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Codimension Reduction for Real Submanifolds of a Complex Hyperbolic Space

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ABSTRACT. We study real submanifolds of a complex hyperbolic space and prove a codimension reduction theorem.

0. INTRODUCTION.

Recently Okumura ([3]) defined holomorphic first normal space for real submanifolds of a Kaehler manifold and proved a codimension reduction theorem for real submanifolds of a complex projective space. Namely, he showed following:

Theorem. *Let M be a connected n-dimensional real submanifold of a real $(n+p)$-dimensional complex projective space $\mathbf{C}P^{(n+p)/2}$ and let $N_0(x)$ be the orthogonal complement of first normal space in $T_x^{\perp}(M)$. We put $H_0(x) = JN_0(x) \cap N_0(x)$ and let $H(x)$ be a J-invariant subspace of $H_0(x)$ where J is complex structure of $\mathbf{C}P^{(n+p)/2}$. If the orthogonal complement $H_2(x)$ of $H(x)$ in $T_x^{\perp}(M)$ is invariant under parallel translation with respect to the normal connection and if q is the constant*

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dimension of $H_2(x)$, then there exists a real $(n+q)$-dimensional totally geodesic complex projective subspace $\mathbf{C}P^{(n+q)/2}$ in $\mathbf{C}P^{(n+p)/2}$ such that $M \subset \mathbf{C}P^{(n+q)/2}$.

The purpose of this paper is to prove that the similar result to the above theorem is still hold in a submanifold of complex hyperbolic space.

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1. CODIMENSION REDUCTION FOR SUBMANIFOLDS OF ANTI-DE SITTER SPACE.

Let $\mathbf{R}_2^{n+1}$ be a real vector space of $(n+1)$ dimension with a pseudo-Riemannian metric $\bar{g}$ of signature $(n-1,2)$ given by

$$\bar{g}(x,y) \;=\; -x_0y_0 - x_1y_1 + \sum_{i=2}^{n} x_iy_i \tag{1.1}$$

where $x = {}^t(x_0, x_1, \ldots, x_n)$, $y = {}^t(y_0, y_1, \ldots, y_n) \in \mathbf{R}^{n+1}$. Let $H_1^n = \{x \in \mathbf{R}_2^{n+1} \mid g(x,x) = -1\}$. Then the hypersurface H_1^n is a Lorentzian manifold with the induced Lorentzian metric $\tilde{g}$ of constant sectional curvature -1. We call it n-dimensional anti-De Sitter space.

Let H_1^{n+p} be an $(n+p)$-dimensional anti-De Sitter space and let $i: M \to H_1^{n+p}$ be an isometric immersion of a connected n-dimensional Lorentzian manifold with the Lorentzian metric g into H_1^{n+p}. Then the tangent bundle $T(M)$ is identified with a subbundle of $T(H_1^{n+p})$ and the normal bundle $T^{\perp}(M)$ is a subbundle of $T(H_1^{n+p})$ consisting of all element in $T(H_1^{n+p})$ which are orthogonal to $T(M)$ with respect to $\tilde{g}$. We denote by ∇ and $\tilde{\nabla}$ the Levi-Civita connection of M and H_1^{n+p} respectively and D the induced normal connection from $\tilde{\nabla}$ to $T^{\perp}(M)$. Then they are related by the following Gauss and Weingarten formulae:

$$\tilde{\nabla}_{iX} iY \;=\; i\nabla_X Y + h(X,Y) \tag{1.2}$$

$$\tilde{\nabla}_{iX}\xi = -iA_\xi X + D_X\xi \tag{1.3}$$

where $\xi \in T^{\perp}(M)$, $h(X,Y)$ is the second fundamental form and A_ξ is a symmetric linear transformation of $T(M)$ which is called the shape operator with respect to ξ. They satisfy

$$\tilde{g}(h(X,Y),\xi) = g(A_\xi X, Y). \tag{1.4}$$

Next let $N_0(x) = \{\xi \in T_x^{\perp}(M) \mid A_\xi = 0\}$. The first normal space $N_1(x)$ is defined to be the orthogonal complement of $N_0(x)$ in $T_x^{\perp}(M)$.

Theorem 1.1. *Let $i : M \to H_1^{n+p}$ be as above. Let $N_2(x)$ be a subspace of $T_x^{\perp}(M)$ such that $N_1(x) \subset N_2(x)$. If $N_2(x)$ is invariant under parallel translation with respect to the normal connection and if q is the constant dimension of $N_2(x)$, then there exists a totally geodesic anti-De Sitter subspace H_1^{n+q} of H_1^{n+p} such that $i(M) \subset H^{n+q}$.*

Proof. We consider H_1^{n+p} as a hypersurface of $\mathbf{R}_2^{n+p+1}$. Let $x \in M$ and let $\xi = \overrightarrow{i(x)}$ be the position vector. Then $\xi(x)$ is normal to H_1^{n+p} and $\bar{g}(\xi(x),\xi(x)) = -1$ where $\bar{g}$ is the metric of R_2^{n+p+1}. Let $\bar{\nabla}$ be the Levi-Civita connection on $\mathbf{R}_2^{n+p+1}$ with respect to $\bar{g}$ and φ be an immersion from H_1^{n+p} to R_2^{n+p+1}. Then

$$\nabla_{\varphi X}\xi = \varphi X \tag{1.5}$$
$$\bar{\nabla}_{\varphi X}\varphi Y = \varphi\tilde{\nabla}_X Y - \tilde{g}(X,Y)\xi \tag{1.6}$$

where $X, Y \in T_x(H_1^{n+p})$. For $x \in M$ let $P(x) = T_x(M) + N_2(x)$. For any $x \in M$ there exist orthonormal normal vector fields $\xi_1, \ldots, \xi_p$ defined in a neighborhood U of x such that:

(a) For any $y \in U$, $\xi_1(y), \ldots, \xi_q(y)$ span $N_2(y)$, and $\xi_{q+1}(y), \ldots, \xi_p(y)$ span $N(y)$ where $N(x)$ is the orthogonal complement of $N_2(x)$ in $T^{\perp}(M)$.

(b) $\tilde{\nabla}_{iX}\xi_\alpha = 0$ in U for $\alpha \geq q+1$ and X tangent to M.

(c) $\{P(y) \mid y \in U\}$ is invariant under parallel translation with respect to the connection $\tilde{\nabla}$ along any curve in U (see [1]). Then $\bar{\nabla}_{\varphi(iX)}\varphi\xi_\alpha = \tilde{\nabla}_{iX}\xi_\alpha$ for X tangent to M. Let D' be the normal connection in the normal bundle $T^{\perp}(M)$ of M in $\mathbf{R}_2^{n+p+1}$. Then

$N_2(x) + \text{span}\{\xi(x)\}$ is invariant under parallel translation with respect to D'. Further,

$$W(x) = T_x(M) + N_2(x) + \text{span}\{\xi(x)\} \tag{1.7}$$

is invariant under parallel translation with respect to $\bar{\nabla}$. Next we shall show that there exists a totally geodesic submanifold H_1^{n+q} of H_1^{n+p} such that $i(M) \subset H_1^{n+q}$. Define functions f_α on U by $f_\alpha = \bar{g}(\overrightarrow{i(x)}, \varphi\xi_\alpha)$ for $\alpha \geq q+1$.

$$\varphi(iX) \cdot f_\alpha = \bar{g}(\bar{\nabla}_{\varphi(iX)} \overrightarrow{i(x)}, \varphi\xi_\alpha) + \bar{g}(\overrightarrow{i(x)}, \varphi\xi_\alpha), \bar{\nabla}_{\varphi(iX)}\varphi\xi_\alpha) = 0$$

Thus $f_{q+1}, \ldots, f_p$ are constant. Put

$$f_\alpha = C_\alpha (= \text{constant}) \ (\alpha \geq q+1). \tag{1.8}$$

And put $\overrightarrow{i(x)} = (x_0, \ldots, x_{n+p})$ and $\varphi\xi_\alpha = (\xi_\alpha^0, \ldots, \xi_\alpha^{n+p})$. Then (1.6) can be written

$$\begin{cases} -\xi_{q+1}^0 x_0 - \xi_{q+1}^1 x_1 + \sum_{i=1}^{n+p} \xi_{q+1}^i x_i = C_{q+1}, \\ \vdots \\ -\xi_p^0 x_0 - \xi_p^1 x_1 + \sum_{i=2}^{n+p} \xi_p^i x_i = C_p. \end{cases} \tag{1.9}$$

Since $\xi_{q+1}, \ldots, \xi_p$ are linearly independent, U lies in the intersection of $p-q$ hyperplanes and the dimension of the hyperplane is $n+q+1$. As the normal vectors of the intersection W' are $\xi_{q+1}, \ldots, \xi_p$, they span $N(x)$. Since W' is affine space, W' is the orthogonal complement of $N(x)$ in $T_x(\mathbf{R}_2^{n+p+1})$. On the other hand, the orthogonal complement of $N(x)$ in $T_x(\mathbf{R}_2^{n+p+1})$ is $T_x(M) + N_2(x) + \text{span}\{\xi(x)\}$ $(= W(x))$. Therefore $W' = W$. We may assume that the point $(1, 0, \ldots, 0)$ is in U. $W(x)$ contains ξ, and if $\xi = (1, 0, \ldots, 0)$, then $W(x)$ passes through the origin of $\mathbf{R}_2^{n+p+1}$. Thus $W(x) = \mathbf{R}^{n+q+1}$. Moreover since M is Lorentzian submanifold and ξ is the position vector, the signature of the induced metric of

$\mathbf{R}^{n+q+1}$ is $(n+q-1,2)$. Then $W' = \mathbf{R}_2^{n+q+1}$. Thus $H_1^{n+p} \cap \mathbf{R}_2^{n+q+1}$ is totally geodesic H_1^{n+p}, that is,

$$i(U) \subset H_1^{n+q} = H_1^{n+p} \cap \mathbf{R}_2^{n+q+1}. \tag{1.10}$$

Hence Theorem 1.1. is true locally. In entirely the same way as in [1], we can get the global result. This completes the proof.

2. REAL SUBMANIFOLDS OF A KAEHLER MANIFOLD AND HOLOMORPHIC FIRST NORMAL SPACE.

Let $\bar{M}$ be a real $(n+p)$-dimensional Kaehler manifold with Kaehler structure $(J, <,>)$, that is, J is the endomorphism of the tangent bundle $T(\bar{M})$ satisfying J^2 = -identity and $<,>$ the Riemannian metric of $\bar{M}$ satisfying the Hermitian condition $< J\bar{X}, J\bar{Y} > = < \bar{X}, \bar{Y} >$ for any $\bar{X}, \bar{Y} \in T(\bar{M})$.

Let M be a connected n-dimensional submanifold and let i be the isometric immersion. For any $X \in T(M)$ the transform JiX is written as a sum of its tangential parts iFX and the normal parts $u(X)$ in the following way:

$$JiX = iFX + u(X) \tag{2.1}$$

Then F is an endmorphism on the tangent bundle $T(M)$ and u is a normal valued 1-form on the tangent bundle. In the same way, for any $\xi \in T^{\perp}(M)$, the transform $J\xi$ is written as

$$J\xi = -iU_{\xi} + P\xi, \tag{2.2}$$

where P defines an endomorphism on the normal bundle $T^{\perp}(M)$. It is easily verified that

$$g(X, U_{\xi}) = < u(X), \xi >, \tag{2.3}$$

where g is the Riemannian metric which is induced from the Riemannian metric $<,>$.

We define the holomorphic first normal space. We put $H_0(x) = JN_0(x) \cap N_0(x)$. Then $H_0(x)$ is the maximal J-invariant subspace of

$N_0(x)$. Since J is isomorphism, we see that $JH_0(x) = H_0(x)$. Making use of (2.2), we can easily prove the following

Proposition 2.1. ([3]) *For any $\xi \in H_0(x)$, we have $A_\xi = 0$ and $U_\xi = 0$.*

Definition ([3]) *The holomorphic first normal space $H_1(x)$ is the orthogonal complement of $H_0(x)$ in $T_x^\perp(M)$.*

Proposition 2.2. ([3]) *If M is a complex submanifold of a Kaehler manifold, then $H_1(x) = N_1(x)$.*

Proposition 2.3. ([3]) *Let $H(x)$ be a J-invariant subspace of $H_0(x)$ and let $H_2(x)$ be the orthogonal complement of $H(x)$ in $T_x^\perp(M)$. Then $T_x(M) + H_2(x)$ is a J-invariant subspace of $T_x(\bar{M})$.*

3. CODIMENSION REDUCTION FOR SUBMANIFOLDS OF COMPLEX HYPERBOLIC SPACE.

In this section, we consider the case that the ambient manifold $\bar{M}$ is a complex hyperbolic space $CH^{(n+p)/2}$ with the Bergmann metric of constant holomorphic sectional curvature -4. Given a real n-dimensional submanifold M of $CH^{(n+p)/2}$, one can construct a Lorentzian submanifold M' with time like totally geodesic fibres and projection $\pi' : M' \to M$ such that the diagram ([2])

$$\begin{array}{ccc} M' & \xrightarrow{\tilde{i}} & H_1^{n+p+1} \\ \pi' \downarrow & & \downarrow \pi \\ M & \xrightarrow[i]{} & CH^{(n+p)/2} \end{array}$$

is commutative ($\tilde{i}$ being the isometric immersion). Let V' be the unit vector field tangent to the fibre of M'. Then $\tilde{i}V'$ is the unit vector field tangent to the fibre of H_1^{n+p+1}. We denote by g' and ∇' the Lorentzian metric and the Levi-Civita connection of M' respectively. Also we denote by F and X^* the fundamental tensor of the submersion

π' and the horizontal lift for $X \in T(M)$ respectively. In the same way, ξ^* is the horizontal lift of the normal field $\xi \in T^{\perp}(M)$. The fundamental equations for the submersion π' are given as following ([4]):

$$\nabla'_X * Y^* = (\nabla_X Y)^* + g'((FX)^*, Y^*)V', \tag{3.1}$$
$$\nabla'_X * V' = \nabla'_{V'} X^* = (FX)^*, \tag{3.2}$$

where ∇ is the Levi-Civita connection of M. The similar equations are valid for the submersion $\pi : H_1^{n+p+1} \to CH^{(n+p)/2}$ when we replace F and V' with J and $\tilde{i}V'$ respectively. Let $\tilde{g}$, $\tilde{\nabla}$, A' and D' be respectively the Lorentzian metric of H_1^{n+p+1}, the Levi-Civita connection for $\tilde{g}$, the shape operator and the normal connection of M', and let A and D be the shape operator and the normal connection of M respectively. Then ([3]) we have

$$A'_\xi * X^* = (A_\xi * X)^* - g(U_\xi, X)^* V', \tag{3.3}$$
$$D'_X * \xi' = (D_X \xi)^*, \tag{3.4}$$
$$A'_\xi * V' = U^*_{\xi'}, \tag{3.5}$$
$$D'_{V'} \xi^* = (P\xi)^*. \tag{3.6}$$

In fact, from the commutativity of the diagram, (2.3) and (3.1) imply

$$\begin{aligned} \tilde{\nabla}_{(iX)} * \xi^* &= (\tilde{\nabla}_{iX}\xi)^* + g((JiX)^*, \xi^*)\tilde{i}V' \\ &= -(iA_\xi X)^* + (D_X\xi)^* + < JiX, \xi >^* \tilde{i}V' \\ &= -\tilde{i}(A_\xi X)^* + < u(X), \xi >^* \tilde{i}V' + (D_X\xi)^* \\ &= -\tilde{i}\{(A_\xi X)^* - g(U_\xi, X)^* V'\} + (D_X\xi)^* \quad . \end{aligned} \tag{3.7}$$

On the other hand, by the Weingarten formula, we get

$$\tilde{\nabla}_{(iX)} * \xi^* = -\tilde{i}A_\xi * X^* + D'_X * \xi^*. \tag{3.8}$$

Comparing (3.7) and (3.8), we have (3.3) and (3.4).

Lemma 3.1. ([3]) *For a point* x' *such that* $\pi(x') = x$, *we have* $N_0'(x') = \{\xi^* \mid A_\xi = 0,\ U_\xi = 0\}$.

Theorem 3.2. *Let* $i: M \to \mathbf{C}H^{(n+p)/2}$ *be an isometric immersion of a connected* n*-dimensional real submanifold into a real* $(n+p)$*-dimensional complex hyperbolic space* $\mathbf{C}H^{(n+p)/2}$ *and let* $H(x)$ *be a* J*-invariant subspace of* $H_0(x)$. *If the orthogonal complement* $H_2(x)$ *of* $H(x)$ *in* $T_x^\perp(M)$ *is invariant under parallel translation with respect to the normal connection and if the dimension* q *of* $H_2(x)$ *is constant, then there exists a real* $(n+q)$*-dimensional totally geodesic complex hyperbolic subspace* $\mathbf{C}H^{(n+q)/2}$ *in* $\mathbf{C}H^{(n+p)/2}$ *such that* $i(M) \subset \mathbf{C}H^{(n+q)/2}$.

Proof. We construct the principal circle bundle M' over M with time like totally geodesic fibre S^1. We shall show that $H_2(x)^*$ is invariant under parallel translation with respect to the normal connection. Assume $\xi \in H(x)$. Then $\xi \in H_0(x)$ and by Proposition 2.1., we have

$$A_\xi = 0 \text{ and } U_\xi = 0. \tag{3.9}$$

From Lemma 3.1., this yields

$$A'_{\xi^*} = 0. \tag{3.10}$$

This shows that, for a point x' such that $\pi(x') = x$, $H(x)^* = \{\xi^* \mid \xi^* \in H(x)\}$ is a subspace of $N_0'(x')$. Hence, the orthogonal complement $H_2(x)^* = \{\xi^* \mid \xi \in H_2(x)\}$ of $H(x)^*$ in $T_{x'}^\perp(M')$ is a subspace of $T_{x'}^\perp(M')$ such that $N_1'(x') \subset H_2(x)^*$. Since $H_2(x)$ is invariant under parallel translation with repect to the normal connection, so is $H(x)$, that is, for $\xi \in H(x)$, $D_X\xi \in H(x)$, hence, from (3.4) and (3.5), we have $D'_X * \xi^* = (D_X\xi)^* \in H(x)^*$ and $D'_{V'}\xi^* = (P\xi)^* \in H(x)^*$. Since $H(x)^*$ is invariant under translation with respect to the normal connection of M', so is $H_2(x)^*$. Here from Theorem 1.1., there exists a totally geodesic submanifold H_1^{n+q+1} such that $\tilde{i}(M') \subset H_1^{n+q+1}$. Let $U(x')$ be a neighborhood of x' which satisfies $\pi(x') = x$. For $y' \in U(x')$ with $y = \pi'(y')$, we get

$$\begin{aligned} T_{y'}(H_1^{n+q+1}) &= T_{y'}(M') + H_2(y)^* \\ &= (T_y(M) + H_2(y))^* + \mathrm{span}\{V'\} \end{aligned} \tag{3.11}$$

The integral curve S^1 of $\tilde{i}V$ is time like totally geodesic fibre in H_1^{n+q+1}. Since H_1^{n+q+1} is totally geodesic in H_1^{n+p+1}, the integral curve S^1 is a geodesic of H_1^{n+q+1}. We denote by $\mathbf{C}H^{(n+q)/2}$ the quotient space H_1^{n+q+1}/S^1. Then the Hopf fibration $H_1^{n+q+1} \to \mathbf{C}H^{(n+q)/2}$ by the geodesic S^1 is compatible with the Hopf fibration $\pi : H_1^{n+p+1} \to \mathbf{C}H^{(n+p)/2}$ and since H_1^{n+q+1} is totally geodesic in H_1^{n+p+1}, $\mathbf{C}H^{(n+q)/2}$ is totally geodesic in $\mathbf{C}H^{(n+p)/2}$. Hence the diagram

$$\begin{array}{ccc} H_1^{n+q+1} & \longrightarrow & H_1^{n+p+1} \\ \downarrow & & \downarrow \\ \mathbf{C}H^{(n+q)/2} & \longrightarrow & \mathbf{C}H^{(n+p)/2} \end{array}$$

is commutative. Since the tangent space of the $\mathbf{C}H^{(n+q)/2}$ at x is $T_x(M) + H_2(x)$, by Proposition 2.3., $\mathbf{C}H^{(n+q)/2}$ is J-invariant subspace of $\mathbf{C}H^{(n+p)/2}$. This completes the proof.

For a complex submanifold M, from Proposition 2.2. and Theorem 3.2., we have

Corollary. *Let M be an $n/2$-dimensional complex submanifold of $\mathbf{C}H^{(n+p)/2}$. Suppose a J-invariant subspace of the first normal space $N_1(x)$ has constant dimension q and $N_1(x)$ is parallel with respect to the normal connection. Then there exists a totally geodesic $(n+q)$-dimensional complex hyperbolic subspace $\mathbf{C}H^{(n+q)/2}$ such that $M \subset \mathbf{C}H^{(n+q)/2}$.*

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