all equalities, and x_{ij} consists of $1+r_{ij} = 1+\tilde{r}_{ij}$ points $x_{ij}^k \in E_{ij}^k$, being $x_{ij} = z_{ij}(0)$ thus proving A_i for $i > 0$.

This argument does not hold for the last step of the induction, case $i = 0$, since then the key assertion (bold letters above) does not hold anymore. In this case $r_{ij} = \tilde{r}_{ij}$ for all $i > 0$ is already assumed. Summing up the two equalities

$$n = \sum (ir_{ij}) + 2(d - \sum r_{ij}) \text{ and } 4d - n = \sum (4-i)\tilde{r}_{ij} + 2(d - \sum \tilde{r}_{ij})$$

we obtain

$$0 = \sum (i-2)r_{ij} - \sum (i-2)\tilde{r}_{ij}$$

so we conclude that also $r_0 = \tilde{r}_0$, which is part of A_0.

Furthermore, whenever $i > 0$, the set $x \cap \tilde{C}_{ij}^k$ consists of only one point of E_{ij}^k so it is the limit $\tilde{z}_{ij}^k(0)$ of the point $\tilde{z}_{ij}^k(t)$, $t \neq 0$ as t goes to zero.

The vertical $V(x \cap \tilde{C}_{ij}^k)$ does not meet C_0, so we can assume (after eventually stretching the interval $(-\epsilon, \epsilon)$, that in fact all $V(\tilde{z}_{ij}^k(t))$, for $t \in (-\epsilon, \epsilon)$ are disjoint with an open neighborhood U_{ij}^k of C_0. Take $U = \cap U_{ij}^k$. For $t \neq 0$, $i > 0$, the schemes $\tilde{Z}_{ij}^k(t)$ lie in the vertical $V(\tilde{z}_{ij}^k(t))$, so they are disjoint with U, thus their limit $\tilde{Z}_{ij}^k(0)$ is also disjoint with U and consequentely with C_0. As a consequence, the scheme X_0, whose support x_0 is C_0, must be contained in the limit $\tilde{Z}_0(0)$ of the $\tilde{Z}_0(t)$. Since the points $C_0^0, \ldots, C_0^{r_0}$ of C_0 are assumed in different verticals, writting vert(T) the minimum number of verticals (counted with multiplicity) containing a finite scheme T, we have

$$1 + r_0 = \#C_0 \leq \text{vert}(\tilde{Z}_0(0)) \leq \text{vert}(\tilde{Z}(t)) = 1 + \tilde{r}_0$$

and being $r_0 = \tilde{r}_0$, these are all equalities. This proves A_0 and the induction is ended.

It will be convenient for the sequel to assume that the set $C_0 = \{C_0^0, \ldots, C_0^k\} = x_0 = z_0(t) = \tilde{z}_0(t)$ has been reindexed so that $C^k = z_0^k(t)$ is precisely the point $\tilde{z}_0^k(t)$.

We prove now the remaining T11 and T13. For $i = 0$, it is easy: $X_0 = \sqcup X_0^k$ has support $x_0 = \{x_0^k | k = 0, \ldots, r_0\}$, being $x_0^k = C_0^k = E_0^k$. Call $Z_0^k(t)$ the subscheme of $Z_0(t)$ supported in x_0^k. Since $z_0(t) = x_0$, it is $Z_0(t) = \sqcup Z_0^k(t)$, and $X_0 = Z_0(0) = \sqcup Z_0^k(0)$ so $X_0^k = Z_0^k(0)$, thus length $X_0^k =$ length $Z_0^k(0) =$ length $Z_0^k(t) = a_0^k$.

On the other hand, the scheme $\tilde{Z}_0^k(t)$ of length $\tilde{a}_0^k$ is supported in the vertical $V(\tilde{z}_0^k(t)) = V(x_0^k) = V(E_0^k)$ so also the limit $\tilde{Z}_0^k(0) \leq X_0$ of same length $\tilde{a}_0^k$, which in fact must be supported in $x_0^k = x_0 \cap V(x_0^k)$, so $\tilde{Z}_0^k(0) \subseteq X_0^k$. Since $X_0 = \sqcup X_0^k$ is equal to $\tilde{Z}(0) = \sqcup Z_0^k(0)$ this implies that $X_0^k = \tilde{Z}_0^k(0)$, so $a_0^k = \tilde{a}_0^k$ and X_0^k is both supported in x_0^k and contained in $V(x_0^k)$, so it is the a_0^k-th neighborhood of x_0^k in $V(x_0^k)$. This proves T11 and T13 for $i = 0$.

Assume now $i > 0$. We know that $Z(t) = \sqcup Z_{ij}^k(t)$, $t \neq 0$ converges to X, and the distinct points $z_{ij}^k(t)$ converge to distinct points $z_{ij}^k(t)$ converge to distinct points $x_{ij}^k = x_{ij} \cap E_{ij}$ of $x = \{x_{ij}^k\}$ none of them in the same vertical. On the other hand, each x_{ij}^k is the limit of the exactely one point of $z_{ij}(t)$, say $z_{ij}^k(t)$, since we saw that $z_{ij}(0) = x_{ij}$ and that both $z_{ij}(t)$ and x_{ij} have the same cardinality $r_{ij} + 1$. Thus the punctual subscheme of X supported in x_{ij}^k, say X_{ij}^k, must be the limit of the punctual subscheme $Z_{ij}^k(t)$ of $Z_{ij}(t)$ supported in $z_{ij}^k(t)$, thus $\text{length}(X_{ij}^k) = \text{length}(Z_{ij}^k(t)) = a_{ij}^k$ for decompositions $X = \sqcup X_{ij}^k$ and $Z(t) = \sqcup Z_{ij}^k(t)$.

On the other hand, the schemes $\tilde{Z}_{ij}^k(t)$ in the decomposition $\tilde{Z} = \sqcup \tilde{Z}_{ij}^k(t)$ are contained in the distinct verticals $V(\tilde{z}_{ij}^k(t))$, so their limit $\tilde{Z}_{ij}^k(0)$ (of same length $\tilde{a}_{ij}^k$) is contained in $V(\tilde{z}_{ij}^k(0)) = V(x_{ij}^k)$. But recall no two points of E lie in the same vertical, so all verticals $V(x_{ij}^k)$ are distinct, thus X_{ij}^k is the subscheme of X supported in $V(x_{ij}^k)$, and all $\tilde{z}_{ij}^k(0)$ are mutually disjoint. As a consequence, the scheme $\tilde{Z}_{ij}^k(0)$ of length $\tilde{a}_{ij}^k$, is contained in the punctual scheme X_{ij}^k. But the unions $\tilde{Z}(0) = \sqcup \tilde{Z}_{ij}(0)$ and $X = \sqcup X_{ij}^k$ are equal, thus $X_{ij}^k = \tilde{Z}_{ij}^k(0)$, so their lengths a_{ij}^k and $\tilde{a}_{ij}^k$ are equal, proving T11. Statement T13 is clear from this proof, since X_{ij}^k is a scheme of length $\tilde{a}_{ij}^k$ concentrated in the point x_{ij}^k, coinciding with the scheme $\tilde{Z}_{ij}^k(0)$ which is contained in the vertical $V(x_{ij}^k)$. This is the $\tilde{a}_{ij}^k$-th infinitesimal neighborhood of x_{ij}^k in $V(x_{ij}^k)$, and we are done. Now, statement T1 is an obvious consequence of T11. Asserts T12 and T13 will help us to prove T2.

2. PROOF OF T2

We know from the proof of T1 that a point of $\overline{\mathcal{Z}^{\underline{a}}} \cap \overline{\tilde{\mathcal{Z}}^{\underline{a}}} = \mathcal{Z}^{\underline{a}} \cap \tilde{\mathcal{Z}}^{\underline{a}}$ corresponds to a scheme $X = \sqcup X_{ij}^k$, being X_{ij}^k the a_{ij}^k-th infinitesimal neighborhood of a point $x_{ij}^k \in E_{ij}^k$ in $V(x_{ij}^k)$. In order to avoid cumbersome notations, we assume only one $a_{ij}^k \neq 0$, say a, and take coordinates $u = u' + \sqrt{-1}u''$, $v = v' + \sqrt{-1}v''$ for S in an analityc neighborhood of $x_{ij}^k = C_{ij} \cap \tilde{C}_{ij}^k$ as origin, say $x \in C \cap \tilde{C}$ (It will be essentially enough to prove our claim in this case, as we will comment at the end) Fix for simplicity the value i, for instance $i = 2$, and, from now on, drop indexes i, j, k, in our previous imitations. The oriented cycles C and $\tilde{C}$ are parametrized near x by differentiable functions

$$C: \; u = \varphi(\lambda_1, \lambda_2) \quad v = \psi(\lambda_1, \lambda_2) \text{ with } (\lambda_1, \lambda_2) \in \overset{\circ}{\mathbb{R}}{}^2$$
(some open set of $\mathbb{R}^2$)

$$\tilde{C}: \; u = \tilde{\varphi}(\lambda_3, \lambda_4) \quad v = \tilde{\varphi}(\lambda_3, \lambda_4) \text{ with } (\lambda_3, \lambda_4) \in \overset{\circ}{\mathbb{R}}{}^2$$

Recalling that both $C, \tilde{C}$ meet transversaly at x with sign $\sigma = \sigma_{ij}$

$$\det(\lambda_1, \lambda_2, \lambda_3, \lambda_4) = \begin{vmatrix} \frac{\partial \varphi'}{\partial \lambda_1} & \frac{\partial \varphi''}{\partial \lambda_1} & \frac{\partial \psi'}{\partial \lambda_1} & \frac{\partial \psi''}{\partial \lambda_1} \\ \frac{\partial \varphi'}{\partial \lambda_2} & \frac{\partial \varphi''}{\partial \lambda_2} & \frac{\partial \psi'}{\partial \lambda_2} & \frac{\partial \psi''}{\partial \lambda_2} \\ \frac{\partial \tilde{\varphi}'}{\partial \lambda_3} & \frac{\partial \tilde{\varphi}''}{\partial \lambda_3} & \frac{\partial \tilde{\psi}'}{\partial \lambda_3} & \frac{\partial \tilde{\psi}''}{\partial \lambda_3} \\ \frac{\partial \tilde{\varphi}'}{\partial \lambda_4} & \frac{\partial \tilde{\varphi}''}{\partial \lambda_4} & \frac{\partial \tilde{\psi}'}{\partial \lambda_4} & \frac{\partial \tilde{\psi}''}{\partial \lambda_4} \end{vmatrix}$$

is nonzero of sign σ, when evaluated at $\lambda_1 = 0, \ldots, \lambda_4 = 0$.

Consider the open neighborhood $\overset{\circ}{H} \subseteq Hilb^a S$ of X parametrizing schemes $Z \subseteq U$ of length a of ideal

$$(u - \mu_{a-1}v^{a-1} - \ldots - \mu_1 v - \mu_0, \; (v - \nu_0) \cdot \ldots \cdot (v - \nu_{a-1}) \subseteq \mathbb{C}[\mathrm{u}, \mathrm{v}]$$

for some complex numbers $\nu_0 = \nu_0' + \sqrt{-1}\nu_0'', \ldots, \nu_{a-1}, \mu_0, \ldots, \mu_{a-1}$ thus all null in case $Z = X$ (Caution: this is not a chart of U since the same Z is defined after permuting $\nu_0, \ldots, \nu_{a-1}$)

Observe $Z \in \overset{\circ}{\mathcal{Z}}{}^{\underline{a}} = \mathcal{Z}^{\underline{a}} \cap \overset{\circ}{H}$ if and only if

$$\begin{cases} \mu_0 = \varphi(\lambda_1, \lambda_2), & \nu_0 = \psi(\lambda_1, \lambda_2) \text{ with } (\lambda_1, \lambda_2) \in \overset{\circ}{\mathbb{R}}{}^2 \\ & \nu_0 = \ldots = \nu_{a-1} \end{cases}$$

and $Z \in \tilde{\mathcal{Z}}^{\underline{a}}$ if and only if, for some $\tilde{\ell} \in \{0, \ldots, a-1\}$ it is

$$\begin{cases} \mu_0 = \tilde{\varphi}(\lambda_3, \lambda_4), & \nu_{\tilde{\ell}} = \tilde{\varphi}(\lambda_3, \lambda_4) \text{ with } (\lambda_3, \lambda_4) \in \overset{\circ}{\mathbb{R}}{}^2 \\ & \mu_0 = \ldots = \mu_{a-1} \end{cases}$$

We now define for $\epsilon \in \mathbb{R}$ near zero, a scheme $\overset{\circ}{\mathcal{Z}}{}^{\underline{a}}_{\epsilon} \subseteq \overset{\circ}{H}$ continuous deformation of $\overset{\circ}{\mathcal{Z}}{}^{\underline{a}}_0 = \overset{\circ}{\mathcal{Z}}$. An element $Z \in \overset{\circ}{\mathcal{Z}}{}^{\underline{a}}_{\epsilon}$ is an scheme of ideal in $\mathbb{C}[\mathrm{u}, \mathrm{v}]$

$$\begin{gathered} ((u - \mu_0) - \mu_{a-1}(v - \nu_0)^{a-1} - \ldots - \mu_1(v - \nu_0), \\ (v - \nu_0)(v - \nu_0 - \epsilon)(v - \nu_0 - 2\epsilon) \ldots (v - \nu_0 - (a-1)\epsilon)) \end{gathered}$$

with complex numbers $\mu_0, \ldots, \mu_{a-1}, \nu_0$ satisfying, for some $\ell \in \{0, \ldots, a-1\}$

$$\begin{aligned} &\mu_0 = \varphi(\lambda_1, \lambda_2) - \mu_{a-1}(\ell\epsilon)^{a-1} - \ldots - \mu_1 \ell\epsilon \\ &\nu_0 = \psi(\lambda_1, \lambda_2) - \ell\epsilon, \qquad\qquad \text{with } (\lambda_1, \lambda_2) \in \overset{\circ}{\mathbb{R}}{}^2 \end{aligned}$$

Since there are a^2 possible choices for $\ell, \tilde{\ell} \in \{0, \ldots, a-1\}$ both $\overset{\circ}{\mathcal{Z}}{}^{\underline{a}}_{\epsilon}, \overset{\circ}{\tilde{\mathcal{Z}}}{}^{\underline{a}} \subset U$ intersect in a^2 schemes $X^{\epsilon}_{\ell\tilde{\ell}} = X^{\epsilon}$ or sets of distinct points

$$X^{\epsilon} = \{(\mu, \nu), (\mu, \nu + \epsilon), (\mu, \nu + 2\epsilon), \ldots, (\mu, \nu + (a-1)\epsilon)\}$$

being

$$\begin{cases} \mu = \varphi(\lambda_1^\epsilon, \lambda_2^\epsilon), & \nu = \psi(\lambda_1^\epsilon, \lambda_2^\epsilon - \ell\epsilon) \\ \mu = \tilde{\varphi}(\lambda_3^\epsilon, \lambda_4^\epsilon), & \nu = \tilde{\psi}(\lambda_3^\epsilon, \lambda_4^\epsilon - \tilde{\ell}\epsilon) \end{cases}$$

for some $\lambda_1^\epsilon, \lambda_2^\epsilon, \lambda_3^\epsilon, \lambda_4^\epsilon \in \mathbb{R}^4$.

We consider the open subset $\overset{\circ}{H}{}' \subseteq \overset{\circ}{H}$ defined by imposing $\nu_0 \in B_0, \ldots, \nu_{a-1} \in B_{a-1}$ where the $B_0, \ldots, B_{a-1}$ are open disks of $\mathbb{C}$ with centers $\nu, \nu+\epsilon, \ldots, \nu+(a-1)\epsilon$ and radium smaller than $\epsilon/2$, so to assure they are mutually disjoint. Thus, schemes Z in $\overset{\circ}{H}{}'$ are just sets of distinct points $P_0, \ldots, P_{a-1}$, unambiguously ordered by the belonging of its second coordinate to one of the disks, and thus the $\nu_0, \ldots, \nu_{a-1}; \mu_0, \ldots, \mu_{a-1}$ are an analytical chart of $\overset{\circ}{H}{}'$. We switch for comodity to the analytical chart $\mu_0, \mu_1, \ldots, \mu_{a-1}; \nu_0, \bar{\nu}_1 = \nu_1 - \nu_0, \ldots, \bar{\nu}_{a-1} = \nu_{a-1} - \nu_0$ **and assume $\tilde{\ell} = 0$ after eventual reordering.**

In this chart of $\overset{\circ}{H}{}'$ the naturaly ordered cycle $\mathcal{Z}_\epsilon^{\underline{a}}$ is locally parametrized by $\lambda_1, \lambda_2, \mu_1', \mu_1'', \ldots, \mu_{a-1}', \mu_{a-1}''$ (recall $\mu_1' + \sqrt{-1}\mu_1'' = \mu_1$, etc...) in the way

$$\mu_0 = \varphi(\lambda_1, \lambda_2) - \mu_{a-1}(\ell\epsilon)^{a-1} - \ldots - \mu_1(\ell\epsilon),\ \mu_1 = \mu_1, \ldots, \mu_{a-1} = \mu_{a-1}$$

$$\nu_0 = \psi(\lambda_1, \lambda_2) - \ell\epsilon,\ \bar{\nu}_1 = \epsilon, \ldots, \bar{\nu}_{a-1} = (a-1)\epsilon$$

and $\tilde{\mathcal{Z}}^{\underline{a}}$ is parametrized by $\lambda_3, \lambda_4, \bar{\nu}_1', \bar{\nu}_1'', \ldots, \bar{\nu}_{a-1}, \bar{\nu}_{a-1}''$ as

$$\mu_0 = \tilde{\varphi}(\lambda_3, \lambda_4), \mu_1 = 0, \ldots, \mu_{a-1} = 0$$

$$\nu_0 = \tilde{\psi}(\lambda_3, \lambda_4), \bar{\nu}_1 = \bar{\nu}_1, \ldots, \bar{\nu}_{a-1} = \bar{\nu}_{a-1}$$

Both intersect at the point X^ϵ of $\overset{\circ}{H}{}'$ and the determinant at this point of the matrix of partial derivatives of the above expressions respect the parameters is easily seen to coincide with $\det(\lambda_1^\epsilon, \lambda_2^\epsilon, \lambda_3^\epsilon, \lambda_4^\epsilon)$ which, for small values of ϵ, is nonzero and has the same sign σ as its limit det(0,0,0,0) as $\epsilon \to 0$. Therefore $\mathcal{Z}^{\underline{a}} \cap \overset{\circ}{H}{}'$ and $\tilde{\mathcal{Z}}^{\underline{a}} \cap \overset{\circ}{H}{}'$ have intersection number σa^2 at their only point of intersection. Since this happens at the $|c\cdot\tilde{c}|$ points of intersection of $\overline{\mathcal{Z}}^{\underline{a}}$ with $\overline{\tilde{\mathcal{Z}}}^{\underline{a}}$, their intersection number is $\sigma a^2|c\cdot\tilde{c}| = a^2(c\cdot\tilde{c})$.

It is now clear that, in the general case, taking coordinates u^k_{ij}, v^k_{ij} in disjoint neighborhoods U^k_{ij} of each x^k_{ij} and repeating the same argument for the open subset of the Hilbert scheme parametrizing subschemes contained in $U = \sqcup U^k_{ij}$, we will end up with a determinant which is the product indexed by i, j, k (as made of diagonal blocks) of determinants as the one we have ended up above, thus proving

$$[\overline{\mathcal{Z}}^{\underline{a}}]\cdot[\overline{\tilde{\mathcal{Z}}}^{\underline{a}}] = \prod_{ijk}(a^k_{ij})^2(c_{ij}\cdot\tilde{c}^k_{ij}) = \prod_{ijk}(a^k_{ij})^2(c_{ij}\cdot\tilde{c}_{ij})^{1+r_{ij}} \neq 0$$

3. PROOF OF T3

Gottsche has found in [G] that, the dimension of the sum of the Betti numbers of $\mathrm{Hilb}^d S$ is the coefficient of t^d in the series development of

$$\left(\prod_{m=1}^{\infty}\left(\frac{1}{1-t^m}\right)\right)\left(\prod_{m=1}^{\infty}(1-t^m)\right)^{b_1}\left(\prod_{m=1}^{\infty}\left(\frac{1}{1-t^m}\right)\right)^{b_2}$$
$$\left(\prod_{m=1}^{\infty}(1\pm t^m)\right)^{b_3}\left(\prod_{m=1}^{\infty}\left(\frac{1}{1-t^m}\right)\right)$$

Now, making $Z = 1$ in the lemma, [G] it asserts that

$$\left(\prod_{m=1}^{\infty}\left(\frac{1}{1-t^m}\right)\right) = \sum_{e=0}^{\infty}\left(\sum_{f\geq 0}p(e,e-f)\right)t^e$$

where $p(e, e-f)$ is the number of partitions of e as a sum of $e-f$ positive integers (not necessarely distinct). Therefore, $P(e) = \sum_{f\geq 0} p(e, e-f)$ is the number of partitions of e as a sum of positive integers. On the other hand, it is clear that

$$\left(\prod_{m=1}^{\infty}(1+t^m)\right) = \sum_{e=0}^{\infty}P[e]t^e$$

where $P[e]$ is the number partitions of e as a sum of **distinct** positive integers. Therefore, the dimension of $H(\mathrm{Hilb}^d(S))_Q$ is the coefficient of t^d in

$$\left(\sum_{e=0}^{\infty} P(e)t^e\right)\left(\sum_{e=0}^{\infty} P[e]t^e\right)^{b_1}\left(\sum_{e=0}^{\infty} P(e)t^e\right)^{b_2}\left(\sum_{e=0}^{\infty} P(e)t^e\right)^{b_3}$$
$$\left(\sum_{e=0}^{\infty} P(e)t^e\right) = \prod_{i=0}^{4}\prod_{j=1}^{b_i} P_{e_{ij}} t^{e_{ij}}$$

where $P_{e_{ij}}$ is $P(e_{ij})$ or $P[e_{ij}]$ according i is even or odd. This coefficient is

$$\sum_{i,j}\{P_{e_{ij}} | \sum_{i,j} e_{ij} = d\}$$

i.e. the total number of elements of each or our two candidates to be a base. Then by T1 and T2, we get T3 proved.

Remark. In fact we have found not only two, but four basis, the other two being very similar to the former. Indeed, we might have defined $[\overline{\mathcal{Z}^{\underline{a}}}]$ by taking elements $Z \in \mathcal{Z}^{\underline{a}}$ to be disjoint unions $\sqcup Z_{ij}^k$ with each Z_{ij}^k punctual and supported in a representant C_{ij}^k of the class c_{ij}, all of them different and mutually transverse. Analogously, we might have defined $[\overline{\tilde{\mathcal{Z}}^{\underline{a}}}]$ by taking $\tilde{Z} = \sqcup \tilde{Z}_{ij}^k \in \tilde{\mathcal{Z}}^{\underline{a}}$ so that $\tilde{Z}_{ij}^k$ is vertical intersecting in just one point a representant $\tilde{C}_{ij}$ of $\tilde{c}_{ij}$ (the same representant for all $k = 0, \ldots, r_{ij}$). The argument with these two new candidates - obviously of the same cardinality as the two old ones - would have been analogous.

References

[F] Fantechi, B., *Base of the homology groups of the Hilbert scheme of points on a surface.* Preprint.

[GH] Griffiths, P. and Harris, J., *Principles of algebraic geometry.* Addison-Wesley, 1977.

[G] Gottsche, L., *The Betti numbers of the Hilbert scheme of points on a smooth protective surface.* Math. Ann. 286 (1993), 235-245.

[GS] Gottsche, L. and Soergel, Perverse sheaves find the cohomology of Hilbert schemes of smooths algebraic surfaces, math. Ann. 296 (1993), 235-225.

[MS] Mallavibarrena, R. and Sols, I., *Bases for the homology groups of the Hilbert scheme of points in the plane.* Compositio Math. 74 (1990), 169-202.

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