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Partial Exact Controllability of a Nonlinear System

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ABSTRACT. In this article, we prove the partial exact controllability of a nonlinear system. We use semigroup formulation together with fixed point approach to study the nonlinear system.

1. INTRODUCTION

In this short article, we study the partial exact controllability, which we will make precise later, of a nonlinear system. Let Ω be a bounded open set in $\mathcal{R}^n$ with sufficiently smooth (say C^2) boundary Γ and $Q = (0,T)\times\Omega, \Sigma = (0,T)\times\Gamma$. Consider the coupled equations:

$$u_{tt} - \Delta u + f(\theta) = 0 \qquad \text{in } Q, \tag{1}$$

$$\theta_t - \Delta\theta + g(u) = 0 \qquad \text{in } Q, \tag{2}$$

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with initial conditions

$$u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x), \tag{3}$$

$$\theta(0,x) = \theta_0(x), \tag{4}$$

and boundary conditions

$$u(t,x) = v \quad \text{in } \Sigma, \tag{5}$$

$$\theta(t,x) = 0 \quad \text{in } \Sigma. \tag{6}$$

Here u, θ are the unknowns and v is the control function; f and g are nonlinear terms.

Definition. (Partial Exact Controllability): *We say the system (1)-(6) is partially exactly controllable if there exists $T > 0$ such that for any given initial data (u_0, u_1, θ_0) in a suitable space, there exists a control function v such that the corresponding solution of the system (1)-(6) satisfies*

$$u(T,\cdot) = u_1(T,\cdot) = 0 \qquad \text{in } \Omega.$$

In this article we prove the partial exact controllability of the above system when f and g are Lipschitz continuous with the further assumption that either the product of the Lipschitz constants of f and g is small enough or g is uniformly bounded on $\mathcal{R}$.

We have the following main theorem.

Theorem 1.1. *Assume that f and g are Lipschitz continuous with constants α, α_1, respectively. Then the system (1)-(6) is partially exactly controllable if the product $\alpha\ \alpha_1$ is sufficiently small and T is sufficiently large.*

We discuss the smallness of $\alpha\alpha_1$ at the end of this article.

There are enormous literature in the field of exact controllability. Here we would like to mention few references which are relevent for this article. In [8] E. Zuazua has studied the exact controllability for the nonlinear wave equation $u_{tt} - \Delta u + f(u) = 0, f$ Lipschitz, with initial conditions and boundary control. Here he does not assume any smallness of the Lipschitz constant. Possibly the same technique can be adapted to our system which we discuss in the discussion section of this paper.

In one dimensional setting, E. Zuazua [9]-[10] has studied the controllability of the equation $u_{tt} - u_{xx} + f(u) = 0$ with super linear f satisfying the hypothesis $\lim \frac{f(s)}{|s| log^2 |s|} = 0$ as $|s| \to \infty$. We hope that our system can also be tackled with such nonlinearities f and g satisfying $\lim \frac{f(s)}{|s| log^2 |s|} = 0$ and $\lim \frac{g(s)}{|s| log |s|} = 0$.

For a good survey in the field of exact controllability using a new technique, the so called HUM method introduced by Lions, one can refer to Lions [6]-[7] and the references therein. Lasiecka and Triggiani [3]-[5] (more references can be seen in their papers) use semigroup formalism to study the control problem which is more an algebraic method while the HUM method uses only PDE. Essentially both the approaches reduce the controllability and/or reachability problems to some energy inequalities. We use a fixed point approach combined with the technique of Lasiecka and Triggiani for studying the system (1)-(6). We will not go into the details of the literature and one can see the above cited references.

2. FIXED POINT APPROACH

We use the notations of standard Sobolev spaces. Let $X = L^2(\Omega)$. Our control space is $H_0^1(0, T, L_2(\Gamma))$. Let $\tilde{\theta}$ be fixed in $C([0,T], X)$. Assume that $f(\tilde{\theta})$ is in $C([0,T], X)$. We proceed as follows:

STEP 1: With the known $f(\tilde{\theta})$, consider the equation (1) with the conditions (3) and (5) which is an exact controllability problem for the unknown u. Solve the problem to obtain the controlled solution $\tilde{u}$ and a steering control v, which of course depends on $\tilde{\theta}$. Observe that this control can be chosen in a unique fashion (see equation (13) and also the Remark 3.2) and denote by N the operator defined as $N(\tilde{\theta}) = \tilde{u}$.

STEP 2: Now take the equation (2) where u is replaced by the known solution $\tilde{u}$ obtained in Step 1 and solve for θ with the conditions (4), (6) and this operator is denoted by K, i.e., $K(\tilde{u}) = \theta$.

STEP 3: If the operator $KN : C([0,T], X) \to C([0,T], X)$ with K and N as above, has a fixed point, then the system (1)-(6) is partially exactly controllable.

First we prove the Step 1, where we employ the technique of Lasiecka and Triggiani to get the exact controllability.

Lemma 2.1. *Let $\tilde{\theta} \in C([0,T],X)$ be fixed, then the system*

$$
\begin{aligned}
u_{tt} - \Delta u = -f(\tilde{\theta}) \qquad & \text{in } Q, \\
u(0,x) = u_0(x),\ u_t(0,x) = u_1(x) \qquad & \text{in } \Omega, \\
u(t,x) = v \qquad & \text{in } \Sigma,
\end{aligned}
\tag{7}
$$

with the initial conditions $u_0 \in L^2(\Omega)$, $u_1 \in H^{-1}(\Omega)$, is exactly controllable with control $v \in L^2(0,T;\Gamma)$. Further, if f is Lipschitz continuous with constant α then the operator N defined in Step 2 is Lipschitz continuous from $C([0,T];X)$ into itself.

Proof: Let $\mathcal{A} : \mathcal{D}(\mathcal{A}) \subset L_2(\Omega) \to L_2(\Omega)$ be the positive self adjoint operator defined by

$$\mathcal{A}h = -\Delta h,\ \mathcal{D}(\mathcal{A}) = H^2(\Omega) \cap H_0^1(\Omega).$$

Then $-\mathcal{A}$ generates a strongly continuous cosine operator $C(t)$ on $L_2(\Omega)$ with the sine operator $S(t) = \int_0^t C(\tau)d\tau$. Define the Dirichlet map D as follows:

$$Dg = h \iff \Delta h = 0 \quad \text{in } \Omega,\ h = g \quad \text{on } \Gamma. \tag{8}$$

Then $D: \ H^s(\Gamma) \to H^{s+\frac{1}{2}}(\Omega),\ s \geq 0$ is continuous. Then (7) can be written as a system of first order equtions as:

$$\frac{d}{dt}\begin{bmatrix} u \\ u_t \end{bmatrix} = \begin{bmatrix} 0 & I \\ -\mathcal{A} & 0 \end{bmatrix}\begin{bmatrix} u \\ u_t \end{bmatrix} + \begin{bmatrix} 0 \\ \mathcal{A}\mathcal{D}v \end{bmatrix} + \begin{bmatrix} 0 \\ -f(\tilde{\theta}) \end{bmatrix}. \tag{9}$$

Then

$$A = \begin{bmatrix} 0 & I \\ -\mathcal{A} & 0 \end{bmatrix} : \ \mathcal{D}(\mathcal{A}) \times \mathcal{D}(\mathcal{A}^{\frac{1}{2}}) \to \mathcal{D}(\mathcal{A}^{\frac{1}{2}}) \times L^2(\Omega),$$

generates a unitary strongly continuous group

$$\phi(t) = e^{At} = \begin{bmatrix} C(t) & S(t) \\ -AS(t) & C(t) \end{bmatrix} \text{ on } H^1(\Omega) \times L^2(\Omega).$$

For a given v, a mild solution of (7) can be written as

$$\begin{bmatrix} u \\ u_t \end{bmatrix} = \begin{bmatrix} C(t) & S(t) \\ -AS(t) & C(t) \end{bmatrix} \begin{bmatrix} u_0 \\ u_1 \end{bmatrix}$$

$$+ \int_0^t \begin{bmatrix} C(t-\tau) & S(t-\tau) \\ -AS(t-\tau) & C(t-\tau) \end{bmatrix} \begin{bmatrix} 0 \\ ADv(\tau) \end{bmatrix} d\tau$$

$$+ \int_0^t \begin{bmatrix} C(t-\tau) & S(t-\tau) \\ -AS(t-\tau) & C(t-\tau) \end{bmatrix} \begin{bmatrix} 0 \\ -f(\tilde{\theta}) \end{bmatrix} d\tau.$$

Now define a mapping $L_T : \ H_0^1(0,T,L^2(\Gamma)) \to L^2(\Omega) \times H^{-1}(\Omega)$ by

$$\begin{aligned} L_T v &= \int_0^T \begin{bmatrix} C(T-\tau) & S(T-\tau) \\ -AS(T-\tau) & C(T-\tau) \end{bmatrix} \begin{bmatrix} 0 \\ ADv \end{bmatrix} d\tau \\ &= \begin{bmatrix} A & \int_0^T S(T-\tau)Dv(\tau) \\ A & \int_0^T C(T-\tau)Dv(\tau) \end{bmatrix}. \end{aligned} \tag{10}$$

Then as a consequence of trace regularity for hyperbolic systems (see [2], [5]) L_T is continuous from $H_0^1(0,T;L^2(\Gamma)) \to H_0^1(\Omega) \times L^2(\Omega)$.

So the question of controllability of (7) reduces to the surjectivity of the mapping L_T which can be obtained from Lasiecka and Triggiani [5]. So, for a given $\bar{\theta}$, we have the controllability for the solution (u, u_t) on the space $H_0^1(\Omega) \times L^2(\Omega)$ within the class of controls $H_0^1(0,T;L^2(\Gamma))$, provided T is large enough or equivalently on the space $L^2(\Omega) \times H^{-1}(\Omega)$

with controls in $L^2(0,T;L^2(\Gamma))$ (See [5]). The operator N is given by $N\tilde{\theta} = u$ and put $U = \begin{bmatrix} u \\ u_t \end{bmatrix}$, where

$$U(t) = \phi(t)U_0 + \int_0^t \phi(t-\tau)F(\tilde{\theta})d\tau + \int_0^t \phi(t-\tau)Bv d\tau, \tag{11}$$

with

$$U_0 = \begin{bmatrix} u_0 \\ u_1 \end{bmatrix}, \; F(\tilde{\theta}) = \begin{bmatrix} 0 \\ f(\tilde{\theta}) \end{bmatrix}, \; Bv = \begin{bmatrix} 0 \\ \mathcal{A}Dv \end{bmatrix}.$$

Let U_1, U_2 be, respectively, the controlled solutions corresponding to $\tilde{\theta}_1$ and $\tilde{\theta}_2$ then

$$U_1(t) - U_2(t) = \int_0^t \phi(t-\tau)[F(\tilde{\theta}_1) - F(\tilde{\theta}_2)]d\tau + \int_0^t \phi(t-\tau)B(v_1 - v_2)d\tau. \tag{12}$$

Since f is Lipschitz, the first term on the righthand side can be estimated as

$$\left\| \int_0^t \phi(t-\tau)[F(\tilde{\theta}_2) - F(\tilde{\theta}_1)]d\tau \right\|_X^2 \le Ct\alpha^2 \int_0^t \|\tilde{\theta}_1 - \tilde{\theta}_2\|_X^2 d\tau.$$

Now we estimate the second term as follows: The operator

$$L_T : \; H_0^1(0,T;L^2(\Gamma)) \to H_0^1(\Omega) \times L^2(\Omega),$$

defined by

$$L_T v = \int_0^T \begin{bmatrix} C(T-\tau) & S(T-\tau) \\ -\mathcal{A}S(T-\tau) & C(T-\tau) \end{bmatrix} \begin{bmatrix} 0 \\ \mathcal{A}Dv \end{bmatrix} d\tau$$

$$= \int_0^T \phi(T-\tau)Bv(\tau)d\tau,$$

is bounded, linear and onto and its adjoint L_T^* is given by

$$(L_T^* z)(t) = B^* \phi^*(T - t) z.$$

More details on those operators L_T, L_T^* etc. can be found in [5]. The following control $v_1 \in N(L_T)^{\perp}$ defined by

$$v_1(t) = B^* \phi^*(T-t)(L_T L_T^*)^{-1} \left[-\phi(T)U_0 - \int_0^T \phi(T-\tau)F(\tilde{\theta}_1)d\tau \right], \quad (13)$$

steers the system to zero state. Hence

$$\int_0^t \phi(t-\tau)(Bv_1 - Bv_2)d\tau = \int_0^t \phi(t-\tau)BB^*\phi^*(T-\tau)(L_T L_T^*)^{-1}$$

$$\cdot \left[\int_0^T \phi(T-s)(F(\tilde{\theta}_2) - F(\tilde{\theta}_1))ds \right] d\tau.$$

Observe that the exact controllability of the linear systems implies the existence of $(L_T L_T^*)^{-1}$. Now it is easy to see that (for suitable choices of constants C and k)

$$\left\| \int_0^t \phi(t-\tau)B(v_1 - v_2)d\tau \right\|_X^2$$

$$\leq C \left\| \int_0^T \phi(T-\tau)(F(\tilde{\theta}_2) - F(\tilde{\theta}_1))d\tau \right\|_X^2$$

$$\leq C\alpha^2 \int_0^T \|\tilde{\theta}_1 - \tilde{\theta}_2\|_X^2 d\tau.$$

Hence,

$$||(N\tilde{\theta}_1)(t) - (N\tilde{\theta}_2)(t)||_X^2 \leq k^2\alpha^2 \int_0^T ||\tilde{\theta}_1 - \tilde{\theta}_2||_X^2 d\tau. \qquad (14)$$

Therefore,

$$||N\tilde{\theta}_1 - N\tilde{\theta}_2||_{C([0,T];X)} \leq C||\tilde{\theta}_1 - \tilde{\theta}_2||_{C([0,T];X)} \, .$$

This proves that the operator N is Lipschitz continuous. This completes the proof of the lemma.

Lemma 2.2. *Suppose that g is Lipschitz continuous. Then the operator K defined in Step 2 is well defined and Lipschitz continuous and compact.*

Proof. The operator K is given by $K\tilde{u} = \theta$, where θ is the solution of

$$\theta_t - \Delta\theta = -g(\tilde{u}) \text{ in } Q,$$

$$\theta(0,x) = \theta_0(x) \text{ in } \Omega, \qquad (15)$$

$$\theta(t,x) = 0 \text{ in } \Sigma.$$

Define $\mathcal{A}$ such that $\mathcal{A}h = -\Delta h$ with $\mathcal{D}(\mathcal{A}) = H^2(\Omega) \cap H_0^1(\Omega)$. Then $\mathcal{A}$ is self adjoint and it generates a compact semigroup $T(t)$, $t > 0$, given by (see [1])

$$T(t)x = \sum_{n=1}^{\infty} e^{\lambda_n t} \sum_{j=1}^{r_n} < x, \phi_{nj} > \phi_{nj},$$

where $\{\phi_{nj}\}$ is the system of complete orthonormal eigenfunctions for Δ with Dirichlet data and $\{\lambda_n\}$ is the corresponding set of eigenvalues, with multiplicity r_n. Hence the solution of (15) can be represented as

$$\theta(t) = T(t)\theta_0 - \int_0^t T(t-\tau)g(\tilde{u})d\tau.$$

If $\theta_1 = K\tilde{u}_1,\ \theta_2 = K\tilde{u}_2$, then

$$||K\tilde{u}_1 - K\tilde{u}_2||_{C([0,T];X)} \leq k_1\alpha_1||\tilde{u}_1 - \tilde{u}_2||_{C([0,T];X)},$$

where $k_1 > 0$ is some constant, α_1 being the Lipschitz constant of g. Hence K is Lipschitz continuous.

Since the semigroup generated by $\mathcal{A}$ is compact, it is not hard to verify that the operator K is compact. This completes the proof of Lemma 2.2.

3. PROOF OF THEOREM 1.1

We prove that under the hypothesis of the theorem, the operator

$$KN:\ C([0,T];X) \to C([0,T];X),$$

defined by $KN\tilde{\theta} = \theta$, where θ is given by

$$\theta(t) = T(t)\theta_0 - \int_0^t T(t-\tau)g(N\tilde{\theta})d\tau,$$

is a contraction, which in turn implies partial exact controllability of the system (1)-(6). For $\tilde{\theta}_1, \tilde{\theta}_2 \in C([0,T];X)$, we have

$$(KN\tilde{\theta}_1 - KN\tilde{\theta}_2)(t) = \int_0^t T(t-\tau)[g(N\tilde{\theta}_2) - g(N\tilde{\theta}_1)]d\tau.$$

Now

$$\begin{aligned} ||g(N\tilde{\theta}_1) - g(N\tilde{\theta}_2)||_X^2 &\leq \alpha_1^2||N\tilde{\theta}_1 - N\tilde{\theta}_2||_X^2\ \forall t \in [0,T] \\ &\leq k^2\alpha^2\alpha_1^2\int_0^T ||\tilde{\theta}_1 - \tilde{\theta}_2||_X^2 d\tau, \end{aligned} \tag{16}$$

so that

$$||KN\tilde{\theta}_1 - KN\tilde{\theta}_2||_X^2 \le C\alpha^2\alpha_1^2||g(N\tilde{\theta}_1) - g(N\tilde{\theta}_2)||_X^2$$
$$\le CT\alpha^2\alpha_1^2||\tilde{\theta}_1 - \tilde{\theta}_2||^2_{C([0,T];X)}. \tag{17}$$

Therefore

$$||KN\tilde{\theta}_1 - KN\tilde{\theta}_2||_{C([0,T];X)} \le C\alpha\alpha_1||\tilde{\theta}_1 - \tilde{\theta}_2||_{C([0,T];X)}.$$

Hence KN is a contraction if $\alpha\alpha_1$ is sufficiently small enough and thus KN has a fixed point. This proves the theorem.

Remark 3.1 In fact, we need not have to apply control on all of Σ. One can partition $\Sigma = \Sigma_0 \cup \Sigma_1$, where $\Sigma_0 = (0,T)\times\Gamma_0$, $\Sigma_1 = (0,T)\times\Gamma_1$ and $\Gamma = \Gamma_0 \cup \Gamma_1$ with $\Gamma_0 = \{x \in \Gamma : (x - x_0).\nu(x) > 0\}$, where $\Gamma_1 = \Gamma \setminus \Gamma_0$, $x^0 \in \mathcal{R}^n$. Here $\nu(x)$ is the unit normal to the boundary at x. Then we can obtain the conclusions of our main theorems by applying controls only on the part Σ_0 ([6]).

Remark 3.2 The control given by (13) can also be uniquely obtained by using HUM method. We can write the controlled solution u of (7) as $u = u^1 + u^2$, where u^1 is the unique solution of the following problem:

$$u^1_{tt} - \Delta u^1 = -f(\theta) \quad \text{in } Q,$$

$$u^1(0,x) = 0,\ u^1_t(0,x) = 0 \quad \text{in } \Omega,$$

$$u^1(t,x) = 0 \quad \text{on } \Sigma.$$

Since we are interested in finding the control v satisfying $u(T) = u_t(T) = 0$ in Ω, u^2 should satisfy the reachability problem

$$u^2_{tt} - \Delta u^2 = 0 \quad \text{in } Q,$$

$$u^2(0,x) = u_0(x), u^2_t(0,x) = u_1(x) \quad \text{in } \Omega,$$

$$u^2(t,x) = v \quad \text{on } \Sigma.$$

with the reachability conditions $u^2(T) = -u^1(T)$, $u^2_t(T) = -u^1_t(T)$. This control can be defined in a unique fashion as the minimization of $f_\Sigma v^2 d\Sigma$, among all possible admissible controls.

Remark 3.3 The restriction made in Theorem 1.1 on the Lipschitz constants of f and g can be relaxed if we assume that g is uniformly bounded, (that is, there exists a positive constant k such that $|g(x)| \leq k$, for all $x \in \mathcal{R}$), as we see in the following theorem.

Theorem 3.1 *Suppose that f and g are Lipschitz continuous and g is uniformly bounded, then the system is partially exactly controllable.*

Proof. Let N and K be operators defined as in Lemma 2.1 and Lemma 2.2. Note that N and K are continuous operators. Following the proof of Theorem 1.1, it suffices to show that the operator $KN : C([0,T],X) \to C([0,T],X)$ defined by $KN(\tilde{\theta}) = \theta$, where θ is given by

$$\theta(t) = T(t)\theta_0 - \int_0^t T(t-\tau)g(N\tilde{\theta})d\tau.$$

has a fixed point. It follows that KN is a continuous and compact operator. The uniform boundedness of g implies that there exists $r > 0$ such that $||KN\tilde{\theta}|| \leq r$ for all $\tilde{\theta}$ in $C([0,T],X)$. Let B_r be a closed ball in $C([0,T],X)$ with center 0 and radius r. Thus KN maps B_r into B_r and hence by Schauder's fixed point theorem there exists a θ such that $KN\theta = \theta$, and hence the system (1)-(6) is partially exactly controllable.

Remark 3.4 Of course, one can reverse the hypotheses on f and g in Theorem 3.1. That is, instead of g being bounded, the uniform boundedness of f is sufficient to conclude the same result. On the other hand, the hypothesis that g is uniformly bounded can be relaxed to g is asymptotically linear, (refer E. Zuazua [11]).

In this short note we only showed that we can achieve partial exact controllability using the classical method together with a fixed point approach. Of course it may be possible to obtain the same result using the HUM method introduced by Lions.

4. DISCUSSION

Here we discuss on the assumption that the product $\alpha\alpha_1$ of the Lipschitz constants is small enough. Of course it would be interesting to know the same result without the above assumption. We indicate some possible directions one can go about it. Recall the proof of Lipschitz (Lemma 2.1) continuity of N. We have proved that (see (14))

$$||(N\tilde{\theta}_1 - N\tilde{\theta}_2)(t)||_X^2 \leq C\alpha^2 \int_0^T ||\tilde{\theta}_1 - \tilde{\theta}_2||_X^2 d\tau.$$

In the estimation of the above inequality we had to estimate two terms on the right hand side of (12) and we obtained that the first term, namely,

$$\left|\left| \int_0^t \phi(t-\tau)[F(\tilde{\theta}_1) - F(\tilde{\theta}_2)]d\tau \right|\right|_X^2 \leq Ct\alpha^2 \int_0^t ||\tilde{\theta}_1 - \tilde{\theta}_2||_X^2 d\tau.$$

Observe that the limit of integration on righthand side is from 0 to t. Suppose we could prove the second term is also bounded by the similar estimate (Note that we have only proved $|| \int_0^t \phi(t-\tau)(Bv_1 - Bv_2)d\tau||_X^2 \leq Ct\alpha^2 \int_0^T ||\tilde{\theta}_1 - \tilde{\theta}_2||_X^2 d\tau$, where the limit of integrations is from 0 to T), then it follows that

$$||N\tilde{\theta}_1 - N\tilde{\theta}_2||_X^2 \leq C\alpha^2 \int_0^t ||\tilde{\theta}_1 - \tilde{\theta}_2||_X^2 d\tau. \quad (18)$$

In this case, without the assumption $\alpha\alpha_1$ is small, we can prove that KN has a fixed point. In fact, we prove that $(KN)^n$ is a contraction for some $n > 1$ and this can be seen as follows.

Recall the operator $KN\tilde{\theta} = \theta$, defined in Section 2, where θ is given by

$$\theta(t) = T(t)\theta_0 - \int_0^t T(t-\tau)g(N\tilde{\theta})d\tau.$$

With the assumption

$$||(N\tilde{\theta}_1 - N\tilde{\theta}_2)(t)||_X^2 \leq Ct\alpha^2 \int_0^t ||\tilde{\theta}_1 - \tilde{\theta}_2||_X^2,$$

we get,

$$||g(N\tilde{\theta}_1) - g(N\tilde{\theta}_2)||_X^2 \leq Ct\alpha^2\alpha_1^2 \int_0^t ||\tilde{\theta}_1 - \tilde{\theta}_2||_X^2,$$

and we can show that

$$\begin{aligned} ||(KN\tilde{\theta}_1 - KN\tilde{\theta}_2)(t)||_X^2 &\leq k\alpha_1^2\alpha^2 \int_0^t \left(\tau \int_0^\tau ||\tilde{\theta}_1 - \tilde{\theta}_2||_X^2\right)d\tau \\ &\leq k\alpha_1^2\alpha^2 \frac{t^2}{2} \left(\int_0^t ||\tilde{\theta}_1 - \tilde{\theta}_2||_X^2\right), \end{aligned} \tag{19}$$

where k is some constant which is fixed now onwards.

By a repeated application of (19) it can be seen that

$$||(KN)^n\tilde{\theta}_1 - (KN)^n\tilde{\theta}_2||_X^2$$

$$\leq (k\alpha_1^2\alpha^2)^n \left(\frac{t^{3n-1}}{2^n \cdot 3^{n-1} \cdot (n-1)!}\right) \left(\int_0^t ||\tilde{\theta}_1 - \tilde{\theta}_2||_X^2\right)d\tau,$$

$$\|(KN)^n\tilde{\theta}_1 - (KN)^n\tilde{\theta}_2\|^2_{C([0,T];X)}$$

$$\leq 3\left(\frac{kT^3\alpha_1^2\alpha^2}{2\cdot 3}\right)^n\left(\frac{1}{(n-1)!}\right)\|\tilde{\theta}_1 - \tilde{\theta}_2\|^2_{C([0,T];X)}.$$

So as $n \to \infty$, we can make the constant on the right hand side of the above inequality, less than 1 for n large enough and $(KN)^n$ is a contradiction for some n and hence KN has a fixed point.

Now we present a different approach adapting the technique of E. Zuazua [8]. In [8] he has studied the nonlinear wave equation $u_{tt} - \Delta u + f(u) = 0$, f Lipschitz, with initial conditions and boundary control. He has proved that the above equation is exactly controllable in suitable spaces under the assumption that f is globally Lipschitz continuous. Adapting the same technique to our present system, one is lead to the study of the uniform exact controllability of the following linear system

$$u_{tt} - \Delta u + W_1\theta = h_1 \qquad \text{in } Q,$$

$$\theta_t - \Delta\theta + W_2 u = h_2 \qquad \text{in } Q,$$

with conditions as in (3)-(6). Here $h_1, h_2 \in L^2(Q)$ and $W_1 = W_1(x,t)$, $W_2 = W_2(x,t)$ are potentials which are in $L^\infty(Q)$.

If the above system is uniformly exactly controllable (uniform with respect to the potentials W_1, W_2, see E. Zuazua [8]), then it is possible to obtain exact controllability of the system (1)-(6) for globally Lipschitz continuous functions f and g.

A brief description of the method is as follows. Fix $(\tilde{u},\tilde{\theta}) \in L^2(Q)\times L^2(Q)$ and define $W_1 = \frac{f(\tilde{\theta})-f(0)}{\tilde{\theta}}$, $W_2 = \frac{g(\tilde{u})-g(0)}{\tilde{u}}$ and $h_1 = -f(0)$, $h_2 = -g(0)$. Obviously, $W_1, W_2 \in L^\infty(Q)$ because f and g are Lipschitz continuous. Now using the uniform exact controllability of the above linear system with potentials W_1 and W_2 defined as above, one can show that the controlled solution (u,θ) satisfies $\|(u,\theta)\|_{L^2(Q)\times L^2(Q)} \leq C$, where the constant C is independent of $(\tilde{u},\tilde{\theta})$. With this estimate in mind, one can define a mapping $F: L^2(Q)\times L^2(Q) \to L^2(Q)\times L^2(Q)$ by $F(\tilde{u},\tilde{\theta}) = (u,\theta)$ with satisfies $\|F(\tilde{u},\tilde{\theta})\| \leq C$ uniformly. Also, it is not difficult to show that the mapping F is compact. Now one can apply the Schauder's fixed point theorem to get the desired result in suitable spaces. So the controllability problem for our system (1)-(6) reduces to the uniform controllability problem of the above linear system with bounded potentials W_1 and W_2 and the uniform controllability of the linear system seems to be an open problem.

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