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The Initial Value Problem for the Equations of Magnetohydrodynamic Type in Non-Cylindrical Domains

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ABSTRACT. By using the spectral Galerkin method, we prove the existence of weak solutions for a system of equations of magnetohydrodynamic type in non-cylindrical domains.

1. INTRODUCTION

In several situations the motion of incompressible electrical conducting fluids can be modelled by the so called equations of magnetohydrodynamics, which correspond to the Navier-Stokes' equations coupled with the Maxwell's equations. In the case where there is free motion of

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heavy ions, not directly due to the electric field (see Schlüter [14] and Pikelner [12]), these equations can be reduced to the following form:

$$\begin{cases} \frac{\partial u}{\partial t} - \frac{\eta}{\rho_m}\,\Delta u + u.\nabla u - \frac{\mu}{\rho_m} h.\nabla h = f - \frac{1}{\rho_m}\nabla(p^* + \frac{\mu}{2}\,h^2), \\ \frac{\partial h}{\partial t} - \frac{1}{\mu\sigma}\Delta h + u.\nabla h - h.\nabla u = -\text{grad}\,\omega, \\ \text{div}\ u = 0, \\ \text{div}\ h = 0, \end{cases} \tag{1.1}$$

together with suitable boundary and initial conditions.

Here, u and h are respectively the unknown velocity and magnetic fields; p^* is the unknown hydrostatic pressure; ω is an unknown function related to the motion of heavy ions (in such way that the density of electric current, j_0, generated by this motion satisfies the relation $rot\ j_0 = -\sigma\nabla\omega$); ρ_m is the density of mass of the fluid (assumed to be a positive constant); $\mu > 0$ is the constant magnetic permeability of the medium; $\sigma > 0$ is the constant electric conductivity; $\eta > 0$ is the constant viscosity of the fluid; f is an given external force field.

In this paper we will consider the problem of existence of weak solutions for the problem (1.1) in a time-dependent domain of $\mathbb{R}^n \times (0,T), n \geq 2, 0 < T < +\infty$.

To (1.1) we append the following initial and boundary conditions

$$u|_{\partial Q} = 0 \quad \text{and} \quad h|_{\partial Q} = 0 \quad \text{for all } t, \tag{1.1}$$

$$u(0) = u_0 \quad \text{and} \quad h(0) = h_0 \quad \text{in } Q(0), \tag{1.2}$$

where u_0 and h_0 are given functions. In (1.1), the diferential operator ∇ and Δ are the usual gradient and Laplace operator, respectively.

The main goal in this paper is to show existence of weak solutions for the initial value problem (1.1)-(1.3). Our strategy for setting this question consists of transforming problem (1.1)-(1.3) into another initial boundary-value problem in a cylindrical domain whose sections are not time-dependent. This is done by means of a suitable change of variable.

Next, this new initial value problem is treated using Galerkin approximation. We conclude returning to Q using the inverse of the above change of variable. This technicality was introduced by Dal Passo and Ughi [4] to study a certain class of parabolic equations in non-cylindrical domains.

We feel that it is appropriate to cite some earlier works on the initial value problem (1.1)-(1.3) and to locate our contribution therein. The cylindrical case of (1.1)-(1.3) has been studied by some authors. Among them, let us mention the paper of Lassner [7], Boldrini and Rojas-Medar [2], Rojas-Medar and Boldrini [13].

Lassner [7] by making the use of semigroup techniques as the ones in Fujita and Kato [5] to show the local existence and uniqueness of strong solution. The more constructive spectral Galerkin method was used by Boldrini and Rojas-Medar [2], [13] to obtain local, global existence and uniqueness of strong solutions. The above authors working in $\mathbb{R}^n$ with $n = 2$ or 3.

A mathematical study of the problem (1.1)-(1.3) in a non-cylindrical domain was not done, however, it has to be pointed out that similar time-dependent problem but for the Navier-Stokes and Boussinesq problems have been studied by many different authors. This is the case, for instance of the works Lions [9], [8], Fujita and Sauer [6], Ōeda [10], Ôtani and Yamada [11], Conca and Rojas-Medar [3]. In particular, we would like to emphasize that the arguments of the mentioned authors demand that $Q(t)$ be nondecreasing with respect to t (see Lions [9], problem 11.9, p. 426); except the work of Conca and Rojas-Medar [3].

In this work, we will adapt the technique by [3] to the system (1.1)-(1.3).

The paper is organized as follows. In Section 2 we introduce various functional spaces and state the theorems. Section 3 and 4 deal with their proofs.

2. FUNCTION SPACES AND PRELIMINAIRES

The functions in the paper are either $\mathbb{R}$ or $\mathbb{R}^n$-valued and we will not distinguish these two situations in our notations. To which case we refer will be clear from the context.

Let Ω be a bounded domain in $\mathbb{R}^n$ ($n \geq 2$) with smooth boundary $\partial\Omega$. We write simply $||u||$ for $L^2(\Omega)$-norm. The inner product in $L^2(\Omega)$ is denoted by (u, v). The solenoidal function space is defined as usual

$$C_{0,\sigma}^{\infty}(\Omega) = \{\varphi \in C_0^{\infty}(\Omega) \mid \text{div } \varphi = 0\},$$

$$H(\Omega) = \text{the completion of } C_{0,\sigma}^{\infty}(\Omega) \text{ under the } L^2(\Omega) - \text{norm},$$

$$V_s(\Omega) = \text{the completion of } C_{0,\sigma}^{\infty}(\Omega) \text{ under the } H^s(\Omega) - \text{norm}$$

where $H^s(\Omega)$ denote the usual Sobolev'space with $s \in \mathbb{R}$.

The norm and inner product in $H(\Omega)$ and $V_s(\Omega)$ are:

$$(f, g) = \sum_{i=1}^{n} \int_{\Omega} f_i g_i dx \ , \quad ||f|| = (f, f)^{1/2}$$

and

$$(u, v)_s = \sum_{i=1}^{n} (u_i, v_i)_{H^s} \ , \quad ||u||_s = (u, u)_s^{1/2}$$

where $(u_i, v_i)_{H^s}$ is the standard inner product of $H^s(\Omega)$; $(V_s(\Omega))'$ denotes the topological dual of $V_s(\Omega)$.

In particular, we denote

$$V_1(\Omega) = V(\Omega) \quad \text{and} \quad ||u||_1 = ||\nabla u||.$$

We will use other standard notations and terminology; for them, we refer to Adams[1] and Temam [15].

Let r be a real-valued function which is of C^1-class on the interval $[0, T]$ such that,

$$r(t_0) = \min\{r(t) \mid 0 \leq t \leq T\} > 0 \qquad (2.1)$$

(this condition is essential).

The time-dependent space domain $Q(t)$ is a bounded set in $\mathbb{R}^n$ defined by

$$Q(t) = \{x \in \mathbb{R}^n \ / \ |x| < r(t),\ 0 \leq t \leq T\}$$

where $|\cdot|$ denotes the usual norm in $\mathbb{R}^n$. Its boundary is

$$\partial Q(t) = \{x \in \mathbb{R}^n \mid |x| = r(t),\ 0 \leq t \leq T\}.$$

It is worth noting that such domains $Q(t)$ $0 \leq t \leq T$, generate a non-cylindrical time-dependent domain of $\mathbb{R}^n \times \mathbb{R}$:

$$Q = \bigcup_{0<t<T} Q(t) \times \{t\}.$$

In such conditions, we can now define a notion of weak solution for (11)-(13):

Definition 2.1. *We shall say that a couple of functions (u, h) defined in Q is a weak solution of (1.1)-(1.3) iff*

(i) $u, h \in L^2(0,T; V(Q(t))) \cap L^\infty(0,T; H(Q(t)))$

(ii) $-\int_Q \left[\alpha u.\varphi_t - \nu \sum\limits_{i=1}^{n} \nabla u_i \nabla \varphi_i + \alpha \sum\limits_{i_j=1}^{n} u_j \frac{\partial \varphi_i}{\partial x_j} u_i - \sum\limits_{i_j=1}^{n} h_j \frac{\partial \varphi_i}{\partial x_j} h_i\right] dxdt =$
$\alpha \int_Q f\varphi dxdt,$

(iii) $-\int_Q \left[h.\tilde{\varphi}_t - \gamma \sum\limits_{i=1}^{n} \nabla h_i \nabla \tilde{\varphi}_i + \sum\limits_{i_j=1}^{n} u_j \frac{\partial \tilde{\varphi}_i}{\partial x_j} h_i - \sum\limits_{i_j=1}^{n} h_j \frac{\partial \tilde{\varphi}_i}{\partial x_j} u_i\right] dxdt = 0$

for all $\varphi, \tilde{\varphi} \in C_0^1(Q)$ with div φ = div $\tilde{\varphi} = 0$, *the suffix t denotes the operator $\frac{\partial}{\partial t}$ (derivatives with respect to t will sometimes also be denoted by a' or simply by d/dt).*

(iv) $u(0) = u_0, \quad h(0) = h_0.$
where put $\alpha = \frac{pm}{\mu}$ $\nu = \frac{\eta}{\mu}$ and $\gamma = \frac{1}{\mu\sigma}$.

Remark 2.2. In this definition the initial conditions (iv) have the usual meaning; see for example, Lions [8].

The main result of our article is

Theorem 2.3. *If $f \in L^2(Q)$, u_0, $h_0 \in H(Q(0))$, then there exists a weak solution of problem (1.1)-(1.3) for any time interval* $[0,T]$.

Theorem 2.4. *If $n = 2$, the solution (u,h) obtained in the Theorem 2.3 is unique. Moreover u and h are almost everywhere equal to functions continuous from $[0,T]$ into H and*

$$u(t) \to u_0 \quad in\ H, \quad as\ t \to 0 \tag{2.2}$$

$$h(t) \to h_0 \quad in\ H, \quad as\ t \to 0. \tag{2.3}$$

3. PROOF OF THEOREM 2.3.

Let us introduce the transformation $\tau : Q \to U$, given by

$$\tau(x,t) = \left(\frac{x}{r(t)}, t\right)$$

where $U \equiv D \times (0,T)$ and $D = \{x \in \mathbb{R}^n \mid |x| \leq 1\}$. Since $r(t)$ is a C^1-function which verifies (2.1), we see easily that τ is a diffeomorphism and that its inverse $\tau^{-1} : U \to Q$ satisfies

$$\tau^{-1}(y,t) = (r(t)y,t). \tag{3.1}$$

We also define

$$\begin{aligned} v(y,t) &= u(yr(t),t), \\ b(y,t) &= h(yr(t),t), \\ q(y,t) &= p(yr(t),t), \\ \xi(y,t) &= \omega(yr(t),t), \\ J(y,t) &= f(yr(t),t), \\ v_0(y) &= u_0(yr(0)), \\ b_0(y) &= h_0(yr(0)). \end{aligned} \tag{3.2}$$

By using (3.2), the system on Q (1.1)-(1.3) is transformed into the system:

$$\alpha v' - \frac{\nu}{r(t)}\,\Delta v + \frac{\alpha}{r(t)}\sum_{i=1}^{h} v_i\,\frac{\partial v}{\partial y_i} - \frac{1}{r(t)}\sum_{i=1}^{n} b_i\frac{\partial b}{\partial y_i} = \alpha J + \frac{\mu}{r(t)} \nabla q + \alpha\,\frac{r'(t)}{r(t)}\sum_{i=1}^{n}\frac{\partial v}{\partial y_i}\,y_i, \tag{3.3}$$

$$b' - \frac{\gamma}{r(t)}\,\Delta b + \frac{1}{r(t)}\sum_{i=1}^{n} v_i\frac{\partial b}{\partial y_i} - \frac{1}{r(t)}\sum_{i=1}^{n} b_i\frac{\partial v}{\partial y_i} = \frac{r'(t)}{r(t)}\sum_{i=1}^{n}\frac{\partial b}{\partial y_i}\,y_i + \frac{1}{r(t)}\,\nabla\xi \tag{3.4}$$

$$v(0,y) = v_0(y) \tag{3.5}$$

$$b(0,y) = b_0(y) \tag{3.6}$$

on the cylindrical domain $U = D \times (0,T)$.

On the other hand, let us set

$$c(v,w) = \sum_{i,j=1}^{n}\int_D \frac{\partial v_i}{\partial y_j}\,y_j\,w_i dy$$

$$a(v,w) = \sum_{i=1}^{n}\int_D \frac{\partial v_j}{\partial y_i}\,\frac{\partial w_i}{\partial y_i}$$

$$B(u,v,w) = \sum_{i,j=1}^{n}\int_D u_j\,\frac{\partial v_i}{\partial y_j}\,w_i dy$$

for vector-valued functions u, v and w for which the integrals are well-defined.

The notation of weak solution for (3.3)-(3.6) is completely similar to the ones for (1.1)-(1.3).

To prove the existence of solutions of the transformed system (3.3)-(3.6) we will use the spectral Galerkin method. That is, we fix $s = \frac{n}{2}$ and we consider the Hilbert special basis $\{w^i(x)\}_{i=1}^{\infty}$ of $V_s(D)$, whose elements we will choose as the solution of the following spectral problem

$$(w^i, v)_s = \lambda_i (w^i, v) \qquad \forall v \in V_s(D).$$

Let V^k be the subspace of $V_s(D)$ spanned by $\{w^1, \dots, w^k\}$. For every $k \geq 1$, we define approximations v^k, b^k of v and b, respectively, by means of the following finite expansions:

$$v^k = \sum_{i=1}^{k} c_{ik}(t) w^i(x) \in V^k \qquad t \in (0, T)$$

and

$$b^k = \sum_{i=1}^{k} d_{ik}(t) w^i(x) \in V^k \qquad t \in (0, T)$$

where the coefficients (c_{ik}) and (d_{ik}) will be calculated in such a way that v^k and b^k solve the following approximations of system (3.3)-(3.6):

$$\alpha(v_t^k, \phi) + \frac{\nu}{[r(t)]^2}\, a(v^k, \phi) + \frac{\alpha}{r(t)}\, B(v^k, v^k, \phi) - \frac{1}{r(t)}\, B(b^k, b^k, \phi)$$

$$= \alpha(J, \phi) + \alpha\, \frac{r'(t)}{r(t)}\, c(v^k, \phi) \tag{3.7}$$

$$(b_t^k, \psi) + \frac{\gamma}{[r(t)]^2}\, a(b^k, \psi) + \frac{1}{r(t)}\, B(v^k, b^k, \psi) - \frac{1}{r(t)}\, B(b^k, v^k, \psi)$$

$$= \frac{r'(t)}{r(t)}\, c(b^k, \psi) \tag{3.8}$$

for all $\phi, \psi \in V^k$,

$$v^k(0) = v_0^k;\ b^k(0) = b_0^k \tag{3.9}$$

where $v_0^k \to v_0$ and $b_0^k \to b_0$ in $H(D)$ as $k \to \infty$.

Equations (3.7), (3.8) and (3.9) is a system of ordinary differential equations for the coefficient functions $c_{ik}(t)$ and $d_{ik}(t)$, which defines v^k and b^k in a interval $[0, t_k[$. We will show some a priori estimates independent of k and t, in order to take $t_k = T$. Also, we will prove that (v^k, b^k) converges in appropriate sense to a solution (v, b) of (3.3)-(3.6). We prove the following lemma.

Lemma 3.1. *The transformed system (3.3)-(3.6) admits at least one weak solution (v, b) in $L^2(0,T; V(D)) \cap L^\infty(0,T; H(D))$.*

Proof. Setting $\phi = v^k$ and $\psi = b^k$ in (3.7) and (3.8), respectively, we have

$$\frac{\alpha}{2}\ \frac{d}{dt}\ ||v^k||^2 + \frac{\nu}{[r(t)]^2}\ ||\nabla v^k||^2 =$$

$$\alpha(J, v_k) + \frac{1}{r(t)}\ B(b^k, b^k, v^k) + \frac{\alpha r'(t)}{r(t)}\ c(v^k, v^k)$$

$$\frac{1}{2}\ \frac{d}{dt}\ ||b^k||^2 + \frac{\gamma}{[r(t)]^2}\ ||\nabla b^k||^2 = \frac{1}{r(t)}\ B(b^k, v^k, b^k) + \frac{r'(t)}{r(t)}\ c(b^k, b^k)$$

since $B(v, w, w) = 0$ for $w \in V^k$.

Adding the above inequalities and observing that $\frac{1}{r(t)}\ [B(b^k, b^k, v^k) + B(b^k, v^k, b^k)] = 0$, we obtain

$$\frac{1}{2}\ \frac{d}{dt}\ (\alpha||v^k||^2 + ||b^k||^2) + \frac{1}{[r(t)]^2}\ (\nu||\nabla v^k||^2 + \gamma||\nabla b^k||^2)$$

$$= \alpha(J, v^k) + \frac{r'(t)}{r(t)}\ [\alpha c(v^k, v^k) + c(b^k, b^k)]$$

$$\equiv F^k.$$

Now, we use Hölder and Young inequalities to estimate the right-hand side of the above inequality, we obtain

$$|F^k| \leq \frac{1}{2\alpha}\, ||J||^2 + \frac{\alpha}{2}\, ||v^k||^2 + C_\varepsilon |r'(t)|^2 \alpha^2 ||y||^2_{L^\infty} ||v^k||^2$$

$$+ C_\delta |r'(t)|^2 ||y||^2_{L^\infty} ||b^k||^2 + \frac{\varepsilon}{[r(t)]^2}\, ||\nabla v^k||^2$$

$$+ \frac{\delta}{[r(t)]^2}\, ||\nabla b^k||^2,$$

whence, taking $\varepsilon = \nu/2$ and $\delta = \gamma/2$, we arrive at the inequality

$$\frac{d}{dt}\, (\alpha ||v^k||^2 + ||b^k||^2) + \frac{1}{[r(t)]^2}\, (\nu ||\nabla v^k||^2 + \gamma ||\nabla b^k||^2)$$

$$\leq \frac{1}{2\alpha}\, ||J||^2 + \left(C|r'(t)|^2 \alpha^2 ||y||^2_{L^\infty} + \frac{\alpha}{2} \right) ||v^k||^2 + C|r'(t)|^2 ||y||^2_{L^\infty} ||b^k||^2$$

$$\leq C||J||^2 + C(\alpha ||v^k||^2 + ||b^k||^2),$$

where C is a positive constant that depends only $\max \{|\gamma'(t)| \mid 0 \leq t \leq T\}$, α, $||y||^2_{L^\infty}$.

By integrating the above inequality between 0 and t with $0 \leq t \leq T$, we conclude:

$$\alpha ||v^k(t)||^2 + ||b^k(t)||^2 + \int_0^t \frac{1}{[r(s)]^2}\, (\nu ||\nabla v^k(s)||^2 + \gamma ||\nabla b^k(s)||^2) ds$$

$$\leq C \int_0^t ||J(s)||^2 ds + C \int_0^t (\alpha ||v^k(s)||^2 + ||b^k(s)||^2) ds +$$

$$\alpha ||v^k(0)||^2 + ||b^k(0)||^2.$$

Due to the choice of v_0^k and b_0^k, there exists C independent of k such that $||v_0^k|| \leq C||v_0||$, $||b_0^k|| \leq C||b_0||$, moreover $\int_0^t ||J(s)||^2 ds$ is finite for $0 \leq t \leq T$, we conclude, by using Gronwall's inequality, that

$$\alpha||v^k(t)||^2 + ||b^k(t)||^2 + \int_0^t \frac{1}{[r(s)]^2} \, (\nu||\nabla v^k(s)||^2 + \gamma||\nabla b^k(s)||^2)ds \leq C_1.$$

Thus, for all k we have that v^k and b^k exists globally in t and (v^k) and (b^k) remains bounded in $L^\infty(0,T;H(D)) \cap L^2(0,T;V(D))$ as $k \to \infty$. The next step of the proof consists of proving that (v_t^k) and (b_t^k) are bounded in $L^2(0,T;(V_s(D))')$. To this end, let us fix some notations

$$\langle A(t)v, u\rangle = \frac{1}{[r(t)]^2} \, a(v,u)$$

$$\langle C(t)v, u\rangle = \frac{r'(t)}{r(t)} \, c(v,u)$$

$$\langle E(t)v, u\rangle = \frac{1}{r(t)} \, B(v,v,u)$$

We will prove that $(A(t)v^k)$, $(C(t)v^k)$, $(E(t)v^k)$ and $(E(t)b^k)$ are bounded in $L^2(0,T;(V_s(D))')$.

Indeed, for all $u \in V_s$, we have

$$|A\langle A(t)v^k, u\rangle| = \frac{1}{[r(t)]^2} \, |a(v^k,u)|$$

$$\leq \frac{C}{[r(t)]^2} \, ||\nabla v^k|| \; ||u||_s,$$

whence, we have

$$||A(t)v^k||_{(V_s(D))'} = \sup_{||u||_s \neq 0} \frac{|\langle A(t)v^k, u\rangle|}{||u||_s} \leq \frac{C}{[r(t)]^2} \, ||\nabla v^k||,$$

consequently

$$\int_0^T ||A(t)v^k||^2_{(V_s(D))'}dt \leq \int_0^T \frac{C}{[r(t)]^2}\,||\nabla v^k(t)||^2 dt.$$

Since $\frac{1}{[r(t)]^2} \in C([0,T])$ and (v^k) is bounded in $L^2(0,T;V(D))$, we deduce that

$$\int_0^T \frac{C}{[r(t)]^2}\,||\nabla v^k(t)||^2 dt \leq L$$

and $(A(t)v^k)$ is therefore bounded in $L^2(0,T;(V_s(D))')$. Analogously, we show that $(C(t)v^k)$ is bounded in $L^2(0,T;V_s(D))')$. To prove the boundedness of $(E(t)v^k)$ in the space $L^2(0,T;(V_s(D))')$ we will use the following interpolation result whose proof can be found in Lions [8]:

Lemma 3.2. *If (v^k) is bounded in*

$$L^2(0,T;V(D)) \cap L^\infty(0,T;H(D)),$$

then (v^k) is also bounded in $L^4(0,T;L^p(D))$, where $\frac{1}{p} = \frac{1}{2} - \frac{1}{2n}$.

Using this Lemma, we conclude that

$$|\langle E(t)v^k, u\rangle| \leq \frac{1}{r(t)} \sum_{i,j=1}^n ||v_i^k||_{L^p}||v_j^k||_{L^p}||\frac{\partial u}{\partial y_i}||_{L^n}$$

$$\leq \frac{1}{r(t)} \sum_{i,j=1}^n ||v_i^k||_{L^p}||v_j||_{L^p}||\frac{\partial u}{\partial y_i}||_{H^{s-1}}$$

$$\leq \frac{C}{r(t)}\,||v^k||^2_{L^p}||u||_s$$

since $\frac{1}{p} + \frac{1}{p} + \frac{1}{n} = 1$ and the Sobolev embedding $H^{s-1}(D) \subseteq L^n(D)$. This imply

$$\int_0^T ||E(t)v^k||^2_{(V_s(D))'}dt \leq \int_0^T \frac{C}{[r(t)]^2}\,||v^k(t)||^4_{L^p}dt.$$

Since $\frac{1}{[r(t)]^2} \in C([0,T])$, we can conclude that $(E(t)v^k)$ is bounded in $L^2(0,T;(V_s(D))')$. Similary, we prove that $(E(t)b^k)$ is bounded in $L^2(0,T;(V_s(D))')$.

Now, we consider $P_k: \ H \to V^k$ defined by

$$P_k u = \sum_{i=1}^{k} (u, \omega^i)\omega^i$$

since $V_s(D) \subset H$ and $V^k \subset V_s(D)$, we can consider $P_k: \ V_s(D) \to V_s(D)$. It is easy to see that $P_k \in L(V_s(D), V_s(D))$, ($L(X,Y)$ denote the space of the bounded operator of X into Y) hence

$$P_k^*: \ (V_s(D))' \to (V_s(D))'$$

defined by $\langle P_k^*(v), \omega\rangle = \langle v, P_k(\omega)\rangle$ lies in $L((V_s(D))', (V_s(D))')$ and $||P_k^*|| \leq ||P_k|| \leq 1$.

We also observe that the autofunctions ω^i are invariants by P_k, i.e.,

$$P_k(\omega^i) = \omega^i.$$

From it and (3.7) we conclude that

$$\alpha(v_t^k, \omega^i) = \langle((-\nu A(t) - \alpha E(t) - \alpha C(t))v^k + \alpha J + E(t)b^k), \omega^i\rangle$$

$$= \langle P_k^*((-\nu A(t) - \alpha E(t) - \alpha C(t))v^k + \alpha J + E(t)b^k), \omega^i\rangle,$$

whence, for all $\omega \in V_k$, we have

$$\alpha(v_t^k, \omega) = \langle P_k^*((-\nu A(t) - \alpha E(t) - \alpha C(t))v^k + \alpha J + E(t)b^k), \omega\rangle.$$

Hence, by taking $\omega = P_k u$, for $u \in V_s(D)$, we obtain

$$\alpha(v_t^k, u) = \langle P_k^*((-\nu A(t) - \alpha E(t) - \alpha C(t))v^k + \alpha J + E(t)b^k), u\rangle,$$

and, consequently

$$\alpha v_t^k = P_k^*((-\nu A(t) - \alpha E(t) - \alpha C(t))v^k + \alpha J + E(t)b^k)$$

belong to $L^2(0,T;(V_s(D))')$ thanks to the previous estimates and $||P_k^*|| \leq 1$.

Working as before, we have

$$b_t^k = P_k^*((-\gamma A(t) - C(t))b^k + H(v^k,b^k) - H(b^k,v^k)),$$

where $H(u,\omega) = -\frac{1}{r(t)}\ \tilde{E}(u,\omega)$ and $\langle \tilde{E}(u,\omega),h\rangle = B(u,\omega,h)$. Consequently, it is sufficient to show that $H(v^k,b^k)$ and $H(b^k,v^k)$ belong to $L^2(0,T;(V_s(D))')$ to conclude that b_t^k is bounded in $L^2(0,T;(V_s(D))')$. We observe that

$$\int_0^T ||H(v^k,b^k)||^2_{(V_s(D))'}dt = \int_0^T \frac{1}{r(t)}\ ||\tilde{E}(v^k,b^k)||^2_{(V_s(D))'}dt.$$

On the other hand,

$$|\langle \tilde{E}(v^k,b^k),h\rangle| \leq \sum_{i,j} ||v_j^k||_{L_p}||b_i^k||_{L^p}||\frac{\partial h}{\partial y_i}||_{L^n}$$

$$\leq C||v^k||_{L^p}||b^k||_{L^p}||h||_s,$$

and therefore

$$||\tilde{E}(v^k,b^k)||^2_{(V_s(D))'} \leq C||v^k||^2_{L^p}||b^k||^2_{L^p}$$

this imply

$$\int_0^T \frac{1}{r(t)} \, \|\tilde{E}(v^k, b^k)\|^2_{(V_s(D))'} \leq$$

$$C\Big(\int_0^T \frac{1}{[r(t)]^2} \|v^k\|^4_{L^p}\Big)^{1/2}\Big(\int_0^T \frac{1}{[r(t)]^2} \|b^k\|^4_{L^p}\Big)^{1/2} . \leq C$$

Similarly, we prove that $H(b^k, v^k)$ is bounded in $L^2(0,T;(V_s(D))')$.

Therefore, arguing as in the book of Lions [8, p. 76] and making use of the Aubin-Lions Lemma with $B_0 = V(D)$, $p_0 = 2$, $B_1 = (V_s(D))'$, $p_1 = 2$ and $B = H(D)$ (see Theorem 1.5.1 and Lemma 1.5.2 of the above book, p. 58), we can conclude that there exist $v, b \in L^2(0,T;V(D))$ such that, up to a subsequence which we shall denote again by the suffix k, there hold

$$\left.\begin{array}{l} v^k \to v \\ b^k \to b \end{array}\right\} \quad \text{in } L^2(0,T;V(D)) \text{ and } L^\infty(0,T;H(D)) \text{ weakly and}$$

$$\left.\begin{array}{l} v^k \to v \\ b^k \to b \end{array}\right\} \quad \text{in } L^2(0,T;H(D)) \text{ strongly, as } k \to \infty$$

$$\left.\begin{array}{l} v^k_t \to v_t \\ b^k_t \to b_t \end{array}\right\} \quad \text{in } L^2(0,T;(V_s(D))' \text{ weakly, as } k \to \infty.$$

Now, the next step is to take the limit. But, once the above convergence results have been established, this is standard procedure and it follows the same patter as in Lions [8, p. 76-77]. Consequently, we shall omit it and we will directly deduce that

$$
\begin{aligned}
&-\int_0^T \alpha(v,\phi') + \int_0^T \frac{\nu}{[r(t)]^2}\, a(v,\phi) + \int_0^T \frac{\alpha}{r(t)}\, B(v,v,\phi) - \\
&\int_0^T \frac{1}{r(t)}\, B(b,b,\phi) = \int_0^T \alpha(J,\phi) + \int_0^T \alpha\, \frac{r'(t)}{r(t)}\, c(v,\phi)
\end{aligned}
\tag{3.10}
$$

and

$$
\begin{aligned}
&-\int_0^T \alpha(v,\psi') + \int_0^T \frac{\gamma}{[r(t)]^2}\, a(b,\psi) + \int_0^T \frac{1}{r(t)}\, B(v,b,\psi) - \\
&\int_0^T \frac{1}{r(t)}\, B(b,v,\psi) = \int_0^T \frac{r'(t)}{r(t)}\, c(b,\phi)
\end{aligned}
\tag{3.11}
$$

for all $\phi, \psi \in C^1(U)$ such that div $\phi =$ div $\psi = 0$. So, the Lemma is proved.

To conclude the proof of Theorem, let us consider a test function $\varphi \in C_0^1(U)$ such that div $\varphi = 0$, and define

$$\phi(y,t) = [r(t)]^n \varphi(yr(t),t).$$

Integrating by parts,

$$-\int_0^T \alpha(v,\phi') - \int_0^T \alpha\, \frac{r'(t)}{r(t)}\, c(v,\phi) = -\int_0^T \alpha[r(t)]^n (v,\varphi')$$

and also

$$\int_0^T \frac{\nu}{[r(t)]^2}\, a(v,\phi) = \sum_{i=1}^{n} \int_0^T \int_D \nu \,\frac{[r(t)]^n}{r(t)}\, \frac{\partial v}{\partial y_i}\, \frac{\partial \varphi}{\partial y_i};$$

$$\int_0^T \frac{\alpha}{r(t)}\, B(v,v,\phi) = -\sum_{i,j=1}^{n} \int_0^T \int_D \alpha [r(t)]^n v_i \,\frac{\partial \varphi_i}{\partial y_i}\, v_j$$

$$\int_0^T \frac{1}{r(t)}\, B(b,b,\phi) = -\sum_{i,j=1}^{n} \int_0^T \int_D [r(t)]^n b_i \,\frac{\partial \varphi_i}{\partial y_i}\, b_j.$$

By using the above identities in (3.11), we obtain,

$$-\int_0^T \int_D \alpha [r(t)]^2 v(y,t) \varphi'(yr(t),t) + \sum_{i=1}^{n} \int_0^T \int_D \nu \,\frac{[r(t)]^n}{r(t)}\, \frac{\partial v}{\partial y_i}\, \frac{\partial \varphi}{\partial y_i}$$

$$-\sum_{i=1}^{n} \int_0^T \int_D \alpha [r(t)]^n v_i v_j \,\frac{\partial \varphi_j}{\partial y_i} + \sum_{i=1}^{n} \int_0^T \int_D [r(t)]^n b_i b_j \,\frac{\partial \varphi_i}{\partial y_i}$$

$$= \int_0^T \int_D \alpha [r(t)]^n J\varphi. \tag{3.12}$$

Analagously, we obtain for b,

$$-\int_0^T \int_D [r(t)]^n b(y,t) \tilde{\varphi}'(yr(t),t) + \sum_{i=1}^{n} \int_0^T \int_D \gamma \,\frac{[r(t)]^n}{r(t)}\, \frac{\partial b}{\partial y_i}\, \frac{\partial \tilde{\varphi}}{\partial y_i}$$

$$-\int_0^T \int_D [r(t)]^n v_i b_j \,\frac{\partial \tilde{\varphi}_j}{\partial y_i} + \int_0^T \int_D [r(t)]^n b_i v_j \,\frac{\partial \tilde{\varphi}_j}{\partial y_i} = 0 \tag{3.13}$$

where $\tilde{\varphi} \in C_0^1(U)$ with div $\tilde{\varphi} = 0$.

Let us now consider the transformation $\tau^{-1} : U \to Q$ which is defined by (3.1). We observe that its Jacobian is $[r(t)]^n$. Consequently, by change of variables in the integrals, (3.13) and (3.14) become

$$-\int_Q \alpha u\varphi' + \sum_{i=1}^n \int_Q \nu \nabla u_i \nabla \varphi_i - \sum_{i,j=1}^n \int_Q u_j \frac{\partial \varphi_i}{\partial x_j} u_i$$

$$+ \sum_{i,j=1}^n \int_Q h_j \frac{\partial \varphi_i}{\partial x_j} h_i = \int_Q \alpha f.\varphi$$

and

$$-\int_a h.\tilde{\varphi}' + \sum_{i=1}^n \int_Q \gamma \nabla h_i \nabla \tilde{\varphi}_i - \sum_{i,j=1}^n \int_Q u_j \frac{\partial \tilde{\varphi}_i}{\partial x_j} h_i + \sum_{i,j=1}^n \int_Q h_j \frac{\partial \tilde{\varphi}_i}{\partial x_j} u_i = 0$$

which proves that (u,h) is a weak solution of the problem; since the mappings

$$L^2(0,T;V(D)) \to L^2(0,T;V(Q(t)))$$

$$v(y,t) \to u(x,t) = v\left(\frac{x}{r(t)}, t\right),$$

$$b(y,t) \to h(x,t) = h\left(\frac{x}{r(t)}, t\right),$$

and

$$L^2(0,T;H(D)) \to L^2(0,T;H(Q(t))$$

$$v(y,t) \to u(x,t) = v\left(\frac{x}{r(t)}, t\right),$$

$$b(y,t) \to h(x,t) = b\left(\frac{x}{r(t)}, t\right)$$

are smooth bijections of class C^1, it follows that

$$u, h \in L^2(0,T; V(Q(t))) \cap L^\infty(0,T; H(Q(t))).$$

Finally, a standard arguments shows that $u(0) = u_0$ and $h(0) = h_0$ (see remark 2.2). This finished the proof of the Theorem.

4. PROOF OF THEOREM 2.4.

We first prove the regularity result. We observe that the proof of the above theorem shown that $u' \in L^2(0,T; V')$; consequently, applying Lemma 1.2 in Temam [15], p. 260, we obtain that u is almost everywhere equal to a function continuous from $[0,T]$ into H.

Thus,

$$u \in C([0,T]; H)$$

and (2.2) follows easily. Analogously it is proved the continuity of h and (2.3).

We also recall that Lemma 1.2 in Temam [15], p. 260-261, asserts that the equations below holds:

$$\frac{d}{dt} \|u(t)\|^2 = 2\langle u'(t), u(t)\rangle,$$

$$\frac{d}{dt} \|h(t)\|^2 = 2\langle h'(t), h(t)\rangle.$$

These results will be used in the following proof of uniqueness which we will start now.

Consider that (u_1, h_1) and (u_2, h_2) are two solutions of the problem (1.1)-(1.3) with the same f and u_0, h_0 and define the differences $\omega =$

$u_1 - u_2$ and $v = h_1 - h_2$. They satisfy

$$\alpha(\omega_t, \phi) + \nu a(\omega, \phi) = -\alpha B(\omega, u_1, \phi) - \alpha B(u_2, \omega, \phi)$$

$$+ B(v, h_1, \phi) + B(h_2, v, \phi)$$

$$(v_t, \psi) + \gamma a(v, \psi) = -B(u_1, v, \psi) - B(\omega, h_2, \psi)$$

$$+ B(v, u_1, \psi) + B(h_2, \omega, \psi)$$

for any $\phi, \psi \in V$; also $\omega(0) = v(0) = 0$.

By the proof of Theorem 2.3, w_t and v_t belong to $L^2(0,T;V')$; consequently by setting $\phi = w$ and $\psi = v$ in the above inequalities, we obtain

$$\frac{\alpha}{2} \frac{d}{dt} \|\omega\|^2 + \nu a(\omega, \omega) = -\alpha B(\omega, u_1, \omega) + B(v, h_1, \omega) + B(h_2, v, \omega)$$

$$\frac{1}{2} \frac{d}{dt} \|v\|^2 + \gamma a(v, v) = -B(\omega, h_2, v) + B(v, u_1, v) + B(h_2, \omega, v)$$

thanks to the above remark.

Adding the above identities, we get

$$\frac{1}{2} \frac{d}{dt} (\alpha\|\omega\|^2 + \|v\|^2) + \nu\|\omega\|_1 + \gamma\|v\|_1$$

$$= -\alpha B(\omega, u_1, \omega) + B(v, h_1, \omega) - B(\omega, h_2, v) + B(v, u_1, v) \quad (3.14$$

since $B(h_2, v, \omega) + B(h_2, \omega, v) = 0$.

Now, we observe that

$$\alpha B(\omega,u_1,\omega)\le \alpha C||\omega||^2_{L^4}||u_1||_1$$

$$\le \alpha^2 C||\omega||\;||\omega||_1\;||u_1||_1$$

$$\le \frac{\nu}{6}\;||\omega||_1^2+C_\nu(\alpha)\alpha||\omega||^2||u_1||_1^2$$

where we used the Lemma 3.3 in Temam [15], p. 291, together with Hölder and Young inequalities.

Analogously, we can prove

$$B(v,u_1,v)\le\frac{\gamma}{6}\;||v||_1^2+C_\gamma||v||^2||u_1||_1^2,$$

$$B(v,h_1,\omega)\le\frac{\nu}{6}\;||\omega||_1^2+\frac{\gamma}{6}\;||v||_1^2+C_{\nu,\gamma}(\alpha)(||v||^2+\alpha||\omega||^2)||h_1||_1^2$$

$$B(\omega,h_2,v)\le\frac{\nu}{6}\;||\omega||_1^2+\frac{\gamma}{6}\;||v||^2+C_{\nu,\gamma}(\alpha)(||v||^2+\alpha||\omega||^2)||h_2||_1^2.$$

By using the above inequalities in (4.1), we get

$$\frac{d}{dt}\;(\alpha||\omega||^2+||v||^2)+\nu||\omega||_1^2+\gamma||v||_1^2$$

$$\le C(\alpha||\omega||^2+||v||^2)(||u_1||_1^2+||h_1||_1^2+||h_2||_1^2),$$

where C is a positive constant that only depend on ν,γ,α.

By integrating in time, the use of Gronwall's inequality, we obtain

$$\alpha||\omega(t)||^2+||v(t)||^2\le(\alpha||\omega(0)||^2+||v(0)||^2)e^{\varphi(t)}$$

where $\varphi(t) = C\int_0^t(||u_1||_1^2+||h_1||_1^2+||h_2||_1^2)ds < +\infty$, for every $t \in [0,T]$. This last inequality, implies $\omega(t) = v(t) = 0$. Hence $u_1 = u_2$ and $h_1 = h_2$ and the uniqueness is proved. This completes the proof of the Theorem.

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