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The Modular Characters of the Twisted Chevalley Group $^2D_4(2)$ *Over* GF_2

Ibrahim A.I. SULEIMAN

ABSTRACT. In this paper we calculate the 2-modular character table of the twisted Chevalley group $^2D_4(2)$ using computer techniques available in an algebra package called "Meat-Axe". This package is now available in Mu'tah University as well as other Universities such Birmingham University in U.K. and Aachen University in Germany. The determination of this character table will be a contribution to modular calculations of various simple groups.

INTRODUCTION

The twisted Chevallery group $^2D_4(2)$ is a simple group of order 197 406 720=$2^{12}.\underline{3}^4$ .5 .7 .17 .

This group is isomorphic to the orthogonal group $O_8^-(2)$ which is the derived group of all 8×8 matrices over GF_2 preserving a quadratic form of Witt defect 1. Its automorphism group $^2D_4(2).2$ is the largest maximal subgroup of the symplectic group $PS_8(2)$. The maximal subgroups for this group are as follows in "ATLAS" notation (see [2]):

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$2^6 : U_4(2)$, $S_6(2)$, 2^{3+6} : $(L_3(2) \times 3)$, 2_+^{1+8} : $(S_3 \times A_5)$ $(3 \times A_8)$: 2 , $L_2(16) : 2$, $S_3 \times S_3 \times A_5$ and $L_2(7)$
The 5-, 7- and 17-modular characters have been determined (see [4]). In this paper we determine the 2-modular character table of ${}^2D_4(2)$.

1. THE 2-MODULAR CHARACTER TABLE OF ${}^2D_4(2)$

The 2-modular central characters give the block distribution of the ordinary irreducible characters. There is one block $B1 = \{4096\}$ of defect zero. This implies that 4096 is a 2-modular irreducible representation for ${}^2D_4(2)$. The remaining thirty-eight ordinary irreducible characters are in the principal block Bo of defect 12. There are sixteen 2-regular classes. Hence there are fifteen 2-modular irreducible representations to find.

To construct the 2-modular irreducible representations for ${}^2D_4(2)$, we started with two generators for the 8-dimensional representation of ${}^2D_4(2)$ over the field GF_2 of order 2. These generators are a and b where:

$$a = \begin{bmatrix} 0&1&0&0&0&0&0&0 \\ 0&0&0&1&0&0&0&0 \\ 1&1&1&0&1&0&0&0 \\ 1&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&1 \\ 1&1&0&0&0&1&1&1 \\ 0&1&0&1&1&1&0&1 \\ 0&0&0&0&1&0&0&1 \end{bmatrix} \qquad b = \begin{bmatrix} 0&0&1&0&0&0&0&0 \\ 0&0&0&0&1&0&0&0 \\ 0&0&0&0&0&1&0&0 \\ 0&0&0&0&0&0&0&1 \\ 1&0&0&0&1&1&0&0 \\ 0&1&1&1&1&0&1&0 \\ 1&1&1&0&1&0&1&0 \\ 0&0&0&0&0&0&0&1 \end{bmatrix}$$

All the other 2-modular irreducible representations for ${}^2D_4(2)$ were obtained by tensoring representations together and using the "Meat-Axe" (see [6]) to chop the resulting representations into irreducibles. By this means, we first determine that the irreducible degrees are as in table 1. We then find representatives of all the 2-regular classes of ${}^2D_4(2)$ as words in our generators a and b. We then work out the character values on these classes using the program "EV" of Meat-Axe which works out the eigenvalues of a matrix [6]. Using the "Meataxe" (see [10], we find that:

i) $8_a \otimes 8_a$ breaks up as $4(1) + 8_a + 2(26)$, which gives a new representation of degree 26.

ii) $8_a \otimes 26 = 48_a + 160_a$, which gives two new 2-modular irreducible representations of degrees 48 and 160.

iii) $8_a \otimes 48_a = 2(1) + 2(8_b) + 2(8_c) + (26) + 246$, wich gives three 2-modular irreducibles of degrees 8, 8, 246.

iv) $8_b \otimes 26 = 48_b + 160_b$, which gives two new 2-modular irreducibles of degrees 48 and 160.

v) $8_c \otimes 26 = 48_c + 160_c$, which gives two new 2-modular irreducibles of degrees 48 and 160.

vi) $26 \otimes 48_a = 4(1) + 2(b_a) + 2(8_b) + 2(8_c) + 48_a + 48_b + 2(160_a) + 784_a$, which gives one new 2-modular irreducible 784_a of degree 784.

vii) $26 \otimes 48_b = 4(1) + 2(8_b) + 3(8_c) + 48_b + 48_c + 2(160_b) + 784_b$, which gives one new 2-modular irreducible 784_b of degree 784.

viii) $26 \otimes 48_c = 4(1) + 2(8_c) + 3(8_b) + 48_b + 48_c + 2(160_c) + 784_c$, which gives the last 2-modular irreducible 784_c of degree 784.

Hence, we constructed the fifteen 2-modular irreducible representations of ${}^2D_4(2)$ which are in the principal block Bo. We then proved that $8_b = \overline{8_c}$, $48_b = \overline{48_c}$, $160_b = \overline{160_c}$ and $784_b = \overline{784_c}$.

2. THE INDICATORS OF THE 2-MODULAR IRREDUCIBLES OF ${}^2D_4(2)$

It is not always easy to tell theoretically if a 2-modular irreducible representation supports an invariant quadratic form or not. The representations of degrees 1 and 4096 lift to ordinary representations of the same degree, which have Schur indicator +, so these have fixed quadratic form mod 2 also. All the other cases were checked by computer calculations using the same method which was explained in detail in [6]. Here is a brief explanation of that method as follows:

Every 2-modular self-dual irreducible supports an invariant symplectic form. Some will also support an invariant quadratic form. The symbol + is used to denote that the 2-modular irreducible representation supports a non-zero invariant quadratic form; if not, we use the symbol -. It is often very difficult to determine theoretically, whether a

2-modular representation supports an invariant quadratic form or not, so we use computer calculations to solve this problem.

Using the programs of the Meat-Axe, Standard-Base "SB", Transpose "TR" and Invert "IV" (to get the dual representation) and Standard-Base again, we find a matrix P such that:

$$P^{-1} g_i P = (g_i^T)^{-1}$$

for each group generator g_i. Hence $g_i P g_i^T = P$ and P is the matrix of a symplectic form invariant under ${}^2D_4(2)$.

Now a quadratic form q can be specified by giving the associated symplectic form, together with the values of q on a basis. Since all the basis vectors produced by "SB" are in the same orbit under the group, there is just one possible quadratic form for each element of the field.

Each quadratic form may be represented by a matrix Q obtained by taking the bottom-left of P (i.e. the part below or on the main diagonal), and adding a scalar matrix. We have to check whether the diagonal of $g_i Q g_i^T$ is equal to the diagonal of Q. If it is, for all the generators g_i of G, then the quadratic form represented by Q is invariant under G.

Using these computational calculations we find that each of the other 2-modular representations supports an invariant quadratic form. Hence all the indicators are +.

3. CALCULATING THE CHARACTER VALUES

We find representatives of all the 2-regular classes of ${}^2D_4(2)$ as words in the two generators a and b. These words are as follows:

$ab^2 \in 9A$

$(ab^2)^2 \in 3C$

$((ab^2)^2ab^3)^2 \in 15A$

$(15A)^3 \in 5A$

$(15A)^5 \in 3B$

$((ab^2)^2ab^3ab^2)^2 \in 21A$

$(21A)^2 \in 21B$

$(21A)^3 \in 7A$

$(21A)^7 \in 3A$

$((ab^2)^2ab^3ab^2)^2(ab^3) \in 17A$

$(17A)^2 \in 17B$

$(17A)(17B) \in 17C$

$(17C)^2 \in 17D$

$((ab^2)((ab^2)^2ab^3ab^2)^2(ab^3)^2)^2 \in 15B$

$(a^2b)^4ab(a^2b)^2ab \in 15C$

We then worked out the character values for all the representations on these classes using the program "EV" of the MEAT-AXE which works out the eigenvalues of a matrix (see [6]). Table 1 gives the complete 2-modular character table of ${}^2D_4(2)$:

;	@	@	@	@	@	@	@	@	@	@	@	@	@	@	@	@	;	;
		60	1															
197	406720	480	080	648	180	21	9	90	90	45	17	17	17	17	21	21		
p power		A	A	A	A	A	C	AB	AB	AA	A	A	A	A	AA	AA		
p' part		A	A	A	A	A	A	AB	AB	AA	A	A	A	A	AA	AA		
ind	1A	3A	3B	3C	5A	7A	9A	15A	B*	15C	17A	B*2	C*3	D*6	21A	B*	fus	ind
+	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	:	+
+	8	5	2	-1	3	1	2	-3	-3	0	*	*	b17	b17	-2	-2	:	+
+	8	-4	2	-1	-2	1	-1	*	1+3b5	1	d17&3	*2	*3	*6	*	b21	·	+
+	8	-4	2	-1	-2	1	-1	1+3b5	*	1	*2	d17&3	6*	*3	b21	*	·	
+	26	8	-1	-1	1	-2	-1	4	4	-2	*	*	-b17	-b17	1	1	:	+
+	48	6	0	3	-2	-1	-3	-5	-5	1	-3	-3	-3	-3	-1	-1	:	+
+	48	-12	0	3	-2	-1	0	1-3b5	*	-2	*6	*3	1-d17	*2	2	2	·	+
+	48	-12	0	3	-2	-1	0	*	1-3b5	-2	*3	*6	*2	1-d17	2	2	·	
+	160	34	-2	-2	5	-1	1	-7	-7	-1	*	*	-1+b17-	1+b17	-1	-1	:	+
+	160	-20	-2	-2	0	-1	1	*	9b5	0	2d17+3&3+&6	*2	*3	*6	*	-2+b21	·	+
+	160	-20	-2	-2	0	-1	1	9b5	*	0	2d17*2+&3+3&6	*2	*3	*6	-2+b21	*	·	
+	246	12	-6	3	-4	1	0	-1	-1	2	*	*	b17	b17	-2	-2	:	+
+	784	10	-8	1	-6	0	1	-3	-3	0	2	2	2	2	3	3	:	+
+	784	-44	-8	1	4	0	1	7	7	1	2+d17*2-&3	*2	*3	*6	*	1-b21	·	+
+	784	-44	-8	1	4	0	1	7	7	1	2+d17-&6	*2	*3	*6	1-b21	*	·	
+	4096	64	-8	-8	-4	1	1	2	2	-1	-1	-1	-1	-1	1	1	:	+

Table 1

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Department of Maths. and Stats
Faculty of Science
Mu'tah University
61710. Al-Kerak, P.O. BOX (7)
JORDAN

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