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On the Total (Non Absolute) Curvature of a Even Dimensional Submanifold X^n Immersed in $\mathbf{R}^{n+2}$

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ABSTRACT. The total curvatures of the submanifolds immersed in the euclidean space have been studied mainly by Santaló and Chern-Kuiper. In this paper we give geometrical and topological interpretation of the total (non absolute) curvatures of the even dimensional submanifolds immersed in $\mathbf{R}^{n+2}$. This gives a generalization of two results obtained by Santaló.

0. INTRODUCTION

The total curvatures of compact manifolds X^n, (without boundary), immersed in the euclidean space $\mathbf{R}^{n+N}$ have been widely studied in the literature. In [CHL. 1,2] the "absolute" total curvatures of X^n were studied. This work of Chern has been followed by the works of several other authors. In this direction L.A. Santaló, [SA. 1], gives the definition

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of some "absolute" total curvatures. He, [SA. 2], later analyses a lot of properties of these curvatures. He gives the local representation for the total, (absolute and non absolute), curvatures and studies this kind of curvatures in some cases; in particular, for the non absolute curvatures when $n = N = 2$, $r = 1$ and for $n = 4$, $N = 2$, $r = 2$, where r is the order of the total curvature. In this note we generalize these two results to the cases $n = 4t$, $N = 2$, $r = 2t$ and $n = 4t+2$, $N = 2$, $r = 2t+1$. For $n = 4t$, $N = 2$, $r = 2t$, we obtain that this absolute curvature has also a topological meaning, generalizing also the topological interpretation given by L.A. Santaló in [SA. 2] for $n = 4$, $N = 2$, $r = 2$.

1. SOME GENERAL RESULTS.

Let X^n be a compact differentiable manifold without boundary. Following the notations that L.A. Santaló had used in [SA. 2] we recall several notations and general properties about the Integral Geometry of a submanifold X^n immersed in the euclidean space $\mathbf{R}^{n+N}$.

Let $(x, e_1, \ldots, e_{n+N})$ be a local field of orthonormal frames, such that, when it is restricted to X^n, the vectors $e_1, \ldots, e_n$ are tangent to X^n and the remaining vectors $e_{n+1}, \ldots, e_{n+N}$ are normal to X^n. We use the following range of indices $1 \leq i, j, k, h, \ldots \leq n < \alpha, \beta, \gamma, \ldots \leq n+N$; $1 \leq A, B, C, \ldots \leq n+N$ and the summation convention will be used throughout.

The following equations for the moving frames in $\mathbf{R}^{n+N}$ are well known

$$dx = \omega_A e_A; \; de_A = \omega_A^B e_B; \; \omega_B^A + \omega_A^B = 0;$$

$$\omega_\alpha = 0, \; \omega_{i\alpha} = A_{\alpha,ij}\omega^j; \; A_{\alpha,ij} = A_{\alpha,ji}; \; d\omega_{ij} = \omega_{ih} \wedge \omega_{hj} + \Omega_{ij}$$

where $\Omega_{ij} = \omega_{i\alpha} \wedge \omega_{\alpha j}$ with $R_{ijkh} = A_{\alpha,ik}A_{\alpha,jh} - A_{\alpha,ih}A_{\alpha,jk}$.

We have also $d\omega_{\alpha\beta} = \omega_{\alpha\gamma} \wedge \omega_{\gamma\beta} + \Omega_{\alpha\beta}$ where $\Omega_{\alpha\beta} = \omega_{\alpha i} \wedge \omega_{i\beta}$ with $R_{\alpha\beta hj} = A_{\alpha,ih}A_{\beta,ij} - A_{\alpha,ij}A_{\beta,ih}$. We recall that $A_{(.,..)}$ are the coefficients of the second fundamental form of the submanifold.

Let L_h denote a h-dimensional linear subspace in $\mathbf{R}^{n+N}$. Let $L_{h[0]}$ denote a h-plane in $\mathbf{R}^{n+N}$ through a fixed point 0. We shall represent the element of volume in the corresponding Grassmann manifold by $dL_{h[0]}$, which expression can be found in [SA. 2] or [CH]. So we have

$$dL_{h[0]} = dL_{n-h[0]} = \bigwedge_{a,u} \omega_{au}$$

$$1 \leq a, b, \ldots \leq h < u, v, \ldots \leq n + N.$$

Also, the density for sets of h-planes, not through 0, in $\mathbf{R}^{n+N}$ is given by, [SA.2].

$$dL_h = dL_{h[0]} \wedge \omega_{h+1} \wedge \ldots \wedge \omega_{n+N}$$

where $\omega_{h+1} \wedge \ldots \wedge \omega_{n+N}$ is the volume element of the $(n + N - h)$-plane spanned by the vectors $e_{h+1}, \ldots, e_{n+N}$ orthogonal to L_h at the intersection point.

The volume of the Grassmann manifold

$$G_{h,n+N-h} = \frac{O(n+N)}{O(h) \times O(n+N-h)}$$

is given by

$$v\Big(G_{h,n+N-h}\Big) = \int_{G_{h,n+N-h}} dL_{h[0]} = \frac{O_h O_{h+1} \times \cdots \times O_{n+N-1}}{O_1 O_2 \times \cdots \times O_{n+N-h-1}} \quad (1.1)$$

where O_i represents the area of the i-dimensional unit sphere.

With the same notations used previously we have the following.

Definition. [SA. 1] *Assuming $1 \leq r \leq n + N - 1$, the r-th total absolute curvature of X^n can be defined as follows:*

Let O be the fixed point in $\mathbf{R}^{n+N}$ and consider a $(n + N - r)$-plane through $0, 1 \leq r \leq n$, $(n \leq r \leq n + N - 1)$. Let Γ_r be the set of all r-planes L_r in $\mathbf{R}^{n+N}$ which are contained in (contain) some $T(x)$, $x \in X^n$, pass through x and are perpendicular to $L_{n+N-r[O]}$. The intersection $\Gamma_r \cap L_{n+N-r[O]}$ is a compact variety in $L_{n+N-r[O]}$ of dimension $\delta_1 = n - rN$, $(\delta_2 = n(r + 1 - n - N)$.

Let $\mu_{\delta_i}(\Gamma_r \cap L_{n+N-r[O]})$ denote the measure of this variety as a subvariety of $L_{n+N-r[O]}$. Then the r-th total absolute curvature of X^n immersed in L_{n+N} is

$$K^*_{r,N}(X^n) = \frac{O_1 \times \cdots \times O_{n+N-r-1}}{O_r \times \cdots \times O_{n+N-1}}$$

$$\int_{G_{r,n+N-r}} \mu_{\delta_i}\Big(\Gamma_r \cap L_{n+N-r[O]}\Big) dL_{n+N-r[O]}.$$

Here we are interested only in the case $\delta_1 = 0$; that is, $n = rN$. Then μ_o means the number of intersection points of Γ_r and $L_{n+N-r[O]}$. In this case, L.A. Santaló had obtained the local representation of the curvatures $K^*_{r,N}(X^n)$ and it is given by

$$K^*_{r,N}(X^n) = \int_X Q^*_{r,N}(x) d\sigma_n$$

where

$$Q^*_{r,N}(X^n) = \frac{O_1 \times \cdots \times O_{n+N-r-1}}{O_r \times \cdots \times O_{n+N-1}} \int_{G_{r,n-r}} \|G(x, L_r(x))\| dL_{r[x]}$$

with

$$G(x, L_{r[x]}) d\sigma_n = \bigwedge_{i,\alpha} \omega_{i\alpha}. \qquad (1.2)$$

$$1 \leq i, j, \ldots \leq r; \; n+1 \leq \alpha, \beta, \ldots \leq n+N.$$

The expression for G is rather complicated depending on the coefficients of the second fundamental form. As L.A. Santaló pointed out, the integration of this fonction over the Grassmann manifold is very difficult. If we consider the "total (non absolute)" curvatures $K_{r,N}(X^n)$, then this integration was made in several cases for low dimensions, [SA. 2].

2. TOTAL (NON ABSOLUTE) CURVATURE OF A EVEN DIMENSIONAL SUBMANIFOLD X^n IMMERSED IN $\mathbf{R}^{n+2}$

If X^n, $n = 2l$ is a submanifold immersed in $\mathbf{R}^{2l+2}$, we consider separately the cases:

A) $n = 4t$, $N = 2$, $r = 2t$.

B) $n = 4t + 2$, $N = 2$, $r = 2t + 1$.

First, we analyse the CASE A). According to (1.2), if we consider the $2t$-plane $L_{2t}(x)$ contained in L_{4t} and spanned by $e_1, \dots, e_{2t}$, we have

$$G(x, [e_1, \dots, e_{2t}])d\sigma_{4t} = \omega_{1,4t+1} \wedge \omega_{1,4t+2} \wedge \dots \wedge \omega_{2t,4t+1} \wedge \omega_{2t,4t+2}. \quad (2.1)$$

It is possible also to give the corresponding expression for the $2t$-plane $L_{2t[x]}$ spanned by the vectors $e'_i = \gamma_{ih} e_h$, but it is not useful.

Instead of evaluating the integral at the right of (2.1) over $G_{2t,2t}$ is easier to observe that for any frame $(e_1, \dots, e_{2t})$ the sum

$$S = \sum_{(i_1, \dots, i_{2t})} \omega_{i_1,4t+1} \wedge \omega_{i_1,4t+2} \wedge \dots \wedge \omega_{i_{2t},4t+1} \wedge \omega_{i_{2t},4t+2} \quad (2.2)$$

is invariant under the orthogonal group $O(2t)$. In consequence, S is equal to its mean value over $G_{2t,2t}$ and according to (1.1) we have

$$Q_{2t,2t[x]} d\sigma_{4t} = \frac{O_1 \times \dots \times O_{2t-1}}{O_{2t+2} \times \dots \times O_{4t+1}} \frac{1}{V_{2t}^{4t}} S \quad (2.3)$$

where V_{2t}^{4t} represents the variations of $4t$ elements taken $2t$ each time.

Although it is possible to give expressions for $Q_{2t,2t}(x)$ depending on the coefficients of the second fundamental form and depending on the Riemannian curvature operator of the submanifold, these are rather complicated; however we can prove the following:

Proposition 1. *With the same conditions as before, the $4t$-differential form S represents the t-th Pontrjagin class of X^{4t}.*

Proof. We know, ([KN] p.313) that

$$p_k(\Omega) = \frac{1}{(2\pi)^{2k}(2k)!}\sum \delta^{j_1\ldots j_{2k}}_{i_1\ldots i_{2k}}\Omega_{i_1 j_1}\wedge\ldots\wedge\Omega_{i_{2k}j_{2k}} \tag{2.4}$$

represent the k-th Pontrjagin class of X^{4t}, where the summation runs over all ordered subsets $(i_1,\ldots,i_{2k})$ of $2k$ elements of $(1,\ldots,4k)$ and all permutations $(j_1,\ldots,j_{2k})$ of $(i_1,\ldots,i_{2k})$. If $i_a = j_a$ for all $a = 1,\ldots,2k$, $\Omega_{i_a i_a} = 0$. So, i_a must be always different of j_a.

There are several posibilities for the partition $(i_1,\ldots,i_{2k})$; in any case, this set remains separated in two or more subsets and each one with their own permutation. Each subset is of the form

$$\sum_{(i_1,\ldots,i_{2\lambda})}\Omega_{i_1 i_{p(1)}}\wedge\ldots\wedge\Omega_{i_{2\lambda}i_{p(2\lambda)}} =$$

$$= \sum_{(i_\lambda)}(\omega_{i_1,4t+1}\wedge\omega_{4t+1,i_{p(1)}}+\omega_{i_1,4t+2}\wedge\omega_{4t+2,i_{p(1)}})\wedge\ldots\wedge$$
$$\wedge(\omega_{i_{2\lambda},4t+1}\wedge\omega_{4t+1,i_{p(2\lambda)}}+\omega_{i_{2\lambda},4t+2}\wedge\omega_{4t+2,i_{p(2\lambda)}}) \tag{2.5}$$

where p belongs to the group of permutations of order 2λ and we have used (2.7) for the expression of the curvature form.

Now, in (2.5) all products are zero except in the case we have exactly the index $4t+1$ λ times and similarly for the index $4t+2$. Then remembering the properties of ω_{ij}, all terms are of the form

$$\sum_{(i_\lambda)}\omega_{i_1,4t+1}\wedge\omega_{i_1,4t+2}\wedge\ldots\wedge\omega_{i_{2\lambda},4t+1}\wedge\omega_{i_{2\lambda},4t+2}. \tag{2.6}$$

Taking the product of all subsets of the form (2.6) which appear in (2.5) see that (2.2) and (2.4) represent the same cohomology class.

CASE B)

Now, $L^{4t+2}_{2t+1[x]}$ is the $2t+1$ plane spanned by $e_1, \dots, e_{2t+1}$. According to (1.2), we have

$$G(x, [e_1, \dots, e_{2t+1}])d\sigma_{4t+2} = \omega_{1,4t+3} \wedge \omega_{1,4(t+1)} \wedge \dots \wedge$$

$$\wedge \omega_{2t+1,4t+3} \wedge \omega_{2t+1,4(t+1)}. \tag{2.7}$$

For the general $(2t+1)$ -space spanned by the vectors $e'_i = \gamma_{ih} e_h$, $(1 \leq i, h, \dots \leq 2t+1)$, we have also a general formula for $G(x, [e'_1, \dots, e'_{2t+1}])$ but it is not manageable. Now, we proceed as in the case a). So, we construct the following orthogonal invariant:

$$S = \sum_{(i_1, \dots, i_{2t+1}) = (i)} \omega_{i_1,4t+3} \wedge \omega_{i_1,4(t+2)} \wedge \dots \wedge \omega_{i_{2t+1},4t+3} \wedge \omega_{i_{2t+1},4(t+2)}. \tag{2.8}$$

In this case we have also the following

Proposition 2. *The local representation of the curvature $Q_{2t+1,2}(x)$ is*

$$Q_{2t+1,2}(x) = \frac{1}{V^{4t+2}_{2t+1}} \frac{O_1 \times \dots \times O_{2t}}{O_{2t+3} \times \dots \times O_{4t+3}}$$

$$R(4t+3, 4t+4;\ 1, 2, \dots, 4t+1, 4t+2) \tag{2.9}$$

where

$$R(4t+3, 4t+4;\ 1, 2, \dots, 4t+1, 4t+2) =$$

$$\sum_{l} \varepsilon(\rho) R(4t+3, 4t+4, \rho(1), \rho(2)) \times \dots$$

$$\dots \times R(4t+3, 4t+4, \rho(4t+1), \rho(4t+2))$$

is a kind of generalized mixed curvature tensor, which generalizes the classical Ricci equation.

Proof. With the same considerations as in the case a), we have

$$Q_{2t+1,2}(x) d\sigma_{4t+2} = \frac{1}{V^{4t+2}_{2t+1}} \frac{O_1 \times \dots \times O_{2t}}{O_{2t+3} \times \dots \times O_{4t+3}} S. \tag{2.10}$$

In the following, by simplicity, we identify the index $4t+3$ with α and $4(t+1)$ with β. Since $\omega_{i\alpha} = A_{\alpha,ij}\omega^j$, we have

$$S = \sum_{(i)} \omega_{i_1\alpha} \wedge \omega_{i_1\beta} \wedge \ldots \wedge \omega_{i_{2t+1}\alpha} \wedge \omega_{i_{2t+1}\beta} =$$

$$= \sum_{(i)} (A_{\alpha,i_1j}\omega^j) \wedge (A_{\beta,i_1k}\omega^k) \wedge \ldots \wedge (A_{\alpha,i_{2t+1}l}\omega^l) \wedge (A_{\beta,i_{2t+1}m}\omega^m) =$$

$$\sum_{\rho} \varepsilon(\rho) A_{\alpha,i_1\rho(1)} A_{\beta,i_1\rho(2)} \cdots A_{\alpha,i_{2t+1}\rho(4t+1)} A_{\beta,i_{2t+1}\rho(4t+2)} d\sigma_{4t+2} =$$

$$\sum_{\rho/\rho(2l-1)<\rho(2l)} \varepsilon(\rho)[A_{\alpha,i_1\rho(1)}A_{\beta,i_1\rho(2)} - A_{\alpha,i_1\rho(2)}A_{\beta,i_1\rho(1)}] \cdots$$

$$[A_{\alpha,i_{2t+1}\rho(4t+1)}A_{\beta,i_{2t+1}\rho(4t+2)} - A_{\alpha,i_{2t+1}\rho(4t+2)}A_{\beta,i_{2t+1}\rho(4t+1)}]d\sigma_{4t+2} =$$

$$= \sum \varepsilon(\rho) R_{\alpha\beta\rho(1)\rho(2)} \cdots R_{\alpha\beta\rho(4t+1)\rho(4t+2)} d\sigma_{4t+2} =$$

$$R_{\alpha\beta\ldots(4t+1)(4t+2)} d\sigma_{4t+2}. \tag{2.11}$$

From (2.10) and (2.11) we obtain (2.9).

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