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Link Homotopy Invariants of Graphs in R^3

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ABSTRACT. In this paper we define a link homotopy invariant of spatial graphs based on the second degree coefficient of the Conway polynomial of a knot.

1. INTRODUCTION

Throughout this paper we work in the piccewise linear category. Let G be a finite graph without loops and multiple edges. Then there are various embeddings of G into the three-dimensional Euclidean space R^3. Two embeddings $f, g : G \to R^3$ are said to be *link homotopic* if g is

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obtained from f by a finite sequence of self-crossing changes (Fig. 1.1) and ambient isotopy.

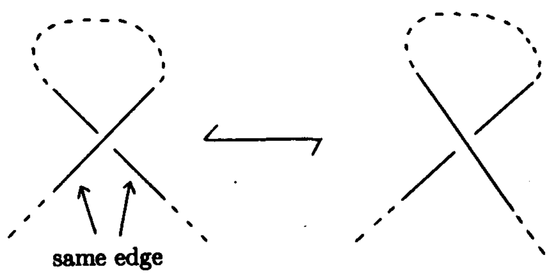


Fig. 1.1

Two edges of G are called *adjacent* if they have a vertex in common. Two embeddings $f, g : G \to R^3$ are called *weakly link homotopic* if g is obtained from f by a finite sequence of crossing changes of adjacent edges (Fig. 1.2) and ambient isotopy.

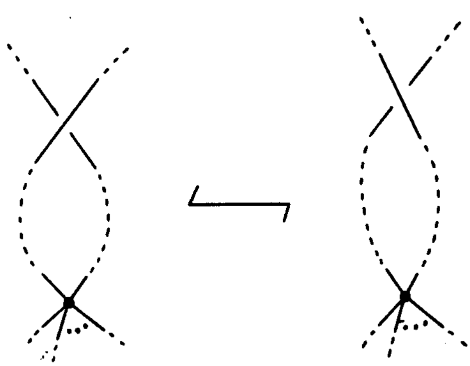


Fig. 1.2

We note that a self-crossing change is replaced by crossing changes of adjacent edges as illustrated in Fig. 1.3. Therefore link homotopy implies weak link homotopy.

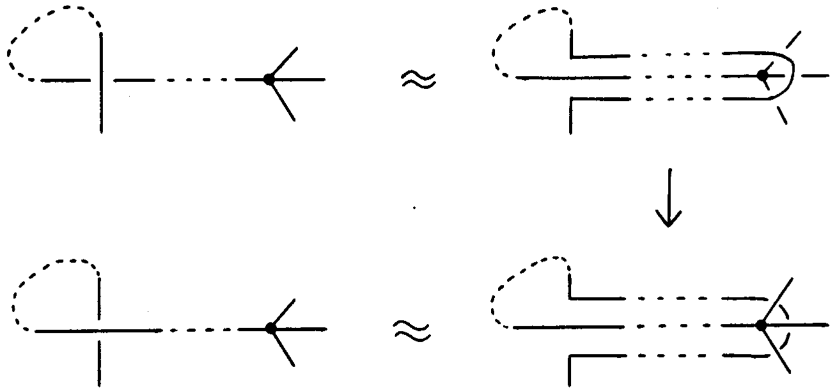


Fig. 1.3

An $n-$*cycle* is a graph with n vertices that is homeomorphic to a circle. When G is a disjoint union of cycles our link homotopy and weak link homotopy coincide with Milnor's link homotopy defined in [4].

The purpose of this paper is to define link homotopy invariants and weak link homotopy invariants for an arbitrary graph G. By the fundamental theorem in [7] we have that a link homotopy invariant is an I-equivalence invariant and hence an isotopy invariant and also a cobordism invariant. Conversely a homology invariant is a link homotopy invariant. Thus Wu's invariant (see [8]) is a weak link homotopy invariant and hence a link homotopy invariant. Except the case that G is a disjoint union of cycles, the author knows no other link homotopy invariants and weak link homotopy invariants.

A *cycle* of a graph G is a subgraph of G that is a cycle. Let $\Gamma = \Gamma(G)$ be the set of all cycles of G. Let Z be the integers. Let n be a non-negative integer. Let $Z_n = \{0,1,2,\ldots,n-1\}$ if $n > 0$. Let $Z_0 = Z$. Let $\omega : \Gamma \to Z_n$ be a map. We call ω a *weight* on Γ. For an embedding $f : G \to R^3$ we define $\alpha_\omega(f) \in Z_n$ by

$$\alpha_\omega(f) \equiv \sum_{\gamma\in\Gamma} \omega(\gamma) a_2(f(\gamma)) \pmod{n}$$

where $a_2(K)$ is the coefficient of z^2 in the Conway polynomial $\nabla_K(z)$ of a knot K. We will show that if a weight ω satisfies certain conditions then α_ω is a (weak) link homotopy invariant.

We remark here that the modulo 2 reduction of $a_2(K)$ equals the Arf invariant of K [3]. Therefore when G is the complete graph K_7, $n = 2$ and

$$\omega(\gamma) = \begin{cases} 1 & \text{if } \gamma \text{ is a 7-cycle} \\ 0 & \text{otherwise} \end{cases}$$

α_ω equals an invariant defined in [2]. In [2] Gordon proved that α_ω is invariant under any crossing change. He found a particular embedding $f: K_7 \to R^3$ such that $\alpha_\omega(f) \equiv 1 \pmod 2$. Therefore $\alpha_\omega(g) \equiv 1 \pmod 2$ for any embedding $g: K_7 \to R^3$. Since $a_2(\text{unknot}) = 0$ he could conclude that every spatial embedding of K_7 contains a nontrivially knotted 7-cycle.

For our purpose it is enough that α_ω is invariant under a self-crossing change or a crossing change of adjacent edges. In this sense the idea in [2] was a great hint of this paper. We also remark here that our definition of $\alpha_\omega(f)$ generalizes Shimabara's generalization of Gordon's invariant [6].

Let e be an edge of G. We give an arbitrary orientation to e. Let Γ_e be a subset of Γ defined by

$$\Gamma_e = \{\gamma \in \Gamma \mid \gamma \supset e\}.$$

We give an orientation to each $\gamma \in \Gamma_e$ by the orientation of e. We say that a weight $\omega: \Gamma \to Z_n$ is *balanced* on e if the homological sum $\Sigma_{\gamma \in \Gamma_e} \omega(\gamma)\gamma$ is zero in $H_1(G; Z_n)$. We remark that this property does not depend on the choice of the orientation of e.

Lemma 1.1. *Let $\omega: \Gamma(G) \to Z_n$ be a weight that is balanced on an edge e of G. If an embedding $g: G \to R^3$ is obtained from an embedding $f: G \to R^3$ by a self-crossing change of the edge e then*

$$\alpha_\omega(f) \equiv \alpha_\omega(g) \pmod n.$$

As an immediate corollary we have:

Theorem 1.2. *Let $\omega : \Gamma(G) \to Z_n$ be a weight that is balanced on each edge of G. Then α_ω is a link homotopy invariant. Namely if two embeddings $f, g : G \to R^3$ are link homotopic then*

$$\alpha_\omega(f) \equiv \alpha_\omega(g) \pmod{n}.$$

Let e_1 and e_2 be adjacent edges of G. We give an arbitrary orientation to e_1. Let Γ_{e_1,e_2} be a subset of Γ defined by

$$\Gamma_{e_1,e_2} = \{\gamma \in \Gamma | \gamma \supset e_1, e_2\}.$$

We give an orientation to each $\gamma \in \Gamma_{e_1,e_2}$ by the orientation of e_1. We say that a weight $\omega : \Gamma \to Z_n$ is *balanced* on a pair of adjacent edges (e_1, e_2) if the homological sum $\Sigma_{\gamma \in \Gamma_{e_1,e_2}} \omega(\gamma)\gamma$ is zero in $H_1(G; Z_n)$.

Lemma 1.3. *Let $\omega : \Gamma(G) \to Z_n$ be a weight that is balanced on a pair of adjacent edges (e_1, e_2) of G. If an embedding $g : G \to R^3$ is obtained from an embedding $f : G \to R^3$ by a crossing change between e_1 and e_2 then*

$$\alpha_\omega(f) \equiv \alpha_\omega(g) \pmod{n}.$$

As an immediate corollary we have:

Theorem 1.4. *Let $\omega : \Gamma(G) \to Z_n$ be a weight that is balanced on each pair of adjacent edges of G. Then α_ω is a weak link homotopy invariant. Namely if two embeddings $f, g : G \to R^3$ are weakly link homotopic then*

$$\alpha_\omega(f) \equiv \alpha_\omega(g) \pmod{n}.$$

This paper is organized as follows. In §2 we prove Lemma 1.1 and Lemma 1.3. In §3 we show some examples. In §4 we show that Milnor's μ-invariant for 3-component homologically unlinked links can be re-defined via a weak link homotopy invariant of a certain graph.

2. PROOFS OF LEMMA 1.1 AND LEMMA 1.3

Proof of Lemma 1.1. We recall the equality

$$(*) \quad a_2(K_+) - a_2(K_-) = \ell k(L_0)$$

where K_+, K_- and L_0 are knots and a two-component link as illustrated in Fig. 2.1 and ℓk denotes the linking number [3].

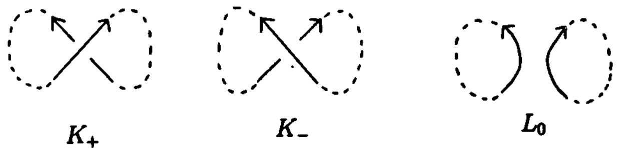


Fig. 2.1

Let γ be a cycle in Γ_e. We recall that γ is oriented by the orientation of e. We may suppose without loss of generally that $f(\gamma)$ and $g(\gamma)$ are related as illustrated in Fig. 2.2 (a) and (b). Let $L_{f,g}(\gamma) = \ell_{f,g}(\gamma) \cup m_{f,g}(\gamma)$ be the 2-component link as illutrated in Fig. 2.2 (c).

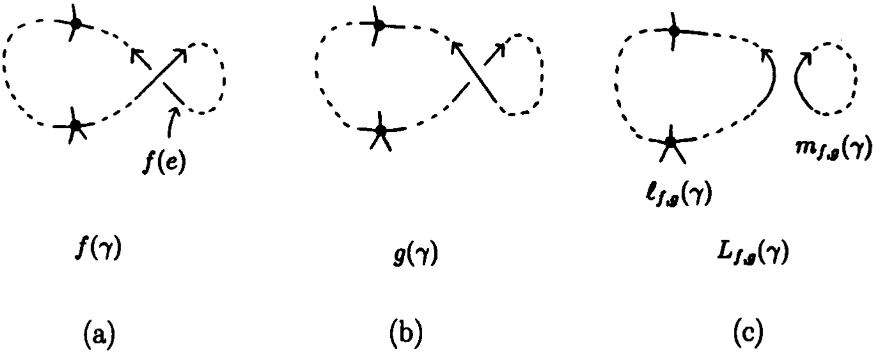


Fig 2.2

Then we have

$$\alpha_\omega(f) - \alpha_\omega(g) \equiv \sum_{\gamma\in\Gamma} \omega(\gamma)a_2(f(\gamma)) - \sum_{\gamma\in\Gamma} \omega(\gamma)a_2(g(\gamma))$$

$$\equiv \sum_{\gamma\in\Gamma} \omega(\gamma)(a_2(f(\gamma)) - a_2(g(\gamma))) \equiv \sum_{\gamma\in\Gamma_e} \omega(\gamma)(a_2(f(\gamma)) - a_2(g(\gamma)))$$

$$\equiv \sum_{\gamma\in\Gamma_e} \omega(\gamma)\ell k(\ell_{f,g}(\gamma), m_{f,g}(\gamma)) \pmod{n}.$$

Since $m_{f,g}(\gamma) = m_{f,g}(\gamma')$ for any $\gamma, \gamma' \in \Gamma_e$ we may write $m_{f,g}(\gamma)$ as $m_{f,g}$. Since linking number is a homological invariant we have

$$\sum_{\gamma\in\Gamma_e} \omega(\gamma)\ell k(\ell_{f,g}(\gamma), m_{f,g}) \equiv \sum_{\gamma\in\Gamma_e} \ell k(\omega(\gamma)\ell_{f,g}(\gamma), m_{f,g})$$

$$\equiv \ell k\Big(\sum_{\gamma\in\Gamma_e} \omega(\gamma)\ell_{f,g}(\gamma), m_{f,g}\Big) \pmod{n}.$$

Since ω is balanced on e we have that the homological sum

$$\sum_{\gamma\in\Gamma_e} \omega(\gamma)\ell_{f,g}(\gamma) \equiv 0 \pmod{n}.$$

Therefore we have

$$\ell k\Big(\sum_{\gamma\in\Gamma_e} \omega(\gamma)\ell_{f,g}(\gamma), m_{f,g}\Big) \equiv \ell k(0, m_{f,g}) \equiv 0 \pmod{n}.$$

This completes the proof. ■

Proof of Lemma 1.3. The proof is similar to that of Lemma 1.1. We note that one of the two components of the smoothed link is common for all $\gamma \in \Gamma_{e_1,e_2}$ as in the case of Lemma 1.1, see Fig. 2.3.

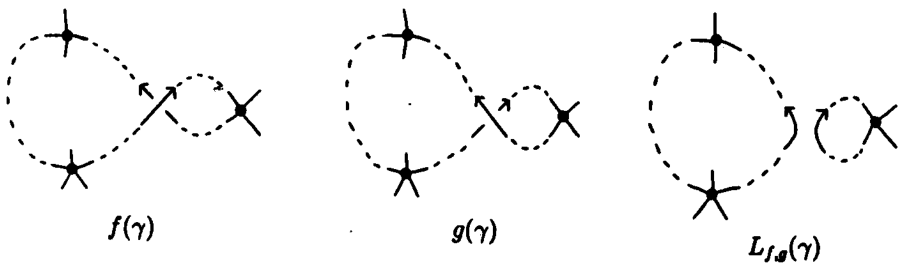


Fig. 2.3

Therefore the same proof works. ■

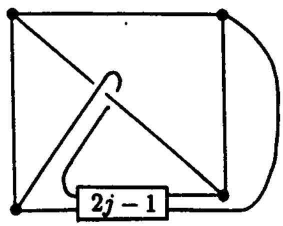
3. EXAMPLES

Example 3.1. Let G be the complete graph K_4. Let $n = 0$ and let $\omega: \Gamma(K_4) \to Z$ be a weight defined by

$$\omega(\gamma) = \begin{cases} 1 & \text{if } \gamma \text{ is a 4-cycle} \\ -1 & \text{if } \gamma \text{ is a 3-cycle.} \end{cases}$$

Then it is easily checked that ω is balanced on each edge of K_4. Therefore α_ω is a link homotopy invariant.

Let j be an integer and let $f_j : K_4 \to R^3$ be an embedding illustrated by Fig. 3.1 where the box denotes $2j-1$ right-handed half twists when $j > 0$, $-2j+1$ left-handed half twists when $j \leq 0$.



$f_j(K_4)$

Fig. 3.1

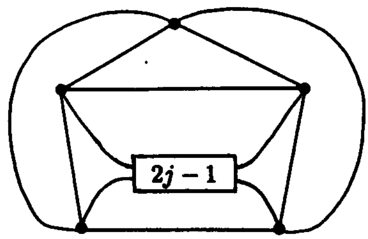
Then $f_j(K_4)$ contains at most one nontrivial knot. The knot is a twisted knot. Since a twisted knot has unknotting number one a_2 is easily calculated by the equality $(*)$. Then we have $\alpha_\omega(f_j) = j$.

Example 3.2. Let $G = K_5$, $n = 0$ and $\omega : \Gamma(K_5) \to Z$ a weight defined by

$$\omega(\gamma) = \begin{cases} 1 & \text{if } \gamma \text{ is a } 5\text{-cycle} \\ -1 & \text{if } \gamma \text{ is a } 4\text{-cycle} \\ 0 & \text{if } \gamma \text{ is a } 3\text{-cycle} \end{cases}$$

Then it is easily checked that ω is balanced on each pair of adjacent edges of K_5. Thus α_ω is a weak link homotopy invariant.

Let j be an integer. Let $f_j:\ K_5 \to R^3$ be an embedding illustrated by Fig. 3.2.



$f_j(K_5)$

Fig. 3.2

Then at most two 5-cycles and a 4-cycle can be nontrivial knots. They are all the $(2,2j-1)$-torus knot. From the equality $(*)$ we have that $a_2((2,2j-1)\text{-torus knot}) = \frac{j(j-1)}{2}$. Therefore we have that $\alpha_\omega(f_j) = \frac{j(j-1)}{2}$.

It is known in [8] that $\{f_j \mid j \in Z\}$ is a complete list of the homology classes of embeddings of K_5 into R^3. In [5] we will show that homology implies weak link homotopy when $G = K_5$. Therefore $\{f_j \mid j \in Z\}$ is also a complete list of weak link homotopy classes. Thus α_ω classifies the embeddings of K_5 into R^3 up to weak link homotopy and mirror image.

3. 3-COMPONENT HOMOLOGICALLY UNLINKED LINKS

Let G be the graph of Fig. 4.1.

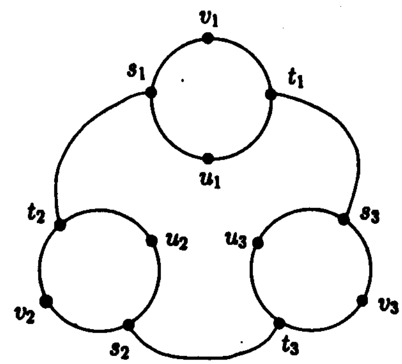


Fig. 4.1

Let $n = 0$ and let $\omega : \Gamma(G) \to Z$ be a weight defined by

$$\omega(\gamma) = \begin{cases} 1 & \text{if } \gamma \text{ is a 9-cycle that contains zero or two of } v_1,\ v_2 \text{ and } v_3 \\ -1 & \text{if } \gamma \text{ is a 9-cycle that contains one or three of } v_1,\ v_2 \text{ and } v_3 \\ 0 & \text{if } \gamma \text{ is a 4-cycle.} \end{cases}$$

Then ω is balanced on each pair of adjacent edges of G. Thus α_ω is a weak link homotopy invariant.

A 3-component ordered oriented link $L = \ell_1 \cup \ell_2 \cup \ell_3$ is called *homologically unlinked* if $\ell k(\ell_1, \ell_2) = \ell k(\ell_2, \ell_3) = \ell k(\ell_3, \ell_1) = 0$. Let H be the subgraph of G that is the disjoint union of three 4-cycles of

G. Let $f: H \to R^3$ be an embedding. Let $\ell_i(f) = f(v_i s_i u_i t_i v_i)$ $(i = 1,2,3)$. Then $L(f) = \ell_1(f) \cup \ell_2(f) \cup \ell_3(f)$ is a 3-component ordered oriented link.

Theorem 4.1. *Let $f, g : G \to R^3$ be embeddings such that both $L(f|_H)$ and $L(g|_H)$ are homologically unlinked. Then f and g are weakly link homotopic if and only if $f|_H$ and $g|_H$ are weakly link homotopic.*

Proof. The 'only if' part is clear. We show 'if' part. Suppose that $f|_H$ is weakly link homotopic to $g|_H$. Then f is weakly link homotopic to an embedding, still denoted by f, so that $f|_H = g|_H$. It is sufficient to show that a crossing change between the edge $s_i t_{i+1}$ and an edge of G is realized by a weak link homotopy (here we consider the suffix modulo 3). By replacing a crossing change by some crossing changes as in Fig. 1.3 we have that a crossing change between $s_i t_{i+1}$ and an edge that is not on the cycle $v_{i+2} s_{i+2} u_{i+2} t_{i+2} v_{i+2}$ is realized by some crossing changes of adjacent edges. Then by the symmetry of G is sufficient to show that a crossing between $s_1 t_2$ and $v_3 s_3$ is realized by a weak link homotopy. We choose a small ball B^3 near the crossing where the crossing change is desired, see Fig. 4.2.

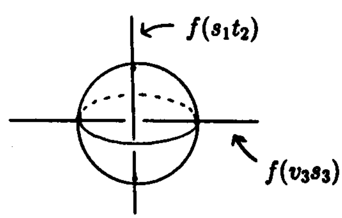


Fig. 4.2

Step 1. By a weak link homotopy outside of B^3 we deform f so that $\ell_1(f) \cup \ell_2(f)$ is a trivial 2-component link.

Step 2. We choose a disk D^2 in general position so that $\partial D^2 = \ell_1(f), D^2 \cap \ell_2(f) = \emptyset$ and $D^2 \cap B^3 = \emptyset$.

Step 3. We remove the intersection if any of D^2 and $f(s_1, t_2)$ by a weak link homotopy outside of B^3.

Step 4. We perform the crossing change by a weak link homotopy as illustrated in Fig. 4.3.

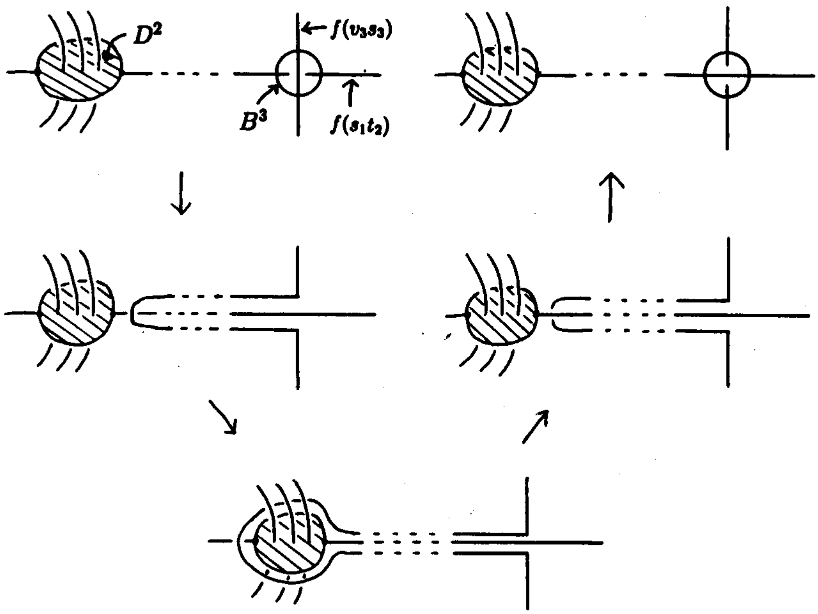


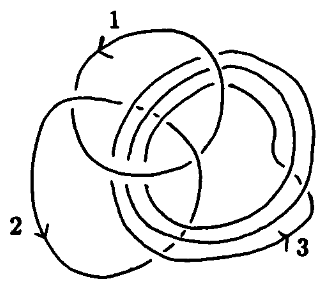
Fig. 4.3

Step 5. We re-fix the 3-ball B^3 and retrace one's steps from Step 3 to Step 1.

Thus we have the desired crossing change. ■

Let L be a homologically unlinked 3-component ordered oriented link. Let $f : G \to R^3$ be an embedding such that $L(f|_H) = L$. Then by Theorem 4.1 $\alpha_\omega(f)$ is a well-defined weak link homotopy invariant of L. Since weak link homotopy equals link homotopy for links $\alpha_\omega(f)$

is a link homotopy invariant of L. It is known in [4] that 3-component homologically unlinked links are classified up to link homotopy by Milnor's μ-invariant. Let j be an integer and let L_j be a link illustrated in Fig. 4.4.

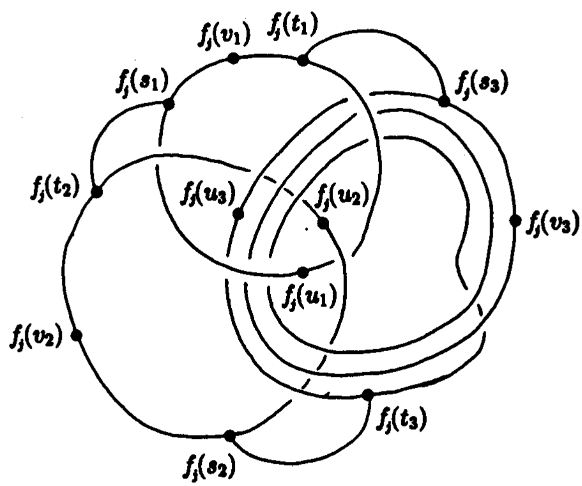


L_j $(j = 3)$

Fig. 4.4

Then $\mu(L_j) = j$ and $\{L_j | j \in Z\}$ is the complete list of link homotopy classes [4]. Let $f_j : G \to R^3$ be an embedding illustrated in

Fig. 4.5.



$f_j(G)$ $(j = 3)$

Fig. 4.5

Then $L(f_j) = L_j$. It is easy to check that $f_j(G)$ contains at most two nontrivial knots that are twisted knots. Then we have $\alpha_\omega(f_j) = j$. Thus Milnor's μ-invariant is re-defined, cf. [1].

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