REVISTA MATEMATICA de la
Universidad Complutense de Madrid
Volumen 8, número 1: 1995

Sequence Spaces Generated by Moduli of Smoothness

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ABSTRACT. There are defined sequential moduli in the remainder form for real sequences. Properties of sequence spaces generated by means of the above moduli are investigated.

1. INTRODUCTION

In many problems of mathematical analysis, one of the important tools form moduli of continuity and smoothness and variations of a function. The modulus of continuity may be defined in spaces of continuous functions and in L^P-spaces. In [6] and [7] we transfered the notion of modulus of continuity to spaces of sequences, by the formula $\omega(x,r) = \sup_{m\geq r}\sup_{i\geq m} |t_{m+i} - t_i|$, where $x = (t_i)_{i=0}^{\infty}$, $r = 0,1,2,\ldots$. We developed a theory of modular spaces of sequences generated by the modulus (see also [3]).

1991 Mathematics Subject Classification: 46A45, 46E30
Servicio publicaciones Univ. Complutense. Madrid, 1995.

In the present paper we transfer the definition of L^P-modulus to the sequential case, introducing the remainder form of the sequential modulus. Moreover, we replace the power p by a sequence of φ-functions, $\varphi = (\varphi_i)_{i=1}^{\infty}$, (for definition of φ-function see for instance [4], 1.9). There are analysed structural properties of modular spaces generated by means of the above notions. In a subsequent paper we shall show application to problems of two modular convergence of sequences with aid of moduli of smoothness and Φ-variations and we shall derive some inequalities.

2. MODULUS OF SMOOTHNESS

We introduce the remainder form of the sequential modulus in the space X of all real sequences. Let $x = (t_i)_{i=0}^{\infty} \in X$, then we denote $(x)_j = t_j$ and we write $(\tau_m x)_j = t_j$ for $j < m$ and $(\tau_m x)_j = t_{m+j}$ for $j \geq m$ where $m, j = 0, 1, 2, \ldots$. The sequence $\tau_m x = ((\tau_m x)_j)_{j=0}^{\infty}$ is called the m-translation of the sequence x (see [6]). Let $\varphi = (\varphi_i)_{i=1}^{\infty}$ be a sequence of φ-functions. The remainder form of the sequential φ-modulus of the sequence x will be defined as

$$\omega_\varphi(x, r) = \sup_{m \geq r} \sum_{i=1}^{\infty} \varphi_i(|(\tau_m x)_i - (x)_i|), \; r = 0, 1, 2, \ldots.$$

Obviously, we have

$$\omega_\varphi(x, r) = \sup_{m \geq r} \sum_{i=m}^{\infty} \varphi_i(|t_{m+i} - t_i|).$$

For any two sequences x and y we have

$$\omega_\varphi(x + y, r) \leq \omega_\varphi(2x, r) + \omega_\varphi(2y, r).$$

Let Ψ be a nonnegative, nondecreasing function of $u \geq 0$ such that $\Psi(u) \to 0$ as $u \to 0_+$, $\Psi(u)$ not vanishing identically, and let (a_r) be sequence of positive numbers with $a = \inf_{r \geq 0} a_r > 0$. We define the set

$$X(\Psi) = \{x \in X : \; a_r \Psi(\omega_\varphi(\lambda x, r)) \to 0 \text{ as } r \to \infty \text{ for a } \lambda > 0\}.$$

3. φ-FUNCTIONS AND THEIR PROPERTIES

We shall need the following conditions concerning the function Ψ and functions φ_i, $i = 1, 2, \ldots$.

The function Ψ is said to satisfy the conditions (Δ_2) for small u (for all u), if there are $u_0 > 0$ and $K > 0$ such that $\Psi(2u) \le K\Psi(u)$ for all $0 < u \le u_0$ (for all $u \ge 0$).

This implies that for every $u_1 > 0$ there exists $K_1 > 0$ such that $\Psi(2u) \le K_1\Psi(u)$ for all $0 < u \le u_1$.

The sequence $\varphi = (\varphi_i)_{i=1}^{\infty}$ will be said to satisfy the condition (A), if for every $\varepsilon > 0$ there exist $A > 0$ and $\alpha > 0$ such that for all $0 \le u \le A$ for all $i = 1, 2, \ldots$

$$\varphi_i(\alpha u) \le \varepsilon \varphi_i(u).$$

The sequence $\varphi = (\varphi_i)_{i=1}^{\infty}$ will be said to satisfy the condition (A'), if there exists an $\alpha > 0$ such that for every $u \ge 0$, for all $i = 1, 2, \ldots$

$$2\varphi_i(\alpha u) \le \varphi_i(u)\ .$$

Let us remark that if the functions φ_i are all s-convex with a fixed $s \in (0, 1\rangle$ then $\varphi = (\varphi_i)_{i=1}^{\infty}$ satisfies both conditions (A) and (A'), (for definition of s-convex function see e.g. [2], [4], [6]). A converse statement is not true. For example, taking

$$\varphi_i(u) = \varphi(u) = 1 - \sqrt{1 + \frac{1}{ln\ u}}$$

for $0 < u < v_0$, with v_0 sufficiently small, we see easily that (A) is satisfied but φ is not equivalent to an s-convex function for $0 < s \le 1$.

We shall say that the function Ψ satisfies the condition (B), if there exists a $v > 0$ such that for every $\delta > 0$ there is an $\eta > 0$ satisfying the inequality $\Psi(\eta u) \le \delta\Psi(u)$ for any $0 \le u \le v$.

The sequence $\varphi = (\varphi_i)_{i=1}^{\infty}$ of φ-functions will be said to satisfy the condition (C), if for every $\eta > 0$ there exists an $\varepsilon > 0$ such that for all $u > 0$ and all indices i, the inequality $\varphi_i(u) < \varepsilon$ implies $u < \eta$.

Let us remark that (C) implies that $\varphi_i(u) > 0$ if $u > 0$.

4. SPACE $X(\Psi)$

We give now some characteristic of the space $X(\Psi)$ defined in 2, and we investigate the vector structure on $X(\Psi)$.

Theorem 1. *Let us suppose that Ψ satisfies the condition (Δ_2) for small u and let the functions φ_i satisfy (Δ_2) for all u with a constant $K > 0$ independent of i. Then $x \in X(\Psi)$ if and only if $a_r\Psi(\omega_\varphi(\lambda x, r)) \to 0$ as $r \to \infty$ for every $\lambda > 0$.*

The easy proof will be omitted.

Remark 1. It is easy to verify that if φ_i satisfy (Δ_2) for small u with K and u_0 independent of i and the sequence x is bounded, then the thesis of Theorem 1 is true.

Theorem 2. *Let one of the following two conditions hold:*

1^o. *Ψ satisfies (Δ_2) for small u,*

2^o. *φ satisfies (A').*

Then $X(\Psi)$ is a vector space.

Proof. Supposing $x, y \in X(\Psi)$ and applying the inequality $\varphi(u + v) \leq \varphi(2u) + \varphi(2v)$, we obtain for $x = (t_i)$, $y = (s_i)$

$$\omega_\varphi(x + y, r) \leq \sup_{m \geq r} \sum_{i=m}^{\infty} [\varphi_i(2|t_{i+m} - t_i|) + \varphi_i(2|s_{i+m} - s_i|)] \leq$$

$$\leq \omega_\varphi(2x, r) + \omega_\varphi(2y, r)$$

for every $r > 0$. Now, by the definition of $X(\Psi)$ there exists a $\lambda > 0$ such that $a_r\Psi(\omega_\varphi(\lambda x, r)) \to 0$ and $a_r\Psi(\omega_\varphi(\lambda y, r)) \to 0$ as $r \to \infty$. We have

$$a_r\Psi\left(\omega_\varphi\left(\frac{1}{2}\lambda(x+y),r\right)\right) \le a_r\Psi[\omega_\varphi(\lambda x,r)+\omega_\varphi(\lambda y,r)] \le$$

$$\le a_r\Psi(2\omega_\varphi(\lambda x,r)) + a_r\Psi(2\omega_\varphi(\lambda y,r)),$$

by monotonicity of the function Ψ.

Now, let us suppose 1^o. By assumptions, there are constants $M,\delta > 0$ such that $0 < \Psi(u) \le \delta$ implies $u \le M$. Since $a_r\Psi(\omega_\varphi(\lambda x,r)) \to 0$ as $r \to \infty$ and $a = \inf_{r\ge 0} a_r > 0$, we have $\Psi(\omega_\varphi(\lambda x,r)) \to 0$ as $r \to \infty$. Hence there exists an $r_1 > 0$ such that $\Psi(\omega_\varphi(\lambda x,r)) \le \delta$ for $r \ge r_1$. Consequently, $\omega_\varphi(\lambda x,r) \le M$ for $r \ge r_1$. Similarly $\omega_\varphi(\lambda y,r) \le M$ for $r \ge r_2$ with some $r_2 > 0$, and we may suppose $r_2 = r_1$. Taking $u_1 = M$, by 1^o there is a $K_1 > 0$ such that $\Psi(2\omega_\varphi(\lambda x,r)) \le K_1\Psi(\omega_\varphi(\lambda x,r))$ and $\Psi(2\omega_\varphi(\lambda y,r)) \le K_1\Psi(\omega_\varphi(\lambda y,r))$ for $r \ge r_1$. Hence for $r \ge r_1$ we obtain

$$a_r\Psi\left(\omega_\varphi\left(\frac{1}{2}\lambda(x+y),r\right)\right) \le K_1[a_r\Psi(\omega_\varphi(\lambda x,r)) + a_r\Psi(\omega_\varphi(\lambda y,r))] \to 0$$

as $r \to \infty$. Hence $x+y \in X(\Psi)$.

Next, let us suppose 2^o. Then

$$\omega_\varphi(\alpha\lambda x,r) = \sup_{m\ge r}\sum_{i=m}^{\infty}\varphi_i(\alpha\lambda|t_{i+m}-t_i|) \le$$

$$\le \frac{1}{2}\sup_{m\ge r}\sum_{i=m}^{\infty}\varphi_i(\lambda|t_{i+m}-t_i|) = \frac{1}{2}\omega_\varphi(\lambda x,r)$$

and similarly

$$\omega_\varphi(\alpha\lambda y,r) \le \frac{1}{2}\omega_\varphi(\lambda y,r)$$

for $r \ge 0,\ \lambda > 0$.

Thus

$$a_r\Psi\left(\omega_\varphi\left(\frac{1}{2}\lambda\alpha(x+y),r\right)\right) \leq a_r\Psi(2\omega_\varphi(\lambda\alpha x,r)) + a_r\Psi(2\omega_\varphi(\lambda\alpha y,r)) \leq$$

$$\leq a_r\Psi(\omega_\varphi(\lambda x,r)) + a_r\Psi(\omega_\varphi(\lambda y,r)) \to 0$$

as $r \to \infty$ for sufficiently small $\lambda > 0$. Hence $x+y \in X(\Psi)$. This proves the theorem.

5. MODULAR STRUCTURE ON $X(\Psi)$

For every $x \in X$ we define the functional

$$\varsigma(x) = \sup_{r\geq 0} a_r\Psi(\omega_\varphi(x,r)) = \sup_{r\geq 0} a_r\Psi\Big[\sup_{m\geq r}\sum_{i=m}^{\infty}\varphi_i(|t_{i+m}-t_i|)\Big].$$

Theorem 3. *Let $\varphi = (\varphi_i)_{i=1}^{\infty}$ and Ψ satisfy one of the following two conditions:*

1^o *Ψ is concave,*

2^o *functions φ_i are convex.*

Then $X(\Psi)$ is a vector space and ς is a pseudomodular in X.

Proof. If Ψ is concave and $\Psi(0) = 0$ then Ψ satisfies the condition (Δ_2) for all $u > 0$, because $\Psi(2u) \leq 2\Psi(u)$. Hence, by Theorem 2, $X(\Psi)$ is a vector space. Moreover, if $x, y \in X$, $x = (t_i)$, $y = (s_i)$, $\alpha, \beta \geq 0$, $\alpha + \beta = 1$, then

$$\varsigma(\alpha x + \beta y) \leq \sup_{r\geq 0} a_r\Psi\Big[\sup_{m\geq r}\sum_{i=m}^{\infty}\varphi_i(\alpha\,|t_{i+m}-t_i| + \beta\,|s_{i+m}-s_i|)\Big] \leq$$

$$\leq \varsigma(x) + \varsigma(y).$$

Consequently, ς is a pseudomodular.

Now, let us suppose φ_i to be convex for $i = 1, 2, \ldots$. Then $\varphi = (\varphi_i)_{i=1}^{\infty}$ satisfies (A') and so, by Theorem 2, $X(\Psi)$ is a vector space. Moreover, with the same notation as above, we have

$$\varsigma(\alpha x + \beta y) \leq \sup_{r \geq 0} a_r \Psi\Big[\sup_{m \geq r} \sum_{i=m}^{\infty} \varphi_i(\alpha\, |t_{i+m} - t_i| + \beta\, |s_{i+m} - s_i|)\Big] \leq$$

$$\leq \sup_{r \geq 0} a_r \Psi\Big[\sup_{m \geq r} \sum_{i=m}^{\infty} \varphi_i(|t_{i+m} - t_i|)\Big] +$$

$$+ \sup_{r \geq 0} a_r \Psi\Big[\sup_{m \geq r} \sum_{i=m}^{\infty} \varphi_i(|s_{i+m} - s_i|)\Big] = \varsigma(x) + \varsigma(y).$$

Hence ς is a pseudomodular in X.

As well-known, the pseudomodular ς defines an F-pseudonorm

$$|x|_\varsigma = \inf\left\{u > 0 :\ \varsigma\left(\frac{x}{u}\right) \leq u\right\}$$

in the modular space

$$X_\varsigma = \{x \in X :\ \varsigma(\lambda x) \to 0 \text{ as } \lambda \to 0_+\}$$

(compare [5], [8]).

We shall investigate ς in case when Ψ is s-convex with $0 < s \leq 1$.

Remark 2. Let Ψ be s-convex with $0 < s \leq 1$ and let φ_i be convex for $i = 1, 2, \ldots$. Then ς is an s-convex pseudomodular, i.e.

$$\varsigma(\alpha x + \beta y) \leq \alpha^s \varsigma(x) + \beta^s \varsigma(y)$$

if $\alpha, \beta \geq 0,\ \alpha^s + \beta^s \leq 1$.

For proof, let us remark that by Theorem 3, ς is a pseudomodular. Moreover, taking $x = (t_i),\ y = (s_i),\ \alpha, \beta \geq 0,\ \alpha^s + \beta^s \leq 1$, we have $\alpha + \beta \leq 1$ and so

$$\varsigma(\alpha x+\beta y)\leq \sup_{r\geq 0} a_r\Psi\Big[\alpha \sup_{m\geq r}\sum_{i=m}^{\infty}\varphi_i(|t_{i+m}-t_i|)+$$

$$+\beta\sup_{m\geq r}\sum_{i=m}^{\infty}\varphi_i(|s_{i+m}-s_i|)\Big]\leq \alpha^s\varsigma(x)+\beta^s\varsigma(y).$$

Theorem 4. *Let the function Ψ be increasing, continuous and s-convex and let the functions φ_i be convex, $i=1,2,\ldots$, where $0<s\leq 1$. Then the s-homogeneous pseudonorm*

$$||x||_\varsigma^s=\inf\{u>0:\ \varsigma(\frac{x}{u^{1/s}})\leq 1\}$$

satisfies the following inequalities:

1^o *if $x\in X_\varsigma$, $||x||_\varsigma^s<1$, then*

$$||x||_\varsigma^s\geq \sup_r\left(\frac{\omega_\varphi(x,r)}{\Psi_{-1}(1/a_r)}\right)^s,$$

2^o *if $x\in X_\varsigma$, $||x||_\varsigma^s>1$, then*

$$||x||_\varsigma^s\leq \sup_r\left(\frac{\omega_\varphi(x,r)}{\Psi_{-1}(1/a_r)}\right)^s,$$

where Ψ_{-1} is the inverse to Ψ.

Proof. Since, by Remark 2, ς is s-convex, so $||\cdot||_\varsigma^s$ is an homogeneous pseudonorm. Let $||x||_\varsigma^s<u<1$, then

$$a_r\Psi\left(\frac{1}{u^{1/s}}\sup_{m\geq r}\sum_{i=m}^{\infty}\varphi_i(|t_{i+m}-t_i|)\right)\leq 1$$

for all $r\geq 0$. Hence

$$\sup_{m\geq r}\sum_{i=m}^{\infty}\varphi_i(|t_{i+m}-t_i|)\leq u^{1/s}\Psi_{-1}(1/a_r),$$

i.e.

$$\omega_\varphi(x,r) \le u^{1/s}\Psi_{-1}(1/a_r) ,$$

which gives the inequality 1°, when we take $u \to ||x||^s_{\varsigma+}$.

Now, if $||x||^s_\varsigma > u > 1$, then we have

$$\sup_r a_r \Psi\left[\frac{1}{u^{1/s}}\,\omega_\varphi(x,r)\right] > 1$$

and we obtain the inequality 2° easily.

Corollary. *By the assumptions of Theorem 4, if*

$$\sup_r \frac{\omega_\varphi(x,r)}{\Psi_{-1}(1/a_r)} = 1 ,$$

then $||x||^s_\varsigma = 1$.

Let $\bar{c}$ be the space of all sequences $x = (t_i)_{i=0}^\infty$ such that $t_i = t_{i+1}$ for $i = 1,2,\ldots$. There holds the following

Remark 3. Let us remark that if $\Psi(u) > 0$ for $u > 0$, then $x \in \bar{c}$ if and only if $|x|_\varsigma = 0$.

6. COMPLETENESS

Taking the assumptions of Theorem 2, we may consider the quotient spaces: $\tilde{X}_\varsigma = X_\varsigma/\bar{c}$ and $\tilde{X}(\Psi) = X(\Psi)/\bar{c}$, with elements $\tilde{x},\ldots$ (see [1]). The F-pseudonorms resp. s-convex pseudonorms may be defined by $|\tilde{x}|_\varsigma = |x|_\varsigma$, $||\tilde{x}||^s_\varsigma = ||x||^s_\varsigma$, where $x \in \tilde{x}$, respectively.

Theorem 5. *Let Ψ be increasing, continuous and satisfying the condition (B). Let $\varphi = (\varphi_i)_{i=1}^\infty$ satisfy conditions (A) and (C). Moreover, let at least one of the following two conditions hold:*

1° *Ψ is concave,*

2° *φ_i are convex.*

Then $\tilde{X}_\varsigma$ is a Fréchet space with respect to the F-norm $|.|_\varsigma$.

Proof. Let $(\tilde{x}_n)$ be a Cauchy sequence in $\tilde{X}_\varsigma$, $x_n \in \tilde{x}_n$, $x_n = (t_i^n)_{i=0}^\infty$. Without loss of generality, we may suppose that $t_1^n = 0$ for

$n = 1, 2, \ldots$. We denote by Ψ_{-1} the inverse function to Ψ. Since $a = \inf_{r \geq 0} a_r > 0$, for every $\varepsilon > 0$ one can find an N such that $|x_p - x_q|_\varsigma < a\Psi(\varepsilon)$ for $p, q > N$. By the definition of $|.|_\varsigma$, there exists u_ε such that $0 < u_\varepsilon < a\Psi(\varepsilon)$ and $\varsigma\left(\frac{x_p - x_q}{u_\varepsilon}\right) \leq u_\varepsilon$ for $p, q > N$. Consequently,

$$a_r \Psi\left(\omega_\varphi\left(\frac{x_p - x_q}{u_\varepsilon}, r\right)\right) \leq u_\varepsilon$$

for $p, q > N$ and $r \geq 0$, whence

$$\omega_\varphi\left(\frac{x_p - x_q}{u_\varepsilon}, r\right) \leq \Psi_{-1}\left(\frac{u_\varepsilon}{a_r}\right) \leq \Psi_{-1}\left(\frac{u_\varepsilon}{a}\right) < \varepsilon$$

for $p, q > N$, $r \geq 0$. By the definition of ω_φ, we obtain in particular

$$\sum_{i=m}^{s} \varphi_i\left(\frac{1}{u_\varepsilon} \mid t_{i+m}^p - t_{i+m}^q - t_i^p + t_i^q|\right) < \Psi_{-1}\left(\frac{u_\varepsilon}{a_r}\right) < \varepsilon \tag{1}$$

for $p, q > N$, $s \geq m$ and $i \geq m \geq r \geq 0$. By condition (C), for every $\eta > 0$ one can find an $\varepsilon > 0$ such that

$$\frac{1}{u_\varepsilon}|t_{i+m}^p - t_{i+m}^q - t_i^p + t_i^q| < \eta \tag{2}$$

for $p, q, > N$, $i \geq m \geq 0$. Hence

$$|t_{i+m}^p - t_{i+m}^q| < |t_i^p - t_i^q| + \eta u_\varepsilon < |t_i^p - t_i^q| + \eta a \Psi(\varepsilon)$$

for $p, q > N$, $i \geq m \geq 0$. Since $t_1^n = 0$ for $n = 1, 2, \ldots$, the above inequalities imply $(t_i^p)_{p=1}^\infty$ to be Cauchy sequences for $i = 1, 2, \ldots$. Hence these sequences are convergent. Let us write $t_i = \lim_{n\to\infty} t_i^n$ for $i = 1, 2, \ldots$, $t_0 = 0$, $x = (t_i)_{i=0}^\infty$. Taking $q \to \infty$ in (1), we obtain

$$\sum_{i=m}^{s} \varphi_i\left(\frac{|t_{i+m}^p - t_{i+m} - t_i^p + t_i|}{u_\varepsilon}\right) \leq \Psi_{-1}\left(\frac{u_\varepsilon}{a_r}\right)$$

for $p > N,\ s \geq m \geq r \geq 0$. Again, taking $s \to \infty$, we get

$$\sum_{i=m}^{\infty} \varphi_i\left(\frac{|t_{i+m}^p - t_{i+m} - t_i^p + t_i|}{u_\varepsilon}\right) \leq \Psi_{-1}\left(\frac{u_\varepsilon}{a_r}\right)$$

for $p > N,\ m \geq r \geq 0$. Thus,

$$\omega_\varphi\left(\frac{x_p - x}{u_\varepsilon}, r\right) \leq \Psi_{-1}\left(\frac{u_\varepsilon}{a_r}\right)$$

for $p > N,\ r \geq 0$. Hence

$$a_r \Psi\left(\omega_\varphi\left(\frac{x_p - x}{u_\varepsilon}, r\right)\right) \leq u_\varepsilon \tag{3}$$

for $p > N$ and $r \geq 0$.

We are going to prove that $x_p - x \in X_\varsigma$ for large p, i.e. $\varsigma(\lambda(x_p - x)) \to 0$ as $\lambda \to 0_+$. Let $\varepsilon > 0$ be fixed and let N be chosen as above. Let $p > N$. We have for $\lambda > 0$

$$\omega_\varphi(\lambda(x_p - x), r) = \omega_\varphi\left(\lambda u_\varepsilon\ \frac{x_p - x}{u_\varepsilon}, r\right) =$$

$$= \sup_{m \geq r} \sum_{i=m}^{\infty} \varphi_i\left(\lambda u_\varepsilon \frac{|t_{i+m}^p - t_{i+m} - t_i^p + t_i|}{u_\varepsilon}\right).$$

Taking $q \to \infty$ in (2) we obtain

$$\frac{|t_{i+m}^p - t_{i+m} - t_i^p - t_i|}{u_\varepsilon} \leq \eta$$

for $i \geq m \geq 0$. We apply the condition (A) with $\bar{\varepsilon}$ in place of ε, $\lambda \leq \alpha/u_\varepsilon$, and we choose $\eta = A$. Then for $u = \frac{1}{u_\varepsilon}|t^p_{i+m} - t_{i+m} - t^p_i + t_i|$ we get

$$\varphi_i\left(\lambda u_\varepsilon \frac{|t^p_{i+m} - t_{i+m} - t^p_i + t_i|}{u_\varepsilon}\right) \leq \bar{\varepsilon}\varphi_i\left(\frac{1}{u_\varepsilon}|t^p_{i+m} - t_{i+m} - t^p_i + t_i|\right) ,$$

for $p > N$, $i \geq m \geq 0$. Hence

$$\omega_\varphi(\lambda(x_p - x), r) \leq \bar{\varepsilon} \sup_{m \geq r} \sum_{i=m}^{\infty} \varphi_i\left(\frac{1}{u_\varepsilon}|t^p_{i+m} - t_{i+m} - t^p_i + t_i|\right) =$$

$$= \bar{\varepsilon}\omega_\varphi\left(\frac{x_p - x}{u_\varepsilon}, r\right) \leq \bar{\varepsilon}\Psi_{-1}\left(\frac{u_\varepsilon}{a_r}\right) \leq \bar{\varepsilon} \cdot \varepsilon .$$

Hence for $0 < \lambda \leq \bar{\alpha}/u_\varepsilon$ we have

$$\varsigma(\lambda(x_p - x)) = \sup_{r \geq 0} a_r \Psi(\omega_\varphi(\lambda(x_p - x), r)) \leq \sup_{r \geq 0} a_r \Psi\left(\bar{\varepsilon}\Psi_{-1}\left(\frac{u_\varepsilon}{a_r}\right)\right) .$$

Now, we apply the condition (B) with $v = \Psi_{-1}(\frac{u_\varepsilon}{a})$, $u = \Psi_{-1}(\frac{u_\varepsilon}{a_r})$. Choosing $\delta > 0$ arbitrarily and taking $\bar{\varepsilon} = \eta$, we obtain

$$\Psi\left(\bar{\varepsilon}\Psi_{-1}\left(\frac{u_\varepsilon}{a_r}\right)\right) \leq \delta\Psi\left(\Psi_{-1}\left(\frac{u_\varepsilon}{a_r}\right)\right) = \delta \frac{u_\varepsilon}{a_r}.$$

Consequently,

$$\varsigma(\lambda(x_p - x)) \leq \sup_{r \geq 0} a_r \delta \frac{u_\varepsilon}{a_r} = \delta u_\varepsilon \quad \text{for } 0 < \lambda \leq \alpha/u_\varepsilon .$$

Since u_ε is fixed, this implies $\varsigma(\lambda(x_p - x)) \to 0$ as $\lambda \to 0_+$. Hence $x_p - x \in X_\varsigma$ for $p > N$. But X_ς is a vector space; thus, $x \in X_\varsigma$.

By (3), we have for arbitrary $\varepsilon > 0$,

$$\varsigma\left(\frac{x_p - x}{u_\varepsilon}\right) \le u_\varepsilon$$

for $p > N$. Thus, $|x_p - x|_\varsigma < u_\varepsilon < a\Psi(\varepsilon)$ for $p > N$, and we get $|x_p - x|_\varsigma \to 0$ as $p \to \infty$. This proves the completeness of the space X_ς.

Theorem 6. *Let the function Ψ and the sequence φ satisfy the assumptions of Theorems 1 and 5. The $\tilde{X}(\Psi) \cap \tilde{X}_\varsigma$ is a Fréchet space with respect to the F-norm $|.|_\varsigma$.*

Proof. It is sufficient to show that $\tilde{X}(\Psi) \cap \tilde{X}_\varsigma$ is a closed subspace of $\tilde{X}_\varsigma$ with respect to the F-norm $|.|_\varsigma$. Let $\tilde{x}_p \in \tilde{X}(\Psi) \cap \tilde{X}_\varsigma$, $\tilde{x}_p \to \tilde{x}$ in $\tilde{X}_\varsigma$. Let $x_p \in \tilde{x}_p$, $x \in \tilde{x}$. By the assumption, we have for every $\lambda > 0$

$$a_r\Psi(\omega_\varphi(\lambda(x - x_p), r)) \to 0 \quad \text{as } p \to \infty$$

uniformly with respect to r. By a property of ω_φ, and the condition (Δ_2) for φ_i, we have

$$\omega_\varphi(\lambda x, r) \le \omega_\varphi(2\lambda(x - x_p), r) + \omega_\varphi(2\lambda x_p, r) \le$$

$$\le K[\omega_\varphi(\lambda(x - x_p), r) + \omega_\varphi(\lambda x_p, r)].$$

By properties of Ψ we have that there exist $M > 0$, $\delta > 0$ such that for every u satisfying the condition $0 < \Psi(u) \le \delta$ there holds the inequality $u \le M$. Taking $\lambda > 0$ fixed we may find a p_1 such that $\Psi[\omega_\varphi(\lambda(x - x_p), r)] < \delta$ for $p \ge p_1$, and in consequence we obtain that $\omega_\varphi(\lambda(x - x_p), r) \le M$ for $p \ge p_1$, with an $M > 0$. Let m be such that $K \le 2^m$. Applying the inequality $\Psi(u + v) \le \Psi(2u) + \Psi(2v)$ and condition (Δ_2) for small u with a constant $K_1 > 0$, we thus obtain

$$\Psi(\omega_\varphi(\lambda x, r)) \le \Psi[2K\omega_\varphi(\lambda(x - x_p), r)] + \Psi[2K\omega_\varphi(\lambda x_p, r)] \le$$

$$\le K_1^{m+1}[\Psi(\omega_\varphi(\lambda(x - x_p), r)) + \Psi(\omega_\varphi(\lambda x_p, r))]$$

for $p \geq p_1$. Let us choose an arbitrary $\varepsilon > 0$. Then there exists a $p_0 \geq p_1$ such that

$$a_r \Psi[\omega_\varphi(\lambda(x - x_{p_0}), r)] < \frac{\varepsilon}{2} K_1^{-m-1} .$$

But $x_{p_0} \in X(\Psi)$ and so, by Theorem 1, we have

$$a_r \Psi[\omega_\varphi(\lambda x_{p_0}, r)] \to 0 \quad \text{as } r \to \infty .$$

Hence there exists an r_0 such that

$$a_r \Psi(\omega(\lambda x_{p_0}, r)) < \frac{\varepsilon}{2} K_1^{-m-1} \quad \text{for } r \geq r_0 .$$

Consequently,

$$a_r \Psi(\omega(\lambda x, r)) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \quad \text{for } r \geq r_0 .$$

This shows that $x \in X(\Psi)$. By Theorem 5, $x \in X_\varsigma$. Hence $x \in X(\Psi) \cap X_\varsigma$, and so $\tilde{x} \in \tilde{X}(\Psi) \cap \tilde{X}_\varsigma$.

Let us remark that Theorems 5 and 6 may be expressed also in a form replacing F-norm convergence by means of modular convergence with respect to the modular $\tilde{\varsigma}(\tilde{x}) = \inf\{\varsigma(y) : y \in \tilde{x}\}$.

Let us recall that a sequence $(\tilde{x}_n)$ of elements of $\tilde{X}_\varsigma$ is said to be $\tilde{\varsigma}$-Cauchy, if there exists a $k > 0$ such that for every $\varepsilon > 0$ there is an N such that $\tilde{\varsigma}(k(\tilde{x}_p - \tilde{x}_q)) < \varepsilon$ for all $p, q > N$. The space $\tilde{X}_\varsigma$ is called $\tilde{\varsigma}$-complete, if any $\tilde{\varsigma}$-Cauchy sequence is $\tilde{\varsigma}$-convergent to an element $\tilde{x} \in \tilde{X}_\varsigma$.

There hold the following theorems, proofs of which are analogous to those of Theorems 5 and 6:

Theorem 7. *Under the assumptions of Theorem 5, the space $\tilde{X}_\varsigma$ is $\tilde{\varsigma}$-complete.*

Theorem 8. *Under the assumptions of Theorem 6, the space $\tilde{X}(\psi) \cap \tilde{X}_\varsigma$ is $\tilde{\varsigma}$-complete.*

The authors are indebted to the Referee for this remarks which helped to improve the paper.

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Recibido: 12 de Febrero de 1993
Revisado: 24 de Enero de 1994