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On the Diophantine Equation $w + x + y = z$, with $wxyz = 2^r 3^s 5^t$

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ABSTRACT. In this paper we complete the solution to the equation $w + x + y = z$, where w, x, y and z are positive integers and $wxyz$ has the form $2^r 3^s 5^t$, with r, s and t non-negative integers. Here we consider the case $1 < w \leq x \leq y$, the remaining case having been dealt with in our paper: On the Diophantine Equation $1 + X + Y = Z$, *Rocky Mountain J. of Math.* (to appear). This work extends earlier work of the authors and J.L. Brenner in the field of exponential Diophantine equations.

1. INTRODUCTION

An exponential Diophantine equation is an equation of the form

$$\sum_{i=1}^{n} x_i = 0, \tag{1.1}$$

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where the x_i are integers and the primes dividing their product belong to some fixed finite set S. Such equations arise quite naturally in connection with characters of finite groups. For example, if G is a group and p is a prime dividing $|G|$ to the first power only, then the degrees $x_1, x_2, \ldots, x_n$ of the ordinary irreducible characters in the principal p-block of G satisfy an equation of the form $\sum_{i=1}^{n} \delta_i x_i = 0$, where $\delta_i = \pm 1$, and S is the set of primes dividing $|G|/p$. Significant information concerning G can be obtained from the solutions to this equation (cf., [1]).

In recent years, the authors (cf., [3]-[6]), J.L. Brenner [7], and B.M.M. de Weger [11] have solved a variety of equations of the form (1.1). Recent results of van der Poorten and Schlickewei [10] and Evertse [9] imply that equations of the form (1.1) with four terms have only finitely many primitive solutions (i.e., solutions with g.c.d. $\{x_1, x_2, \ldots, x_n\} = 1$). In particular, in [5] the authors determined all solutions to

$$1 + x + y = z, \tag{1.2}$$

where x, y and z are positive integers divisible only by the primes 2, 3 and 5, so that $xyz = 2^r 3^s 5^t$. Here we determine all primitive solutions to the more general equation

$$w + x + y = z, \tag{1.3}$$

where w, x, y and z are all integers greater than one and $wxyz = 2^r 3^s 5^t$, using only elementary congruence-type arguments. In Section 2 we develop several preliminary lemmas which reduce the number of cases to be considered later. All solutions to (1.3) are then determined in Section 3. In Section 4 all solutions to (1.3) with $1 < w \le x \le y$ are listed (Table 4.1). Tables 4.2 and 4.3, listing orders and indices for various moduli used, are included for reference. All variable exponents in this paper represent non-negative integers.

Remark 1.1. The maximum values of z appearing in Table 4.1 are 2,125,764 and 2,097,152. The latter value occurs twice.

Remark 1.2. Brenner and Foster [7] gave examples of exponential Diophantine equations which cannot be solved by congruence methods alone. Thus it was not obvious at the outset that the methods used here would succeed. (There is no guarantee that finite sequences of prime power moduli which "pin down" every solution must exist. Some very large prime are of necessity involved here, the largest of these being the eleven-digit prime $3^5 2^{21} 5^3 + 1$).

2. PRELIMINARY LEMMAS

We let T denote the set of integers of the form $\pm 2^s 3^t 5^u$.

Lemma 2.1. *Let (w, x, y, z) be a primitive solution to (1.3) (with $w, x, y > 1$). Then precisely two of x, y, z and w are even.*

Proof. It suffices to show that the equations

$$3^a - (-1)^e 3^b + 5^c + (-1)^e 5^d = 0, e = \pm 1, abcd \neq 0, \qquad (2.1)$$

have no solutions. We suppose therefore that (2.1) holds for some (a, b, c, d). It follows that $3^a \equiv (-1)^e 3^b \pmod 5$, so that $a \equiv 2e + b \pmod 4$ and hence $a \equiv b \pmod 2$. Thus $(-1)^a(1 - (-1)^e) + (1 + (-1)^e) \equiv 0 \pmod 4$, an impossibility. ■

Lemma 2.2. *Let $p = 3$ or 5 and let k_1, k_2 and k_3 be fixed integers such that $k_2 \in T$ and k_1 is odd. Suppose that (a, b) is a solution to*

$$k_1 2^a + k_2 p^b + k_3 = 0 \qquad (2.2)$$

for which $a \leq 17$. Let (a', b') be any solution to (2.2) for which $a' \equiv a$ (mod 32). Then in fact $(a', b') = (a, b)$.

Proof. Since $k_1 2^{a'} + k_2 p^{b'} + k_3 \equiv 0 \pmod{65537}$, referring to Table 4.2, we see that $k_1 2^a + k_2 p^{b'} + k_3 \equiv 0 \pmod{65537}$. It follows immediately that $b' \equiv b \pmod{2^{16}}$. Thus (again from Table 4.2) $k_1 2^{a'} + k_2 p^b + k_3 \equiv 0 \pmod{2^{18}}$. Since $a \leq 17$ and k_1 is odd it follows that in fact $a = a'$ and hence $b = b'$. ■

Lemma 2.3. *Let k_1, k_2, k_3 and k_4 be fixed integers such that $k_1, k_2, k_3 \in T$ and $5 \nmid k_3$. Suppose that (a,b,c) is a solution to*

$$k_1 2^a + k_2 3^b + k_3 5^c + k_4 = 0 \tag{2.3}$$

with $c = 4$. Let (a', b', c') be any solution to (2.3) with $(a', b', c') \equiv (a,b,c)$ (mod M), where $M = 30^3$. Then $(a', b') \equiv (a, b)$ (mod $35M$) and $c' = 4$.

Proof. From Table 4.2, $k_1 2^a + k_2 3^b + k_3 5^{c'} + k_4 \equiv 0 \pmod{15121}$. Thus, $c' \equiv c \pmod{7M}$. Hence, considering our equations modulo 631 and 29, successively, we conclude that $(a', b') \equiv (a, b) \pmod{7M}$. Further, applying the moduli 708751, 52501 and 22501, we have $(a', b', c') \equiv (a, b, c) \pmod{35M}$. Hence $k_1 2^{a'} + k_2 3^b + k_3 5^c + k_4 \equiv 0 \pmod{5^5}$ so that in fact $c' = 4$. ■

Lemma 2.4. *Let k_1, k_2, k_3 and k_4 be fixed integers such that $k_1, k_2, k_3 \in T$ and k_1 is odd. Suppose that (a,b,c) is a solution to (2.3) with $a \leq 14$. Let (a', b', c') be any solution to (2.3) with $(a', b', c') \equiv (a, b, c)$ (mod $2^7 N$), where $N = 3^4 5^2$. Then $(b', c') \equiv (b, c)$ (mod $2^{13} N$) and $a' = a$.*

Proof. We successively consider our equations relative to the moduli 25601, 331777, 12289, 40961 and 147457. It follows that $(a', b', c') \equiv (a, b, c) \pmod{(2^{11}N, 2^{13}N, 2^{14}N)}$. From the modulus 2^{15} we conclude that $a' = a$. ■

Lemma 2.5. *Let k_1, k_2 and k_3 be fixed integers such that $k_1, k_2 \in T$ and $3 \nmid k_1$. Suppose that (a, b) is a solution to*

$$k_1 3^a + k_2 5^b + k_3 = 0 \tag{2.4}$$

with $a \leq 9$. Let (a', b') be any solution for which $(a', b') \equiv (a, b)$ (mod $10 \cdot 3^5$). Then $(a', b') = (a, b)$.

Proof. Consideration of our equations modulo 39367, 196831 and 3^{10} successively produces the desired conclusion. ■

3. DETERMINATION OF ALL SOLUTIONS TO 1.3.

It follows from Lemma 2.1 that each solution to (1.3) must arise from precisely one solution to the following family of twenty-two equations:

$$2^a + 2^b + 3^c5^d = 3^e5^f, c+d \neq 0, e+f \neq 0, 0 < a \leq b; \quad (3.1)$$
$$2^a + 3^b5^c + 3^d5^e = 2^f, af \neq 0, b+c \neq 0, d+e \neq 0, b \leq d,$$
$$b = d \Rightarrow c \leq e; \quad (3.2)$$
$$3^a + 3^b + 2^c5^d = 2^e5^f, abce \neq 0, d+f \neq 0, a \leq b; \quad (3.3)$$
$$3^a + 2^b5^c + 2^d5^e = 3^f, abdf \neq 0, c+e \neq 0, b \leq d, b = d \Rightarrow c \leq e; \quad (3.4)$$
$$5^a + 5^b + 2^c3^d = 2^e3^f, abce \neq 0, d+f \neq 0, a \leq b; \quad (3.5)$$
$$5^a + 2^b3^c + 2^d3^e = 5^f, abdf \neq 0, c+e \neq 0, b \leq d, b = d \Rightarrow c \leq e; \quad (3.6)$$
$$2^a + 3^b + 2^c5^d = 3^e5^f, abcdf \neq 0; \quad (3.7)$$
$$2^a + 3^b + 3^c5^d = 2^e5^f, abdef \neq 0; \quad (3.8)$$
$$3^a + 2^b5^c + 3^d5^e = 2^f, abcef \neq 0; \quad (3.9)$$
$$2^a + 2^b5^c + 3^d5^e = 3^f, abcef \neq 0; \quad (3.10)$$
$$2^a + 5^b + 2^c3^d = 3^e5^f, abcde \neq 0; \quad (3.11)$$
$$2^a + 5^b + 3^c5^d = 2^e3^f, abcef \neq 0; \quad (3.12)$$
$$5^a + 2^b3^c + 3^d5^e = 2^f, abcdf \neq 0; \quad (3.13)$$
$$2^a + 2^b3^c + 3^d5^e = 5^f, abcdf \neq 0; \quad (3.14)$$
$$3^a + 5^b + 2^c3^d = 2^e5^f, abcdef \neq 0; \quad (3.15)$$
$$3^a + 5^b + 2^c5^d = 2^e3^f, abcdef \neq 0; \quad (3.16)$$
$$5^a + 2^b3^c + 2^d5^e = 3^f, abcdef \neq 0; \quad (3.17)$$
$$3^a + 2^b3^c + 2^d5^e = 5^f, abcdef \neq 0; \quad (3.18)$$
$$2^a + 3^b + 5^c = 2^d3^e5^f, abcdef \neq 0; \quad (3.19)$$
$$2^a + 3^b + 2^c3^d5^e = 5^f, abcdef \neq 0; \quad (3.20)$$
$$2^a + 5^b + 2^c3^d5^e = 3^f, abcdef \neq 0; \quad (3.21)$$
$$3^a + 5^b + 2^c3^d5^e = 2^f, abcdef \neq 0; \quad (3.22)$$

Let $S_1 = \{217, 671, 13, 41, 241, 17, 73, 703, 181, 601, 151, 401, 271, 109, 433, 577, 1601, 193, 1153, 641, 769, 163, 811, 1621, 251, 2251, 3001,$

3889, 4861, 487}, $S_2 = \{2^r, 3^s, 5^t \mid r \leq 9, s \leq 6, t \leq 4\}$ and $S = S_1 \cup S_2$. Further, define $m = 2^7 3^5 5^3$ and let $\alpha = (a, b, c, d, e, f)$ represent a sextuple of exponents satisfying (3.i). It follows from a computer consideration of each equation (3.i) relative to the moduli in S_1 successively, using conditions arising from the moduli on S_2, that there are 561 solutions α to the equations (3.i) with exponents in Z_m. (Lemmas given below, which in some cases involve additional moduli, substantially reduce the amount of machine calculation for $1 \leq i \leq 6$.) These sextuples are precisely the solutions to the equations (3.i) listed in Tables 3.1-3.22 below. All but 77 of these α are subsequently completely determined (i.e., determined in Z) by the moduli in S_2. The remaining solutions are completely determined in the proofs of the theorems that follow, using the lemmas in Section 2, a few additional moduli and only trivial calculations.

Theorem 3.1. *The solutions to (3.1) are given in Table 3.1.*

a	b	c	d	e	f	a	b	c	d	e	f
1	1	0	1	2	0	2	5	2	0	2	1
1	2	1	0	2	0	2	5	2	1	4	0
1	2	1	2	4	0	2	7	1	0	3	1
1	2	2	0	1	1	2	7	5	0	1	3
1	3	0	1	1	1	2	11	3	1	7	0
1	3	0	3	3	1	3	3	2	0	0	2
1	3	1	1	0	2	3	4	1	0	3	0
1	4	2	0	3	0	3	5	0	1	2	1
1	4	2	2	5	0	3	6	1	0	1	2
1	4	3	0	2	1	3	6	2	0	4	0
1	6	1	1	4	0	3	9	0	5	6	1
1	6	2	0	1	2	4	5	3	0	1	2
1	7	0	1	3	1	4	6	2	1	0	3
2	3	1	0	1	1	4	7	4	0	2	2
2	3	1	1	3	0	5	9	4	0	0	4
2	4	0	1	0	2	5	15	0	1	8	1
2	4	0	2	2	1	6	11	1	2	7	0

Table 3.1. The solutions to (3.1).

Proof. Let α be a solution to (3.1). We first prove:

Lemma 3.1. *(i)* $cdef = 0$. *(ii) Min* $\{c, e\} \leq 4$. *(iii) Min* $\{d, f\} \leq 2$.

Proof of lemma. Define $r = a - b$, $s = c - e$ and $t = d - f$. (i) Assume the contrary. Then $2^a + 2^b \equiv 0 \pmod{15}$, an impossibility. (ii) Suppose that $c, e \geq 5$. Then $2^r + 1 \equiv 0 \pmod{243}$ so that $r \equiv 81 \pmod{162}$. Since $2^{81} + 1 \equiv 0 \pmod{p}$, where $p = 19$ or 163, it follows that $3^s5^t \equiv 1 \pmod{p}$. Referring to Table 4.3, we conclude that $13s + 16t \equiv 0 \pmod{18}$ and $101s + 15t \equiv 0 \pmod{162}$. It follows easily that $s \equiv t \equiv 0 \pmod{6}$. Thus (from (3.1)) $2^r + 1 \equiv 0 \pmod{7}$, again a contradiction. (iii) Suppose that $d, f \geq 3$. Then $2^r + 1 \equiv 0 \pmod{125}$, $r \equiv 50 \pmod{100}$ and $3^s5^t \equiv 1 \pmod{q}$, where $q = 41$ or 101. Thus (referring to Table 4.3 again) $15s + 22t \equiv 0 \pmod{40}$ and $69s + 24t \equiv 0 \pmod{100}$. It follows that $s \equiv t \equiv 0 \pmod{10}$ so that $2^r + 1 \equiv 0 \pmod{11}$ and $r \equiv 5 \pmod{10}$, again a contradiction. ■

The moduli in S completely determine all but five of the solutions listed in Table 3.1. These five distinguished cases are given by: $\alpha \equiv (2^*, 11, 3^*, 1^*, 7, 0^*)$, $(3^*, 9, 0^*, 5, 6, 1^*)$, $(5^*, 9, 4^*, 0^*, 0^*, 4)$, $(5^*, 15, 0^*, 1^*, 8, 1^*)$ and $(6^*, 11, 1^*, 2^*, 7, 0^*) \pmod{m}$, where asterisks indicate exponents which are determined in Z. In the second case, by Lemma 2.4, $b = 9$. Thus, by Lemma 2.5, $e = 6$ and the solution is thus completely determined. In the remaining cases the exponent b (and hence the entire solution) is determined immediately by Lemma 2.2. ■

Theorem 3.2. *The solutions to (3.2) are given in Table 3.2.*

Proof. Let α be a solution to (3.2). We first prove:

Lemma 3.2. *(i) Min* $\{b, d\} \leq 3$. *(ii) Min* $\{c, e\} \leq 1$. *(iii) If* $ce \neq 0$ *and* $bd = 0$ *then* $(a, b, c, d, e, f) = (1, 0, 2, 0, 1, 5)$.

Proof of lemma. Let $r = a - f$, $s = b - d$ and $t = c - e$. (i) Assume the contrary. Then $2^r \equiv 1 \pmod{81}$ so that $r \equiv 0 \pmod{54}$. Since $2^{54} \equiv 1 \pmod{p}$, where $p = 73$ or 262657, we conclude that

$3^s5^t + 1 \equiv 0 \pmod{p}$. Hence (see Table 4.3): $6s + t \equiv 4 \pmod{72}$ and $32166s + t \equiv 131328 \pmod{262656}$. Thus $6s + t \equiv 0 \pmod{8}$ and $6s + t \equiv 4 \pmod{8}$! (ii) Suppose that $c, e \geq 2$. Then $r \equiv 0 \pmod{20}$ and $3^s5^t + 1 \equiv 0 \pmod{11}$, so that $8s + 4t \equiv 5 \pmod{10}$, again an impossibility. (iii) Suppose that $ce \neq 0$, $bd = 0$. If $b = 0$ and $d \neq 0$ then, we reach a contradiction by considering (3.2) modulo 15. Hence $b = d = 0$. From (ii), $c = 1$. Since $2^r \equiv 1 \pmod{5}$, we conclude that $4 \mid r$. Hence $2^r + 1 \equiv 0 \pmod{3}$ so that t is odd and e is even. Considering our equation modulo 8, we conclude that $a = 1$ and hence $7 + 5^e = 2^f$. From [2], Lemma 3.3, p. 90, $(e, f) = (2, 5)$. ■

a	b	c	d	e	f	a	b	c	d	e	f
1	0	1	0	2	5	3	0	1	5	0	8
1	0	1	2	0	4	3	1	1	2	0	5
1	1	0	1	0	3	3	1	1	4	2	11
1	1	0	3	0	5	3	1	2	2	1	7
1	1	1	1	1	5	4	1	0	2	1	6
1	1	3	3	1	9	4	1	1	2	2	8
1	2	1	4	0	7	5	0	1	3	0	6
2	0	2	1	0	5	5	1	1	4	0	7
2	1	0	2	0	4	5	1	2	4	1	9
2	1	1	2	1	6	7	0	3	1	0	8
2	2	0	5	0	8	7	1	3	2	0	9
2	2	2	3	0	8	7	1	4	2	1	11
3	0	1	1	0	4						

Table 3.2. The solutions to (3.2).

From the moduli in S, there is but one case to consider, $\alpha \equiv (7, 1^*, 4^*, 2^*, 1^*, 11) \pmod{m}$, which is immediately eliminated by Lemma 2.2. ■

Theorem 3.3. *The solutions to (3.3) are given in Table 3.3.*

Proof. Let α be a solution to (3.3). We first prove:

Lemma 3.3. *(i) Min* $\{a, e\} \leq 2$. *(ii) Min* $\{d, f\} \leq 2$.

Proof of lemma. (i) follows immediately from (3.3) considered modulo 8. To prove (ii), let $r = a - b$, $s = c - e$ and $t = d - f$. Suppose that $d, f \geq 3$. Then $r \equiv 50 \pmod{100}$ and $3^r + 1 \equiv 0 \pmod{p}$, where $p = 101$ or 1181. Thus $2^s 5^t \equiv 1 \pmod{p}$. It follows that $s + 24t \equiv 0 \pmod{100}$ and $835s + 914t \equiv 0 \pmod{1180}$. Hence $5 \mid t$ and $10 \mid s$. Thus $3^r + 1 \equiv 0 \pmod{11}$, a contradiction. ■

a	b	c	d	e	f	a	b	c	d	e	f
1	1	1	1	4	0	1	5	2	0	1	3
1	1	1	3	8	0	2	2	1	0	2	1
1	1	2	0	1	1	2	2	5	0	1	2
1	2	2	1	5	0	2	3	2	0	3	1
1	2	2	3	9	0	2	3	6	0	2	2
1	2	3	0	2	1	2	4	1	1	2	2
1	3	1	1	3	1	2	4	5	1	1	3
1	3	1	2	4	1	2	6	9	0	1	4
1	3	1	4	8	1	3	3	1	1	6	0
1	3	2	1	1	2	3	4	2	1	7	0
1	4	4	0	2	2	3	5	1	2	6	1
1	5	1	1	8	0						

Table 3.3. The solutions to (3.3)

From the moduli in S, there is a single case to consider: $\alpha \equiv (2^*, 6, 9, 0^*, 1^*, 4) \pmod{m}$. Immediately from Lemma 2.3, $f = 4$, so that this solution is also completely determined. ■

Theorem 3.4. *The solutions to (3.4) are given in Table 3.4.*

a	b	c	d	e	f	a	b	c	d	e	f
1	2	0	2	1	3	3	1	2	2	0	4
1	3	1	3	2	5	3	3	2	4	0	5
1	4	1	5	1	5	3	4	3	5	1	7
2	1	1	3	0	3	4	1	0	5	1	5
2	3	1	5	0	4	4	3	0	7	1	6
2	4	1	7	1	6	4	4	1	8	2	8
2	4	2	6	1	6						

Table 3.4. The solutions to (3.4)

Proof. Let α be a solution to (3.4). We first prove:

a	b	c	d	e	f	a	b	c	d	e	f
1	1	1	0	2	1	1	3	5	0	1	4
1	1	1	1	4	0	1	4	1	2	3	4
1	1	1	3	6	0	2	2	2	0	1	3
1	1	3	0	1	2	2	3	2	1	1	4
1	2	1	1	2	2	2	5	1	6	9	2
1	2	1	2	4	1	3	3	1	1	8	0
1	2	1	4	6	1	3	4	1	2	8	1
1	2	3	1	1	3						

Table 3.5. The solutions to (3.5).

Lemma 3.4. *(i) Min* $\{c, e\} \leq 1$. *(ii) Min* $\{b, d\} \leq 9$.

Proof of lemma. Let $r = a - f$, $s = b - d$ and $t = c - e$. (i) Assume the contrary. Then $r \equiv 0 \pmod{20}$ and $2^s 5^t + 1 \equiv 0 \pmod{p}$, where $p = 11$ or 61. It follows that $s + 4t \equiv 5 \pmod{10}$ and $s + 22t \equiv 30 \pmod{60}$, which produces a contradiction modulo 2. (ii) Suppose that $b, d \geq 10$. Then $3^r \equiv 1 \pmod{1024}$ so that $r \equiv 0 \pmod{256}$. Hence $2^s 5^t + 1 \equiv 0 \pmod{p}$, where $p = 257, 17$ or 41. It follows that $48s + 55t \equiv 128 \pmod{256}$, $14s + 5t \equiv 8 \pmod{16}$ and $26s + 22t \equiv 20 \pmod{40}$. From the first congruence, $8 \mid t$ so that, from the second, $4 \mid s$. Hence the third cannot hold. ■

The moduli in S completely determine all solutions in this case. ■

a	b	c	d	e	f	a	b	c	d	e	f
1	1	0	1	2	2	1	4	1	10	1	5
1	2	1	2	3	3	2	2	0	5	1	3
1	2	1	3	0	2	2	2	2	6	0	3
1	2	3	9	0	4	2	3	1	6	2	4
1	3	1	5	1	3	2	3	3	7	1	4
1	3	2	4	1	3	2	4	1	6	5	6

Table 3.6. The solutions to (3.6)

Theorem 3.5. *The solutions to (3.5) are given in Table 3.5.*

Proof. Let α be a solution to (3.5). We first prove:

a	b	c	d	e	f	a	b	c	d	e	f
1	1	1	1	1	1	5	1	1	1	2	1
1	1	2	1	0	2	5	1	2	2	3	1
1	1	3	1	2	1	5	1	3	1	1	2
1	1	4	2	4	1	5	1	7	1	3	2
1	5	5	1	4	1	5	5	2	2	1	3
2	4	1	2	3	1	5	5	4	2	3	2
2	4	3	1	0	3	5	5	6	2	1	4
2	4	6	1	4	1	6	4	4	1	2	2
3	3	1	1	2	1	7	3	1	3	4	1
3	3	2	2	3	1	9	1	5	1	3	2
3	3	3	1	1	2	10	4	2	1	2	3
3	3	7	1	3	2	11	3	5	3	5	2
4	2	1	2	1	2	12	6	1	4	5	2
4	2	2	1	2	1	12	6	5	2	2	4
4	2	2	2	0	3	13	5	2	4	7	1
4	2	3	2	2	2	13	9	1	3	2	5
4	2	4	3	4	2	13	9	2	4	5	3
4	6	8	1	4	2	15	3	1	1	8	1

Table 3.7. The solutions to (3.7)

Lemma 3.5. *(i) Min* $\{c,e\}=1$, *(ii) Min* $\{d,f\}\leq 3$.

Proof of lemma. (i) follows immediately from (3.5) considered modulo 4. To prove (ii), let $r=a-b$, $s=c-d$ and $t=d-f$. Observe that if, $d,f\geq 4$ then $r\equiv 27 \pmod{54}$ and hence $2^s3^t\equiv 1 \pmod{p}$, where $p=163$ or 487. Thus $s+101t\equiv 0 \pmod{162}$ and $238s+t\equiv 0 \pmod{486}$. It follows that $s\equiv t\equiv 0 \pmod{18}$. Hence $5^r+1\equiv 0$ (mod 19), an impossibility. ■

One case, $\alpha\equiv(2^*,5,1^*,6,9,2^*) \pmod{m}$ survives the moduli in S. It is completely determined immediately by Lemma 2.4. ■

Theorem 3.6. *The solutions to (3.6) are given in Table 3.6.*

Proof. Let α be a solution to (3.6). We first prove:

Lemma 3.6. *(i) Min* $\{c, e\} \leq 2$. *(ii) Min* $\{b, d\} \leq 7$.

a	b	c	d	e	f	a	b	c	d	e	f
1	1	0	1	1	1	5	1	0	3	5	1
1	1	1	1	2	1	5	1	1	1	1	2
1	1	1	2	4	1	5	1	2	1	4	1
1	1	2	1	1	2	5	1	5	1	1	4
1	5	0	1	1	3	5	5	0	3	4	2
1	5	1	2	6	1	5	5	2	1	6	1
2	4	1	1	2	2	5	5	2	2	2	3
2	4	1	2	5	1	6	4	1	1	5	1
3	3	0	1	3	1	6	8	1	5	7	3
3	3	0	3	5	1	6	8	3	3	4	4
3	3	1	1	1	2	7	3	0	1	5	1
3	3	2	1	4	1	7	3	2	1	3	2
3	3	5	1	1	4	7	3	2	3	8	1
4	2	0	2	1	2	8	2	1	3	7	1
4	2	1	1	3	1	8	2	3	1	4	2
4	2	1	2	2	2	8	6	1	1	3	3
4	2	1	3	4	2	9	1	0	3	7	1
4	2	2	2	1	3	9	5	2	1	5	2
4	2	3	1	5	1	11	3	2	3	7	2
5	1	0	1	3	1	11	9	3	3	1	6

Table 3.8. The solutions to (3.8)

Proof of lemma. Let $r = a - f$, $s = b - d$ and $t = c - e$. (i) Assume the contrary. Then $r \equiv 0 \pmod{18}$ so that $2^s 3^t + 1 \equiv 0 \pmod{p}$, where $p = 19, 829$ or 5167. It follows that $s + 13t \equiv 9 \pmod{18}$, $s + 376t \equiv 414 \pmod{828}$ and $1086s + 4081t \equiv 2583 \pmod{5166}$. With a little effort, we conclude that $(s, t) \equiv (18, 9) \pmod{(36,18)}$. It follows that $5^r \equiv 1 \pmod{13}$ so that $r \equiv 0 \pmod{36}$. This produces a contradiction upon consideration of (3.6) modulo 37. ■

Two cases are not completely determined by the moduli in S : $\alpha \equiv (1^*, 2^*, 3^*, 0, 0^*, 4)$ and $(1^*, 4^*, 1^*, 10, 1^*, 5) \pmod m$. These are dispatched by Lemma 2.2. ■

a	b	c	d	e	f	a	b	c	d	e	f
1	1	2	1	2	7	2	7	3	1	3	14
1	2	1	4	2	11	3	4	1	4	1	9
1	2	2	0	2	7	4	1	2	0	3	8
1	4	1	2	1	7	4	2	2	1	2	8
1	4	3	2	1	11	4	3	1	3	1	8
2	1	1	2	1	6	4	5	1	1	1	8
2	1	2	0	1	6	4	5	3	1	1	12
2	1	5	4	3	14	4	7	1	3	3	12
2	3	1	1	1	6	6	1	3	2	1	10
2	3	3	1	1	10	6	5	1	3	1	10
2	7	1	1	3	10						

Table 3.9. The solutions to (3.9)

a	b	c	d	e	f	a	b	c	d	e	f
1	1	1	1	1	3	4	2	1	2	1	4
1	2	1	0	1	3	4	3	1	0	2	4
1	5	1	4	2	7	6	3	1	0	4	6
2	1	2	3	2	6	6	7	1	0	2	6
2	2	2	0	4	6	7	2	2	1	1	5
2	2	3	2	2	6	7	3	1	1	2	5
2	6	1	4	1	6	9	3	3	3	2	7
3	1	1	2	2	5	9	6	2	1	2	7
3	2	2	3	1	5	12	1	4	5	1	8
3	5	1	1	2	5	17	6	4	5	2	11
4	1	2	1	1	4						

Table 3.10. The solutions to (3.10)

Theorem 3.7. *The solutions to (3.7) are given in Table 3.7.*

Proof. Here four cases remain: $\alpha \equiv (12,6,5^*,2^*,2^*,4)$, $(13,5^*,2^*,4,7,1^*)$, $(13,9,1^*,3^*,2^*,5)$ and $(15,3^*,1^*,1^*,8,1^*) \pmod m$. We apply Lemmas 2.3 and 2.2 in the first and last cases, respectively. In the remaining cases, by Lemma 2.4, $a = 13$. Thus by Lemma 2.5 we are finished. ■

a	b	c	d	e	f
1	2	1	2	2	1
1	2	1	3	4	0
1	2	2	3	3	1
1	2	3	3	5	0
1	2	3	4	3	2
1	2	4	1	1	2
1	4	2	7	1	5
1	4	4	1	3	2
2	1	1	1	1	1
2	1	1	2	3	0
2	1	2	2	2	1
2	1	3	2	4	0
2	1	3	3	2	2
2	3	1	1	3	1
2	3	5	1	2	2
3	2	2	1	2	1
3	2	4	1	4	0
3	2	6	1	2	2
3	4	5	1	6	0
3	6	4	7	4	4
3	6	5	4	6	2
4	1	1	1	3	0
4	1	1	3	1	2
4	1	3	1	2	1
4	1	7	1	4	1
5	2	1	2	1	2
5	2	3	1	4	0
5	4	1	2	3	2
5	4	3	2	6	0
5	4	6	7	2	6
6	1	1	1	1	2
6	1	2	1	4	0
6	3	1	3	5	0
6	3	1	5	3	2
6	3	2	2	2	2
6	3	3	3	4	1
6	3	7	3	6	1
7	2	2	5	2	3
7	2	3	2	2	2
7	2	6	2	6	0
8	1	4	2	4	1
8	1	5	3	2	3
8	3	3	1	4	1
9	2	6	1	6	0
10	1	5	1	2	3
11	4	2	5	6	1
11	4	4	5	8	0
11	4	6	5	6	2
14	1	1	5	3	4
15	2	2	1	8	1

Table 3.11. The solutions to (3.11)

Theorem 3.8. *The solutions to (3.8) are given in Table 3.8.*

Proof. Here there are two distinguished cases:

$\alpha \equiv (6^*, 8, 3^*, 3^*, 4^*, 4)$ and $(13, 9, 3^*, 3^*, 1^*, 6)$ (mod m). Lemma 2.5 immediately dispatches the first case. In the second case we apply Lemmas 2.4 and 2.5 successively. ■

Theorem 3.9. *The solutions to (3.9) are given in Table 3.9.*

Proof. There are three cases requiring special treatment here: $\alpha \equiv (2^*, 1^*, 5, 4^*, 3^*, 14)$, $(6, 1^*, 3^*, 2^*, 1^*, 10)$ and $(6, 5^*, 1^*, 3^*, 1^*, 10)$ (mod m). All of these are completely determined by Lemma 2.2. ■

Theorem 3.10. *The solutions to (3.10) are given in Table 3.10.*

Proof. There are six distinguished cases here:
$\alpha \equiv (2^*, 2^*, 2^*, 0^*, 4, 6)$, $(6^*, 3^*, 1^*, 0^*, 4, 6)$, $(9, 3^*, 3^*, 3^*, 2^*, 7)$, $(9, 6^*, 2^*, 1^*, 2^*, 7)$, $(12, 1^*, 4, 5^*, 1^*, 8)$ and $(17, 6^*, 4, 5^*, 2^*, 11)$ (mod m). The first two cases disappear by Lemma 2.5. In the third and fourth cases we apply Lemma 2.2. In the remaining cases, by Lemma 2.3, $c = 4$. Lemma 2.2 is then applicable. ■

Theorem 3.11. *The solutions to (3.11) are given in Table 3.11.*

Proof. There are twelve solutions α (mod m) to consider here which are listed in Table 3.11.1. In cases 1, 3 and 6 it follows immediately from modulus 390001 that $d \equiv 7$ (mod $39 \cdot 5^4$). Hence from modulus 5^5 in case 3 we have $f = 4$ (and are finished) and in cases 1 and 6 we have $b = 4$, so that Lemma 2.5 applies. Cases 2, 4 and 5 are dispatched immediately by Lemma 2.5. Cases 7, 11 and 12 are eliminated by Lemma 2.2. In the remaining cases (8-10) we apply Lemmas 2.3 and 2.2 successively. ■

	a	b	c	d	e	f
1.	1^*	4	2^*	7	1^*	5
2.	3^*	4	5^*	1^*	6	0^*
3.	3^*	6	4^*	7	4^*	4
4.	3^*	6	5^*	4^*	6	2^*
5.	5^*	4	3^*	2^*	6	0^*
6.	5^*	4	6^*	7	2^*	6

	a	b	c	d	e	f
7.	9	2^*	6^*	1^*	6	0^*
8.	11	4	2^*	5^*	6	1^*
9.	11	4	4^*	5^*	8	0^*
10.	11	4	6^*	5^*	6	2^*
11.	14	1^*	1^*	5^*	3^*	4
12.	15	2^*	2^*	1^*	8	1^*

Table 3.11.1

Theorem 3.12. *The solutions to (3.12) are given in Table (3.12)*

a	b	c	d	e	f	a	b	c	d	e	f
1	2	2	0	2	2	5	2	1	1	3	2
1	2	2	1	3	2	5	2	1	3	4	3
1	2	2	3	7	2	5	2	3	1	6	1
1	2	3	0	1	3	6	1	1	0	3	2
1	2	3	1	1	4	6	1	1	2	4	2
1	2	4	0	2	3	6	1	1	4	3	5
1	2	4	1	4	3	6	1	3	0	5	1
2	1	1	0	2	1	6	1	5	2	11	1
2	1	1	1	3	1	6	3	1	0	6	1
2	1	1	3	7	1	6	3	3	0	3	3
2	1	2	0	1	2	6	3	3	1	2	4
2	1	2	1	1	3	6	3	3	2	5	3
2	1	3	0	2	2	6	3	5	0	4	3
2	1	3	1	4	2	7	2	2	0	1	4
2	3	1	1	4	2	7	2	3	1	5	2
3	2	1	0	2	2	7	4	1	1	8	1
3	2	1	1	4	1	8	1	2	2	1	5
3	2	1	2	2	3	8	1	3	0	5	2
3	4	1	1	3	4	8	3	1	0	7	1
3	4	3	1	8	1	8	5	1	2	7	3
4	1	1	0	3	1	9	4	1	1	7	2
4	1	1	1	2	2	10	3	1	0	7	2
4	1	1	2	5	1	10	5	2	2	1	7
4	1	3	0	4	1	11	4	5	0	2	6
4	3	1	0	4	2	11	4	5	1	4	5
4	3	1	2	3	3	11	4	5	2	2	7
4	3	5	0	7	1	14	1	3	7	2	12
4	5	5	2	10	2						

Table 3.12. The solutions to 3.12

Proof. Here there are five distinguished cases:
$\alpha \equiv (4^*, 5, 5^*, 2^*, 10, 2^*)$, $(10, 5, 2^*, 2^*, 1^*, 7)$, $(11, 4, 5^*, 0^*, 2^*6)$, $(11, 4, 5^*, 2^*, 2^*, 7)$ and $(14, 1^*, 3^*, 7, 2^*, 12)$ (mod m). Lemma 2.2 dispatches the first case. In cases 2, 3 and 4 we successively apply

a	b	c	d	e	f	a	b	c	d	e	f
1	1	2	2	0	5	2	3	3	1	1	8
1	1	2	4	2	11	2	5	1	3	1	8
1	2	1	1	1	5	2	5	3	3	1	10
1	2	3	1	1	7	3	1	4	2	2	9
1	3	1	1	0	5	3	2	1	1	3	9
1	4	1	1	2	7	3	4	1	1	4	11
1	4	3	1	2	9	3	4	2	5	0	9
1	5	1	3	0	7	3	7	1	1	0	9
2	1	1	2	2	8	4	2	4	1	2	10
2	2	1	3	0	6	4	3	1	1	3	10
2	2	2	1	0	6	4	5	1	3	3	12
2	2	4	3	2	10	4	7	1	1	1	10
2	2	5	3	0	10	4	7	3	1	1	12
2	3	1	1	1	6	6	7	1	1	3	14

Table 3.13. The solutions to (3.13)

a	b	c	d	e	f	a	b	c	d	e	f
1	2	3	1	1	3	4	1	1	1	0	2
1	4	1	1	2	3	4	7	1	2	2	4
1	5	1	3	0	3	5	1	2	1	2	3
2	1	1	1	1	2	5	2	1	4	0	3
2	1	2	1	0	2	5	4	1	2	1	3
2	1	5	3	1	4	6	1	5	1	2	4
2	2	1	2	0	2	6	6	5	2	0	6
2	3	3	4	1	4	7	2	5	4	2	5
2	6	2	2	1	4	7	2	6	4	0	5
3	2	2	4	0	3	7	5	4	4	1	5
3	2	3	2	0	3	8	2	4	2	1	4
3	3	2	2	1	3	8	4	2	2	2	4
3	10	1	2	1	5	8	5	2	4	0	4

Table 3.14. The solutions to (3.14)

Lemmas 2.4 and 2.5. In the last case, from Lemma 2.4, $a = 14$, so that $607 + 5^d = 4 \cdot 3^g$, where $g \equiv f - 3 \equiv 9 \pmod{m}$. Lemma 2.5 then applies. ■

Theorem 3.13. *The solutions to (3.13) are given in Table 3.13.*

Proof. Here there are seven distinguished cases:
$\alpha \equiv (3^*, 4^*, 1^*, 1^*, 4, 11)$, $(4, 2^*, 4^*, 1^*, 2^*, 10)$, $(4, 3^*, 1^*, 1^*, 3^*, 10)$, $(4, 5^*, 1^*, 3^*, 3^*, 12)$, $(4, 7^*, 1^*, 1^*, 1^*, 10)$, $(4, 7^*, 3^*, 1^*, 1^*, 12)$ and $(6, 7^*, 1^*, 1^*, 3^*, 14) \pmod{m}$. All of these cases are immediately dispatched by Lemma 2.2. ■

Theorem 3.14. *The solutions to (3.14) are given in Table 3.14.*

Proof. Here there are two distinguished cases,
$\alpha \equiv (3^*, 10, 1^*, 2^*, 1^*, 5)$ and $(7^*, 2^*, 6, 4^*, 0^*, 5) \pmod{m}$, for which we apply Lemmas 2.2 and 2.5, respectively. ■

a	b	c	d	e	f	a	b	c	d	e	f
1	1	2	1	2	1	2	4	1	9	6	4
1	1	3	2	4	1	3	1	1	2	1	2
1	1	6	1	3	2	3	1	4	1	4	1
1	2	2	1	3	1	3	1	5	2	6	1
1	2	2	5	3	3	3	1	8	1	5	2
1	2	3	2	2	2	3	2	2	3	5	1
1	3	3	2	3	2	3	2	4	1	2	2
1	3	6	1	6	1	3	3	3	4	5	2
1	3	7	2	8	1	3	3	4	1	3	2
1	3	10	1	7	2	3	5	4	1	7	2
1	4	2	1	7	1	4	2	1	3	5	1
1	4	2	5	6	2	4	2	4	2	1	3
1	5	3	2	7	2	4	2	11	1	1	5
2	1	1	1	2	1	5	1	3	2	6	1
2	1	1	5	2	3	5	3	4	3	5	2
2	1	2	2	1	2	7	2	5	2	2	4
2	2	1	1	3	1	9	3	6	1	5	4
2	2	3	3	1	3	11	1	10	3	13	2
2	4	1	1	7	1						

Table 3.15. The solutions to (3.15)

Theorem 3.15. *The solutions to (3.15) are given in Table 3.15.*

Proof. Here there are five distinguished cases:
$\alpha \equiv (2^*, 4, 1^*, 9, 6^*, 4)$, $(4^*, 2^*, 11, 1^*, 1^*, 5)$, $(7, 2^*, 5^*, 2^*, 2^*, 4)$, $(9, 3^*, 6^*, 1^*, 5^*, 4)$ and $(11, 1^*, 10, 3^*, 13, 2^*)$ (mod m). In the first case, immediately from the modulus 390001, we have $d \equiv 9 \pmod{39 \cdot 5^4}$. Since $9 + 2 \cdot 3^9 \not\equiv 0 \pmod{5^5}$, it follows that $b = f = 4$. In cases 2, 3 and 4 we apply Lemmas 2.2, 2.5 and 2.5, respectively. In the last case, using modulus 65537 we conclude that $a \equiv 11 \pmod{2^{16}}$. Since $3^{11} + 5 \not\equiv 0 \pmod{2^{14}}$ we conclude that $(c, e) = (10, 13)$. ■

a	b	c	d	e	f	a	b	c	d	e	f
1	1	1	1	1	2	3	2	2	1	3	2
1	1	2	2	2	3	3	3	1	1	1	4
1	1	3	1	4	1	3	3	3	1	6	1
1	1	7	1	3	4	3	3	3	3	7	2
1	2	2	1	4	1	3	4	2	3	7	2
1	2	4	1	2	3	3	4	6	1	2	5
1	3	5	1	5	2	4	1	1	1	5	1
1	3	6	2	6	3	4	1	4	2	1	5
1	3	7	1	8	1	4	3	1	1	3	3
1	3	11	1	7	4	4	5	1	3	7	3
1	4	2	1	3	4	5	1	3	1	5	2
2	1	1	1	3	1	5	1	4	2	3	4
2	1	3	1	1	3	5	2	2	1	5	2
2	2	2	1	1	3	5	2	2	3	8	1
2	3	1	1	4	2	5	3	4	2	8	1
2	3	1	3	7	1	5	3	4	4	7	4
3	1	3	1	3	2	6	3	1	1	5	3
3	1	4	2	4	3	7	1	4	2	5	4
3	1	5	1	6	1	7	5	6	2	8	3
3	1	9	1	5	4	7	5	11	1	6	3

Table 3.16. The solutions to (3.16)

Theorem 3.16. *The solutions to (3.16) are given in Table 3.16.*

Proof. Here the moduli in S completely determine all solutions. ■

Theorem 3.17. *The solutions to (3.17) are given in Table 3.17.*

a	b	c	d	e	f	a	b	c	d	e	f
1	2	1	1	1	3	2	7	1	6	1	6
1	2	2	3	1	4	2	9	1	3	4	8
1	2	4	4	2	6	3	1	2	2	2	5
2	1	1	1	2	4	3	2	2	8	2	8
2	1	2	3	2	5	3	2	3	1	1	5
2	1	4	4	3	7	4	1	3	1	2	6
2	2	2	2	1	4	4	3	1	4	1	6

Table 3.17. The solutions to (3.17)

Proof. There are three cases to eliminate: $\alpha \equiv (2^*, 9, 1^*, 3^*, 4, 8)$, $(4, 1^*, 3^*, 1^*, 2^*, 6)$ and $(4, 3^*, 1^*, 4^*, 1^*, 6) \pmod{m}$. The last two cases are immediately dispatched by Lemma 2.5 while in the first case we use Lemmas 2.3 and 2.2. ■

Theorem 3.18. *The solutions to (3.18) are given in Table 3.18.*

a	b	c	d	e	f	a	b	c	d	e	f
1	1	8	2	4	6	2	5	1	2	1	3
1	2	1	1	1	2	2	6	2	3	1	4
1	3	2	1	2	3	2	10	2	8	2	6
1	10	1	1	2	5	3	1	2	4	1	3
2	1	1	1	1	2	3	4	1	1	2	3
2	2	2	4	1	3	4	3	1	2	1	3
2	2	6	3	2	5	4	4	2	4	2	4
2	3	3	4	2	4	4	7	1	5	1	4

Table 3.18. The solutions to (3.18)

Proof. There are four cases to eliminate: $\alpha \equiv (1^*, 1^*, 8, 2^*, 4, 6)$, $(1^*, 10, 1^*, 1^*, 2^*, 5)$, $(2^*, 2^*, 6, 3^*, 2^*, 5)$ and $(2^*, 10, 2^*, 8^*, 2^*, 6)$. In cases 2, 3 and 4 we apply Lemmas 2.2, 2.5 and 2.2, respectively. In the first case, it follows immediately from the modulus 390001 that $c \equiv 8 \pmod{39 \cdot 5^4}$. Thus, using modulus 5^5 we have $e = 4$. Lemma 2.5 then applies. ■

Theorem 3.19. *The solutions to (3.19) are given in Table 3.19.*

a	b	c	d	e	f	a	b	c	d	e	f
1	1	2	1	1	1	5	5	4	2	2	2
1	5	2	1	3	1	6	4	1	1	1	2
2	4	1	1	2	1	6	4	3	1	3	1
3	3	2	2	1	1	6	8	3	1	3	3
4	2	1	1	1	1	7	3	2	2	2	1
4	2	3	1	1	2	8	2	1	1	3	1
4	6	1	1	1	3	9	1	2	2	3	1
5	1	2	2	1	1	11	3	4	2	3	2
5	5	2	2	1	2	11	7	4	2	5	1

Table 3.19. The solutions to (3.19)

Proof. Here all cases are completely determined by the moduli in S. ■

Theorem 3.20. *The solutions to (3.20) are given in Table 3.20.*

a	b	c	d	e	f	a	b	c	d	e	f
1	1	3	1	1	3	6	8	3	2	3	6
1	5	6	2	1	5	6	8	10	1	3	8
2	4	2	3	1	4	7	7	1	4	1	5
3	3	1	2	1	3	8	2	3	2	1	4
4	2	3	1	2	4	8	2	10	1	1	6
5	1	1	2	1	3	12	2	8	2	1	6
6	4	5	1	1	4	12	6	4	3	2	6

Table 3.20. The solutions to (3.20)

Proof. There are six cases to consider: $\alpha \equiv (6^*, 8, 3^*, 2^*, 3^*, 6)$, $(6^*, 8, 10, 1^*, 3^*, 8)$, $(7^*, 7, 1^*, 4^*, 1^*, 5)$, $(8^*, 2^*, 10, 1^*, 1^*, 6)$,

$(12, 2^*, 8^*, 2^*, 1^*, 6)$ and $(12, 6, 4^*, 3^*, 2^*, 6)$ (mod m). In the first and third cases, we apply Lemma 2.5. In the fourth and fifth cases, we apply Lemma 2.2. The second and sixth cases are dispatched by Lemmas 2.4 and 2.5. ■

Theorem 3.21. *The solutions to (3.21) are given in Table 3.21.*

a	b	c	d	e	f		a	b	c	d	e	f
1	2	4	3	1	7		4	1	2	1	1	4
2	1	4	2	1	6		6	3	2	3	1	6
2	3	3	1	2	6		7	2	1	2	1	5
3	6	1	4	2	9							

Table 3.21. The solutions to (3.21)

Proof. Here there is one distinguished case, $\alpha \equiv (3^*, 6, 1^*, 4^*, 2^*, 9)$ (mod m), which is eliminated by Lemma 2.5. ■

Theorem 3.22. *The solutions to (3.22) are given in Table 3.22.*

a	b	c	d	e	f		a	b	c	d	e	f
1	1	3	1	1	7		5	1	3	2	2	11
1	3	7	1	1	11		5	3	4	4	2	15
2	2	1	1	1	6		6	2	1	3	1	10
2	6	1	1	3	14		6	6	1	1	1	14
3	1	5	1	1	9		7	1	4	1	3	13
3	3	3	2	1	9		7	5	6	2	1	13
3	9	7	2	3	21		9	3	5	4	1	15
4	2	1	1	2	8		11	1	10	1	4	21

Table 3.22. The solutions to (3.22)

Proof. Here there are eight cases to consider, including the most challenging subcases of (1.3). These α (mod m) are listed in Table 3.22.1. The odd-numbered cases are immediately dispatched by Lemma 2.2, while cases 4 and 6 are determined by Lemma 2.4. In the second case, considering (3.22) relative to modulus 2^{21}, we conclude that $b \equiv 9$

(mod 2^{19}). Thus from mod p, where $p = 63700992001$, we have $f \equiv 21$ (mod $135 \cdot 2^{20}$). It follows from mod q, where $q = 113246209$, that $b \equiv 9$ (mod $27 \cdot 2^{21}$). Hence from the modulus 2^{22}, we have $f = 21$. In the final case it is immediate from the moduli 414721 and 2^{11} that $c = 10$. Thus from Lemma 2.3, $e = 4$. Applying modulus 2^{21} we have $a \equiv 11$ (mod 2^{19}). Thus, from mod q, $f \equiv 21$ (mod $9 \cdot 2^{20}$). Hence, from mod $p, a \equiv 11$ (mod $3^4 5^3 2^{20}$) so that, again using the modulus 2^{22}, we conclude that $f = 21$. ■

	a	b	c	d	e	f
1.	2*	6	1*	1*	3*	14
2.	3*	9	7*	2*	3*	21
3.	6	2*	1*	3*	1*	10
4.	6	6	1*	1*	1*	14

	a	b	c	d	e	f
5.	7	1*	4*	1*	3*	13
6.	7	5	6*	2*	1*	13
7.	9	3*	5*	4*	1*	15
8.	11	1*	10	1*	4*	21

Table 3.22.1

4. APPENDIX

w	x	y	z	w	x	y	z
2	2	5	9	2	15	64	81
2	3	3	8	2	15	108	125
2	3	4	9	2	16	27	45
2	3	5	10	2	16	225	243
2	3	10	15	2	18	25	45
2	3	15	20	2	25	27	54
2	3	25	30	2	25	45	72
2	3	20	25	2	25	48	75
2	3	27	32	2	25	54	81
2	3	40	45	2	25	81	108
2	3	45	50	2	25	108	135
2	3	75	80	2	25	135	162
2	3	120	125	2	25	216	243
2	3	400	405	2	25	243	270
2	4	9	15	2	25	405	432
2	4	75	81	2	25	648	675
2	5	5	12	2	25	1125	1152
2	5	8	15	2	25	2160	2187
2	5	9	16	2	27	96	125
2	5	18	25	2	45	81	128
2	5	20	27	2	48	75	125
2	5	25	32	2	48	625	675
2	5	128	135	2	75	243	320
2	5	243	250	2	81	160	243
2	8	15	25	2	135	375	512
2	8	125	135	2	160	243	405
2	9	9	20	2	160	2025	2187
2	9	16	27	2	243	2880	3125
2	9	25	36	2	625	8748	9375
2	9	64	75	3	3	4	10
2	10	15	27	3	3	10	16
2	15	15	32	3	3	250	256

Table 4.1. The solutions to (1.3) with $1 < w \leq x \leq y$.

w	x	y	z
3	4	5	12
3	4	8	15
3	4	9	16
3	4	18	25
3	4	20	27
3	4	25	32
3	4	128	135
3	4	243	250
3	5	8	16
3	5	10	18
3	5	12	20
3	5	16	24
3	5	24	32
3	5	32	40
3	5	40	48
3	5	64	72
3	5	72	80
3	5	100	108
3	5	120	128
3	5	192	200
3	5	640	648
3	6	16	25
3	8	9	20
3	8	16	27
3	8	25	36
3	8	64	75
3	9	20	32
3	9	500	512
3	10	12	25
3	10	27	40
3	10	32	45
3	10	243	256
3	12	25	40
3	12	625	640
3	15	32	50
3	16	45	64
3	16	81	100
3	16	125	144
3	20	25	48
3	20	27	50
3	20	625	648
3	20	2025	2048
3	25	32	60
3	25	36	64
3	25	72	100
3	25	80	108
3	25	100	128
3	25	512	540
3	25	972	1000
3	27	50	80
3	27	1250	1280
3	32	40	75
3	32	45	80
3	32	90	125
3	32	100	135
3	32	125	160
3	32	640	675
3	32	1215	1250
3	40	200	243
3	45	80	128
3	45	2000	2048
3	50	72	125
3	50	75	128
3	50	3072	3125

Table 4.1. The solutions to (1.3) with $1 < w \le x \le y$ (continued).

w	x	y	z	w	x	y	z
3	64	125	192	4	5	216	225
3	72	125	200	4	5	375	384
3	72	3125	3200	4	5	720	729
3	80	160	243	4	6	15	25
3	125	128	256	4	6	125	135
3	125	160	288	4	8	15	27
3	125	192	320	4	9	12	25
3	125	256	384	4	9	27	40
3	125	384	512	4	9	32	45
3	125	512	640	4	9	243	256
3	125	640	768	4	15	45	64
3	125	1024	1152	4	15	81	100
3	125	1152	1280	4	15	125	144
3	125	1600	1728	4	16	25	45
3	125	1920	2048	4	25	25	54
3	125	3072	3200	4	25	96	125
3	125	10240	10368	4	27	50	81
3	160	512	675	4	27	225	256
3	625	972	1600	4	32	45	81
3	2500	13122	15625	4	40	81	125
4	5	6	15	4	45	576	625
4	5	9	18	4	50	81	135
4	5	15	24	4	50	675	729
4	5	16	25	4	75	81	160
4	5	18	27	4	81	320	405
4	5	27	36	4	81	540	625
4	5	36	45	4	96	125	225
4	5	45	54	4	100	625	729
4	5	72	81	4	125	600	729
4	5	81	90	4	128	243	375
4	5	135	144	4	135	486	625

Table 4.1. The solutions to (1.3) with $1 < w \leq x \leq y$ (continued).

w	x	y	z	w	x	y	z
4	135	2048	2187	5	16	27	48
4	216	405	625	5	16	54	75
4	225	500	729	5	16	60	81
4	320	405	729	5	16	75	96
5	5	6	16	5	16	384	405
5	5	8	18	5	16	729	750
5	5	54	64	5	18	25	48
5	6	9	20	5	18	27	50
5	6	16	27	5	18	625	648
5	6	25	36	5	18	2025	2048
5	6	64	75	5	24	25	54
5	8	12	25	5	24	96	125
5	8	27	40	5	25	162	192
5	8	32	45	5	27	32	64
5	8	243	256	5	27	40	72
5	9	10	24	5	27	48	80
5	9	16	30	5	27	64	96
6	9	18	32	5	27	96	125
5	9	36	50	5	27	128	160
5	9	40	54	5	27	160	192
5	9	50	64	5	27	256	288
5	9	256	270	5	27	288	320
5	9	486	500	5	27	400	432
5	10	12	27	5	27	480	512
5	10	81	96	5	27	768	800
5	12	15	32	5	27	2560	2592
5	12	64	81	5	32	125	162
5	12	108	125	5	32	32768	32805
5	15	16	36	5	36	40	81
5	15	108	128	5	40	243	288
5	16	24	45	5	48	72	125

Table 4.1. The solutions to (1.3) with $1 < w \le x \le y$ (continued).

w	x	y	z
5	48	75	128
5	48	3072	3125
5	64	75	144
5	64	81	150
5	64	1875	1944
5	64	6075	6144
5	72	243	320
5	75	432	512
5	81	400	486
5	96	1024	1125
5	108	512	625
5	144	256	405
5	225	256	486
5	243	400	648
5	243	1800	2048
5	256	864	1125
5	324	400	729
5	400	2187	2592
5	486	16384	16875
5	2187	6000	8192
5	16384	2109375	2125764
5	27648	177147	204800
5	177147	1920000	2097152
6	9	10	25
6	9	25	40
6	9	625	640
6	25	50	81
6	25	225	256
6	125	125	256
8	8	9	25
8	9	10	27
8	9	15	32
8	9	64	81
8	9	108	125
8	10	27	45
8	10	225	243
8	12	25	45
8	15	25	48
8	15	27	50
8	15	625	648
8	15	2025	2048
8	25	27	60
8	25	48	81
8	25	75	108
8	25	192	225
8	27	40	75
8	27	45	80
8	27	90	125
8	27	100	135
8	27	125	160
8	27	640	675
8	27	1215	1250
8	36	81	125
8	45	72	125
8	45	75	128
8	45	3072	3125
8	75	160	243
8	81	640	729
8	96	625	729
8	100	135	243
8	135	625	768
8	512	3125	3645

Table 4.1. The solutions to (1.3) with $1 < w \le x \le y$ (continued).

w	x	y	z	w	x	y	z
8	2592	15625	18225	9	64	15552	15625
8	4050	15625	19683	9	80	640	729
8	15625	34992	50625	9	81	160	250
9	9	32	50	9	125	250	384
9	10	45	64	9	128	375	512
9	10	81	100	9	135	256	400
9	10	125	144	9	200	2916	3125
9	15	16	40	9	216	400	625
9	15	40	64	9	256	360	625
9	15	1000	1024	9	256	375	640
9	16	20	45	9	256	15360	15625
9	16	25	50	9	320	400	729
9	16	50	75	9	375	640	1024
9	16	75	100	9	375	16000	16384
9	16	100	125	9	512	729	1250
9	16	125	150	9	625	39366	40000
9	16	135	160	9	750	15625	16384
9	16	200	225	9	4096	11520	15625
9	16	225	250	9	6250	10125	16384
9	16	375	400	9	6400	9216	15625
9	16	600	625	10	27	27	64
9	16	2000	2025	10	27	125	162
9	20	25	54	10	27	32768	32805
9	20	96	125	10	81	125	216
9	25	30	64	10	108	125	243
9	25	128	162	10	125	729	864
9	25	216	250	12	25	27	64
9	27	64	100	12	25	125	162
9	32	40	81	12	25	32768	32805
9	36	80	125	12	32	81	125
9	40	576	625	12	125	375	512

Table 4.1. The solutions to (1.3) with $1 < w \leq x \leq y$ (continued).

w	x	y	z	w	x	y	z
15	16	50	81	18	100	125	243
15	16	225	256	18	125	625	768
15	24	25	64	20	24	81	125
15	25	32	72	20	25	27	72
15	25	216	256	20	25	36	81
15	32	81	128	20	25	243	288
15	64	81	160	20	27	81	128
15	81	160	256	20	81	1024	1125
15	81	4000	4096	24	25	32	81
15	100	128	243	24	25	576	625
15	128	625	768	24	80	625	729
15	256	729	1000	24	125	256	405
15	384	625	1024	24	375	625	1024
15	512	625	1152	25	27	48	100
15	625	3456	4096	25	27	108	160
16	20	45	81	25	27	128	180
16	25	40	81	25	27	972	1024
16	27	32	75	25	32	135	192
16	27	200	243	25	32	243	300
16	45	64	125	25	32	375	432
16	75	125	216	25	36	64	125
16	81	128	225	25	48	15552	15625
16	125	243	384	25	54	81	160
16	225	384	625	25	64	640	729
16	729	1280	2025	25	72	128	225
16	3125	6075	9216	25	81	144	250
18	25	32	75	25	81	150	256
18	25	200	243	25	81	6144	6250
18	27	80	125	25	90	128	243
18	32	75	125	25	96	135	256
18	32	625	675	25	128	135	288

Table 4.1. The solutions to (1.3) with $1 < w \leq x \leq y$ (continued).

w	x	y	z	w	x	y	z
25	128	576	729	27	2048	4000	6075
25	128	972	1125	27	144000	1953125	2097152
25	135	864	1024	30	729	15625	16384
25	162	2000	2187	32	45	48	125
25	192	512	729	32	45	243	320
25	216	384	625	32	72	625	729
25	243	500	768	32	75	405	512
25	270	729	1024	32	81	512	625
25	288	2187	2500	32	100	243	375
25	320	384	729	32	125	243	400
25	324	675	1024	32	225	243	500
25	1458	3125	4608	32	243	400	675
25	1526	5000	6561	32	243	625	900
27	40	125	192	32	243	1600	1875
27	45	128	200	32	625	139968	140625
27	48	50	125	36	64	125	225
27	48	125	200	36	125	6400	6561
27	48	3125	3200	40	64	625	729
27	50	243	320	40	75	128	243
27	64	125	216	40	81	135	256
27	80	405	512	45	128	1875	2048
27	125	360	512	45	243	512	800
27	125	648	800	45	250	729	1024
27	125	1000	1152	45	256	324	625
27	128	250	405	48	125	1875	2048
27	128	1125	1280	50	54	625	729
27	160	2000	2187	50	81	125	256
27	320	625	972	54	64	125	243
27	500	625	1152	64	75	486	625
27	625	2048	2700	64	75	2048	2187
27	1125	2048	3200	64	80	81	225

Table 4.1. The solutions to (1.3) with $1 < w \leq x \leq y$ (continued).

w	x	y	z	w	x	y	z
64	81	125	270	125	243	10000	10368
64	81	480	625	125	243	32400	32768
64	125	135	324	125	12960	19683	32768
64	125	216	405	128	405	2592	3125
64	125	243	432	128	810	2187	3125
64	125	486	675	128	972	2025	3125
64	125	540	729	135	160	729	1024
64	125	675	864	144	225	256	625
64	125	3456	3645	225	1024	3125	4374
64	125	6561	6750	243	625	2048	2916
64	3375	6561	10000	243	2500	8192	10935
64	6561	9000	15625	250	8192	19683	28125
64	6561	9375	16000	375	384	15625	16384
64	6561	384000	390625	512	675	1000	2187
75	81	100	256	625	972	2048	3645
75	256	3125	3456	625	1215	2048	3888
75	324	625	1024	625	2048	2187	4860
75	512	1600	2187	625	2048	3888	6561
80	81	6400	6561	625	2048	6075	8748
81	128	2916	3125	625	2048	15552	18225
81	144	400	625	729	800	4096	5625
81	160	384	625	729	1250	4096	6075
81	250	3125	3456	729	4096	10800	15625
81	256	288	625	1215	1250	4096	6561
81	640	3375	4096	1600	2187	3125	6912
96	625	3375	4096	2187	2880	3125	8192
125	144	243	512	2187	3125	10240	15552
125	162	225	512	2500	8192	19683	30375
125	192	19683	20000	3375	8192	19683	31250
125	243	400	768	6075	40000	131072	177147
125	243	432	800				

Table 4.1. The solutions to (1.3) with $1 < w \leq x \leq y$ (continued).

n	$\mathrm{ord}_n 2$	$\mathrm{ord}_n 3$	$\mathrm{ord}_n 5$
$2^k, k \geq 4$	$--$	2^{k-2}	2^{k-2}
$3^k, k \geq 1$	$2 \cdot 3^{k-1}$	$--$	$2 \cdot 3^{k-1}$
$5^k, k \geq 1$	$4 \cdot 5^{k-1}$	$4 \cdot 5^{k-1}$	$--$
7	3	6	6
11	10	5	5
13	12	3	4
17	8	16	16
19	18	18	9
29	28	28	14
37	36	18	36
41	20	8	20
73	9	12	72
109	36	27	27
151	15	50	75
163	$2 \cdot 3^4$	$2 \cdot 3^4$	$2 \cdot 3^3$
181	$2^2 3^2 5$	45	15
193	$2^5 3$	16	$2^6 3$
217	15	30	6
241	24	120	40
251	50	125	25
271	$3^3 5$	30	27
401	200	400	25
433	72	27	$2^4 3^3$
487	3^5	$2 \cdot 3^5$	$2 \cdot 3^3$
577	$2^4 3^2$	$2^4 3$	$2^6 3^2$
601	25	75	12
631	45	630	35
641	64	640	64
671	60	10	30
703	36	18	36
769	$2^7 3$	$2^4 3$	2^7
811	$2 \cdot 3^3 5$	$2 \cdot 3^4 5$	$3^4 5$

Table 4.2. The orders of 2, 3 and 5 mod n for various n used above.

n	$\text{ord}_n 2$	$\text{ord}_n 3$	$\text{ord}_n 5$
1153	$2^5 3^2$	$2^6 3^2$	$2^7 3^2$
1601	$2^4 5^2$	$2^6 5^2$	$2^4 5^2$
1621	$2^2 3^4 5$	$3^2 5$	$3^4 5$
2251	$2 \cdot 3 \cdot 5^3$	$2 \cdot 5^3$	$3^2 5^3$
3001	$2^2 \cdot 3 \cdot 5^3$	$2^2 5^3$	$2 \cdot 5^3$
3889	$2^3 3^4$	3^4	$2^2 3^5$
4861	$2^2 3^5$	$3^5 5$	3^4
12289	$2^{11} 3$	2^9	$2^{11} 3$
15121	$2 \cdot 3^3 5$	$2^3 3^3$	$2^3 3^3 5 \cdot 7$
22501	$2^2 3^2 5^4$	$2 \cdot 3^2 5^4$	$3^2 5^4$
25601	$2^4 5^2$	$2^{10} 5^2$	$2^4 5$
39367	3^7	$2 \cdot 3^9$	$2 \cdot 3^5$
40961	$2^{11} 5$	$2^{13} 5$	$2^{11} 5$
52501	$2^2 5^3 7$	$5^4 7$	$3 \cdot 5^4 7$
65537	2^5	2^{16}	2^{16}
147457	$2^{11} 3^2$	$2^{11} 3^2$	$2^{14} 3$
196831	$3^9 5$	$2 \cdot 3^9 5$	$3^9 5$
331777	$2^6 3^4$	$2^6 3^3$	$2^{12} 3^4$
390001	$2 \cdot 5^4 13$	$3 \cdot 5^4 13$	$2^3 3$
414721	$2^5 5$	$2^9 3^3$	$2^7 3^3$
708751	$3^4 5^3 7$	$2 \cdot 3^4 5^4 7$	$3^4 5^3 7$
113246209	$2^{20} 3^2$	$2^{19} 3^3$	$2^{21} 3^3$
63700992001	$2^{20} 3^3 5$	$2^{20} 3^4 5^3$	$2^{19} 3^5 5^2$

Table 4.2. The orders of 2, 3 and 5 mod n for various n used above (continued).

p	g	$\text{ind}_g 2$	$\text{ind}_g 3$	$\text{ind}_g 5$
11	2	1	8	4
13	2	1	4	9
17	3	14	1	5
19	2	1	13	16
37	2	1	26	23
41	6	26	15	22
61	2	1	6	22
73	5	8	6	1
101	2	1	69	24
163	2	1	101	15
257	3	48	1	55
487	3	238	1	99
829	2	1	376	92
1181	7	835	177	914
5167	6	1086	4081	3157
262657	5	165376	32166	1

Table 4.3. Table of indices for selected primes p relative to the primitive roots g.

References

[1] Alex, L.J.: *Simple groups of order* $2^a 3^b 5^c 7^d p$, Trans. Am. Math. Soc., **173** (1972), 389-399.

[2] Alex, L.J.: *Diophantine equations related to finite groups,* Comm. in Algebra, **4** (1), (1976), 77-100.

[3] Alex, L.J. and Foster, L.L.: *On Diophantine equations of the form* $1+2^a = p^b q^c + 2^d p^e q^f$, Rocky Mountain J. of Math., **13** (1983), 321-331.

[4] Alex, L.J. and Foster, L.L.: *On the Diophantine equation* $1 + p^a = 2^b + 2^c p^d$, Rocky Mountain J. of Math., **15** (1985), 739-761.

[5] Alex, L.J. and Foster, L.L.: *On the Diophantine Equation* $1+X+Y = Z$, Rocky Mountain J. of Math. (to appear).

[6] Alex, L.J. and Foster, L.L.: *On the Diophantine equation* $1+x+y=z$, *with* $xyz = 2^r 3^s 7^t$ (to appear).

[7] Brenner, J.L. and Foster, L.L.: *Exponential Diophantine equations,* Pacific J. Math., **101** (1982), 263-301.

[8] Brillhart, J., Lehmer, D.H., Selfridge, J.L., Tuckerman, B. and Wagstaff, S.S. Jr.: *Contemporary Mathematics*, Vol. 22, American Math. Soc., Providence, Rhode Island, 1980.

[9] Evertse, J.H.: *On sums of S units and linear recurrences,* Compositio Math., **53** (1984), 225-244.

[10] Van der Poorten, A.J. and Schlickewei, H.P.: *The growth conditions for recurrence sequences,* Macquarie Univ. -Rep., 82-0041, North-Ryde, Australia.

[11] De Weger, B.M.M.: *Solving exponential equations using lattice basis reduction algorithms,* J. Number Theory, **26** (1987), 325-367.

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