

# *A Generalized Mixed Topology on Orlicz Spaces*

MARIAN NOWAK

**ABSTRACT.** Let  $L^\varphi$  be an Orlicz space defined by an arbitrary Orlicz function  $\varphi$  over a positive measure space  $(\Omega, \Sigma, \mu)$  and provided with its usual  $F$ -norm  $\|\cdot\|_\varphi$ . In  $L^\varphi$  a natural convergence can be defined as follows: a sequence  $(x_n)$  in  $L^\varphi$  is said to be  $\gamma_\varphi$ -convergent to  $x \in L^\varphi$  whenever  $x_n \rightarrow x$  ( $\mu - \Omega$ ) and  $\sup \|x_n\|_\varphi < \infty$ . In this paper we examine some kind of generalized inductive-limit topology (in the sense of Turpin)  $\mathcal{J}_I^\varphi$  in  $L^\varphi$  that generates our  $\gamma_\varphi$ -convergence in  $L^\varphi$ . The main aim of the paper is to obtain a description of the topology  $\mathcal{J}_I^\varphi$  in terms of some family of  $F$ -norms defined by other Orlicz functions. As an application we obtain a topological characterization of the  $\gamma_\varphi$ -convergence in  $L^\varphi$ .

## 1. INTRODUCTION AND PRELIMINARIES

Every Orlicz space  $L^\varphi$  defined by an Orlicz function  $\varphi$  (not necessarily convex) over a measure space  $(\Omega, \Sigma, \mu)$  can be equipped with two

$F$ -norms:  $\|\cdot\|_\varphi$ - the usual  $F$ -norm on  $L^\varphi$  and  $\|\cdot\|_\mu$ - the  $F$ -norm of convergence in measure ( on  $\Omega$  ) restricted to  $L^\varphi$ . Thus a natural sequential convergence in  $L^\varphi$  can be defined as follows: a sequence  $(x_n)$  in  $L^\varphi$  is said to be  $\gamma_\varphi$ -convergent to  $x \in L^\varphi$ , in symbols  $x_n \xrightarrow{\gamma_\varphi} x$ , whenever

$$x_n \rightarrow x(\mu - \Omega) \text{ (i.e., } \|x_n - x\|_\mu \rightarrow 0 \text{ ) and } \sup_n \|x_n\|_\varphi < \infty.$$

When we replace in the above definition the condition:  $\sup \|x_n\|_\varphi < \infty$  with the boundedness of the set  $\{x_n : n \geq 0\}$  for the topology  $\mathcal{J}_{\|\cdot\|_\varphi}$ , then this new convergence comes under the definition of the so-called two-norm convergence or  $\gamma$ -convergence in the sense of Alexiewicz ([1,1954]). The general theory of two-norm convergence has been extensively developed by A.Alexiewicz [1], W. Orlicz [19], A. Alexiewicz and Z. Semadeni [2], A. Wiweger [23], [24], [25].

It is well known that the theory of two-norm convergence is closely related to the Wiweger's theory of mixed topologies [23], [24]. Indeed, in case when  $\|\cdot\|$  is a homogenous norm and  $\|\cdot\|^*$  is an  $F$ -norm on a linear space  $X$  and  $\|x_n - x\|^* \rightarrow 0$  implies  $\liminf \|x_n\| \geq \|x\|$ , then the sequential  $\gamma$ -convergence in  $X$  is generated by the so-called mixed topology  $\gamma[\mathcal{J}_{\|\cdot\|}, \mathcal{J}_{\|\cdot\|^*}]$ .

The notion of the mixed topology was a starting point for the theory of generalized inductive-limit topologies. There are many kinds of such topologies introduced for different reasons by A. Persson [21], D.J.H. Garling [7], J.B. Cooper [3], P. Turpin [22] and others.

The question arises whether our  $\gamma_\varphi$ -convergence in  $L^\varphi$  is topologized by some linear topology. It turns out that there is a positive answer to this question when we take into account an appropriate generalized inductive-limit topology in the sense of Turpin. This topology will be called here a generalized mixed topology and denoted by  $\mathcal{J}_I^\varphi$ . This term is justified by the fact that  $\mathcal{J}_I^\varphi$  coincides with the usual mixed topology  $\gamma[\mathcal{J}_{\|\cdot\|_\varphi}, \mathcal{J}_{\mu|_{L^\varphi}}]$  (in the sense of Wiweger) when the space  $(L^\varphi, \mathcal{J}_{\|\cdot\|_\varphi})$  is locally bounded.

In this paper we investigate the generalized mixed topology  $\mathcal{J}_I^\varphi$ . Our main aim is to obtain a description of  $\mathcal{J}_I^\varphi$  in terms of some family of  $F$ -norms defined by other Orlicz functions. As application we obtain

a topological characterization of our  $\gamma_\varphi$ -convergence in  $L^\varphi$ . Moreover, for  $\varphi$  being a convex Orlicz function we establish the general form of  $\mathcal{J}_I^\varphi$ -continuous linear functionals on  $L^\varphi$ .

In some special cases the topology  $\mathcal{J}_I^\varphi$  was examined by P. Turpin [22] and the author [14], [15], [16].

Given a linear topological space  $(X, \xi)$  by  $Bd(X, \xi)$  we will denote the collection of all  $\xi$ -bounded subsets of  $X$ . As usual  $\mathcal{N}$  stands for the set of all natural numbers. We assume that  $0 \cdot \infty = 0$ .

Now we recall some notation and terminology concerning Orlicz spaces (see [9], [11], [12], [22] for more details).

By an Orlicz function we mean a function  $\varphi : [0, \infty) \rightarrow [0, \infty]$  which is non-decreasing, left continuous, continuous at 0 with  $\varphi(0) = 0$ , and not identically equal to 0.

An Orlicz function  $\varphi$  is called convex whenever  $\varphi(\alpha u + \beta v) \leq \alpha\varphi(u) + \beta\varphi(v)$  for  $\alpha, \beta \geq 0$ ,  $\alpha + \beta = 1$  and  $u, v \geq 0$ . A convex Orlicz function is usually called a Young function.

For a Young function  $\varphi$  we denote by  $\varphi^*$  the function complementary to  $\varphi$  in the sense of Young, i.e.,

$$\varphi^*(v) = \sup\{uv - \varphi(u) : u \geq 0\} \text{ for } v \geq 0.$$

For a set  $\Psi$  of Young functions we will write:  $\Psi^* = \{\psi^* : \psi \in \Psi\}$ .

Let  $\varphi$  and  $\psi$  be a pair of Orlicz functions vanishing only at zero (resp. taking only finite values). We say that  $\varphi$  increases essentially more rapidly than  $\psi$  for small  $u$  (resp. for large  $u$ ) in symbols  $\psi \overset{s}{\ll} \varphi$  (resp.  $\psi \overset{l}{\ll} \varphi$ ) whenever for any  $c > 0$ ,  $\psi(cu)/\varphi(u) \rightarrow 0$  as  $u \rightarrow 0$  (resp.  $u \rightarrow \infty$ ).

We will write  $\psi \overset{a}{\ll} \varphi$  when  $\psi \overset{s}{\ll} \varphi$  and  $\psi \overset{l}{\ll} \varphi$  hold.

For  $\varphi$  and  $\psi$  being Young functions the condition  $\psi \overset{s}{\ll} \varphi$  (resp.  $\psi \overset{l}{\ll} \varphi$ ) implies  $\varphi^* \overset{s}{\ll} \psi^*$  (resp.  $\varphi^* \overset{l}{\ll} \psi^*$ ) (see [9, Lemma 13.1]).

Let  $(\Omega, \Sigma, \mu)$  be a positive measure space, and let  $L^0$  denote the set of equivalence classes of all real valued measurable functions defined

and finite a.e. on  $\Omega$ . For a subset  $A$  of  $\Omega$  and  $x \in L^0$  we will write  $x_A = x \cdot \chi_A$ , where  $\chi_A$  stands for the characteristic function of  $A$ .

An Orlicz function  $\varphi$  determines a functional  $m_\varphi : L^0 \rightarrow [0, \infty]$  by

$$m_\varphi(x) = \int_{\Omega} \varphi(|x(t)|) d\mu.$$

The Orlicz space generated by  $\varphi$  is the ideal of  $L^0$  defined by

$$L^\varphi = \{x \in L^0 : m_\varphi(\lambda x) < \infty \text{ for some } \lambda > 0\}.$$

The functional  $m_\varphi$  restricted to  $L^\varphi$  is an orthogonally additive semi-modular.

$L^\varphi$  can be equipped with the complete metrizable topology  $\mathcal{J}_\varphi$  of the  $F$ -norm

$$\|x\|_\varphi = \inf\{\lambda > 0 : m_\varphi(x/\lambda) \leq \lambda\}.$$

Moreover, if  $\varphi$  is a Young function, then the topology  $\mathcal{J}_\varphi$  can be generated by the Luxemburg norm

$$\|x\|_\varphi = \inf\{\lambda > 0 : m_\varphi(x/\lambda) \leq 1\}.$$

For  $r > 0$  let

$$B_\varphi(r) = \{x \in L^\varphi : \|x\|_\varphi \leq r\}$$

and let

$$B_{(\varphi)}(r) = \{x \in L^\varphi : \|x\|_\varphi \leq r\}$$

whenever  $\varphi$  is a Young function.

We shall need the following lemma.

**Lemma 1.1.** *Let  $\varphi_1, \varphi_2$  be Orlicz functions, and let  $\varphi(u) = \varphi_1(u) \vee \varphi_2(u)$  for  $u \geq 0$ . Then  $\varphi$  is an Orlicz function and the following statements hold:*

(i)  $L^\varphi = L^{\varphi_1} \cap L^{\varphi_2}$ .

(ii)  $\|x\|_{\varphi_1} \vee \|x\|_{\varphi_2} \leq \|x\|_\varphi \leq \|x\|_{\varphi_1} + \|x\|_{\varphi_2}$  for  $x \in L^\varphi$ .

(iii)  $\mathcal{J}_\varphi = \mathcal{J}_{\varphi_1|L^\varphi} \vee \mathcal{J}_{\varphi_2|L^\varphi}$ .

and  $Bd(L^\varphi, \mathcal{J}_\varphi) = Bd(L^\varphi, \mathcal{J}_{\varphi_1|L^\varphi}) \cap Bd(L^\varphi, \mathcal{J}_{\varphi_2|L^\varphi})$ .

**Proof.** (i) See [8, Theorem 1].

(ii) It follows from the definition of  $\|\cdot\|_\varphi$ .

(iii) It follows from (ii).

Let

$$E^\varphi = \{x \in L^0 : m_\varphi(\lambda x) < \infty \text{ for all } \lambda > 0\}.$$

It is known that  $L^\varphi = E^\varphi$  whenever  $\varphi$  satisfies the  $\Delta_2$ -condition, i.e.,  $\limsup \varphi(2u)/\varphi(u) < \infty$  as  $u \rightarrow 0$  and  $u \rightarrow \infty$ .

Let

$$\varphi_0(u) = \begin{cases} 0 & \text{for } 0 \leq u \leq 1, \\ 1 & \text{for } u > 1. \end{cases}$$

It is known that  $L^{\varphi_0}$  is the largest Orlicz space and consists of all those  $x \in L^0$  that are bounded outside of some set of finite measure, and

$$\|x\|_{\varphi_0} = \inf\{\lambda > 0 : \mu(\{t \in \Omega : |x(t)| > \lambda\}) \leq \lambda\}.$$

It is seen that  $\|x_n - x\|_{\varphi_0} \rightarrow 0$  in  $L^{\varphi_0}$  iff  $x_n \rightarrow x$  in measure on  $\Omega$  (in symbols  $x_n \rightarrow x$  ( $\mu - \Omega$ )). Therefore we will write  $\|\cdot\|_\mu$  instead of  $\|\cdot\|_{\varphi_0}$ , and by  $\mathcal{J}_\mu$  we will denote the topology of the  $F$ -norm  $\|\cdot\|_{\varphi_0}$ .

For  $\varepsilon > 0$  let

$$B_\mu(\varepsilon) = \{x \in L^{\varphi_0} : \|x\|_\mu \leq \varepsilon\}.$$

We shall need the following lemma.

**Lemma 1.2.** *Let  $\varphi$  be an Orlicz function such that  $\varphi(u) \rightarrow \infty$  as  $u \rightarrow \infty$ . Then for  $r > 0$ ,  $B_\varphi(r) \in Bd(L^\varphi, \mathcal{J}_\mu|L^\varphi)$ .*

**Proof.** Let  $x_n \in B_\varphi(r)$  ( $n = 1, 2, \dots$ ) and let  $\lambda_n \rightarrow 0$ . For  $\varepsilon > 0$  let  $\Omega_n(\varepsilon) = \{t \in \Omega : |\lambda_n x_n(t)| > \varepsilon\}$ . Then

$$\begin{aligned} \mu(\Omega_n(\varepsilon))\varphi\left(\frac{\varepsilon}{r|\lambda_n|}\right) &\leq \int_{\Omega_n(\varepsilon)} \varphi\left(\frac{|x_n(t)|}{r}\right) d\mu \\ &\leq m_\varphi\left(\frac{x_n}{r}\right) \leq r. \end{aligned}$$

Since  $\varphi(u) \rightarrow \infty$  as  $u \rightarrow \infty$  we get  $\mu(\Omega_n(\varepsilon)) \rightarrow 0$ , and this means that  $\|\lambda_n x_n\|_\mu \rightarrow 0$ .

## 2. A GENERALIZED MIXED TOPOLOGY ON $L^\varphi$ - GENERAL PROPERTIES

In this section we consider some kind of generalized inductive limit topology on  $L^\varphi$ .

Let  $\varphi$  be an arbitrary Orlicz function, and let

$$F_n = B_\varphi(2^n) \text{ and } \mathcal{J}_n = \mathcal{J}_{\mu|_{F_n}} \text{ for } n \geq 0.$$

Then the family  $\mathcal{B}_\varphi = \{F_n : n \geq 0\}$  forms a base of metric bounded sets in  $(L^\varphi, \|\cdot\|_\varphi)$ .

Moreover, the sequence  $(F_n, \mathcal{J}_n)$  ( $n \geq 0$ ) of balanced topological spaces satisfies the following conditions:

(i)  $L^\varphi = \bigcup_{n \geq 0} F_n$ .

(ii)  $F_n + F_n \subset F_{n+1}$ , and the function

$$F_n \times F_n \ni (x, y) \rightarrow x + y \in F_{n+1}$$

is continuous.

(iii) The function  $[-1, 1] \times F_n \ni (\lambda, x) \mapsto \lambda \cdot x \in F_n$  is continuous.

(iv)  $\mathcal{J}_{n+1|_{F_n}} = \mathcal{J}_n$  for  $n \geq 0$ .

Thus the space  $L^\varphi$  with the system  $\{(F_n, \mathcal{J}_n) : n \geq 0\}$  comes under the conditions of the strict inductive limit of balanced topological spaces in the sense of P. Turpin [22, Ch. 1].

**Definition 2.1.** *The family of all sets of the form*

$$\bigcup_{N=0}^{\infty} \left( \sum_{n=0}^N (B_\varphi(2^n) \cap B_\mu(\varepsilon_n)) \right) \tag{2.1}$$

where  $(\varepsilon_n : n \geq 0)$  is a sequence of positive numbers, forms a base of neighbourhoods of zero for a linear topology on  $L^\varphi$  (in the sense of Turpin) which will be denoted by  $\mathcal{J}_I^\varphi$ .

According to [22, Theorem 1.1.6]  $\mathcal{J}_I^\varphi$  is the finest of all linear topologies  $\xi$  on  $L^\varphi$  which satisfy the conditions:

$$\xi|_{F_n} \subset \mathcal{J}_{\mu|_{F_n}} \text{ for } n \geq 0. \tag{2.2}$$

Moreover, in view of [22, Theorem 1.1.8] we have

$$\mathcal{J}_{I|_{F_n}}^\varphi = \mathcal{J}_{\mu|_{F_n}} \text{ for } n \geq 0. \tag{2.3}$$

Since  $\mathcal{J}_{\mu|_{L^\varphi}} \subset \mathcal{J}_\varphi$  we have  $\mathcal{J}_I^\varphi \subset \mathcal{J}_\varphi$ ; hence  $\mathcal{J}_{\mu|_{L^\varphi}} \subset \mathcal{J}_I^\varphi \subset \mathcal{J}_\varphi$ .

Henceforth in this section we assume that  $\varphi(u) \rightarrow \infty$  as  $u \rightarrow \infty$ .

The basic properties of the topology  $\mathcal{J}_I^\varphi$  are included in the following theorems.

**Theorem 2.1.** *The space  $(L^\varphi, \mathcal{J}_I^\varphi)$  is complete.*

**Proof.** It is known that the balls  $B_\varphi(2^n)$  are closed subsets of  $(L^\varphi, \mathcal{J}_\mu)$  (see [22, 0.3.6]), so the spaces  $(B_\varphi(2^n), \mathcal{J}_{\mu|_{B_\varphi(2^n)}})$  ( $n \geq 0$ ) are complete. Hence, by [22, Theorem 1.1.10] the space  $(L^\varphi, \mathcal{J}_I^\varphi)$  is complete.

**Theorem 2.2.** *For a subset  $Z \subset L^\varphi$  the following statements are equivalent:*

(i)  $\sup\{\|x\|_\varphi : x \in Z\} < \infty$ .

(ii)  $Z$  is bounded for  $\mathcal{J}_I^\varphi$ .

**Proof.** By Lemma 1.2 the balls  $B_\varphi(2^n)$  are bounded subsets of  $(L^\varphi, \mathcal{J}_{\mu|_{L^\varphi}})$ . Moreover, the balls  $B_\varphi(2^n)$  are also closed in  $(L^\varphi, \mathcal{J}_{\mu|_{L^\varphi}})$  (see [22, 0.3.6]). In view of (2.2) and (2.3)  $\mathcal{J}_I^\varphi$  is the finest of all linear topologies  $\xi$  on  $L^\varphi$  such that  $\xi|_{F_n} = \mathcal{J}_{\mu|_{F_n}}$ . Hence by [22, Corollary 1.1.12] the equivalence (i)  $\Leftrightarrow$  (ii) holds.

**Theorem 2.3.** For a subset  $Z \subset L^\varphi$  the following statements are equivalent:

(i)  $Z$  is relatively compact for  $\mathcal{J}_I^\varphi$ .

(ii)  $Z$  is relatively compact for  $\mathcal{J}_{\mu|_{L^\varphi}}$  and

$$\sup\{\|x\|_\varphi : x \in Z\} < \infty.$$

**Proof.** It follows from Theorem 2.2 and (2.3).

Let us recall that a sequence  $(x_n)$  in  $L^\varphi$  is said to be  $\gamma_\varphi$ -convergent to  $x \in L^\varphi$ , in symbols  $x_n \xrightarrow{\gamma_\varphi} x$ , whenever

$$x_n \rightarrow x \ (\mu - \Omega) \quad \text{and} \quad \sup_n \|x_n\|_\varphi < \infty.$$

**Theorem 2.4.** For a sequence  $(x_n)$  in  $L^\varphi$  the following statements are equivalent:

(i)  $x_n \rightarrow 0$  for  $\mathcal{J}_I^\varphi$ .

(ii)  $x_n \xrightarrow{\gamma_\varphi} 0$ .

Moreover,  $\mathcal{J}_I^\varphi$  is the finest of all linear topologies  $\xi$  on  $L^\varphi$  which satisfy the condition:

$$x_n \xrightarrow{\gamma_\varphi} 0 \quad \text{implies} \quad x_n \rightarrow 0 \text{ for } \xi. \quad (+)$$



**Proof.** The equivalence (i)  $\Leftrightarrow$  (ii) follows from Theorem 2.2 and (2.3).

Now let  $\xi$  be a linear topology on  $L^\varphi$  for which the condition (+) holds. Then  $\xi|_{B_\varphi(r)} \subset \mathcal{J}_\mu|_{B_\varphi(r)}$  for  $r > 0$ , because  $\mathcal{J}_\mu$  is a metrizable linear topology. Hence by (2.2) we get that  $\xi \subset \mathcal{J}_I^\varphi$ .

**Definition 2.2.** Let  $(X, \eta)$  be a linear topological space. A linear mapping  $A : L^\varphi \rightarrow X$  is said to be  $\gamma_\varphi$ -linear, if

$$x_n \xrightarrow{\gamma_\varphi} 0 \quad \text{implies} \quad A(x_n) \rightarrow 0 \text{ for } \eta.$$

The next theorem gives a characterization of  $\gamma_\varphi$ -linear functionals on  $L^\varphi$ .

**Theorem 2.5.** For a linear topological space  $(X, \eta)$  and a linear mapping  $A : L^\varphi \rightarrow X$  the following statements are equivalent:

- (i)  $A$  is  $(\mathcal{J}_I^\varphi, \eta)$ -continuous.
- (ii)  $A$  is  $\gamma_\varphi$ -linear.
- (iii) For every  $r > 0$ , the restriction  $A|_{B_\varphi(r)}$  is  $(\mathcal{J}_\mu|_{B_\varphi(r)}, \eta)$ -continuous.

**Proof.** (i)  $\Rightarrow$  (ii) It follows from Theorem 2.4.

(ii)  $\Rightarrow$  (iii) it is obvious.

(iii)  $\Rightarrow$  (i) Let  $W$  be a neighbourhood of zero in  $X$  for  $\eta$ . Then there exists a sequence  $(W_n : n \geq 0)$  of neighbourhoods of zero for  $\eta$  such that  $\sum_{n=0}^N W_n \subset W$  for every  $N \geq 0$ . Thus by our assumption there exists a sequence  $(\varepsilon_n : n \geq 0)$  of positive numbers such that  $A(B_\varphi(2^n) \cap B_\mu(\varepsilon_n)) \subset W_n$ . Thus for  $N \geq 0$

$$A\left(\sum_{n=0}^N (B_\varphi(2^n) \cap B_\mu(\varepsilon_n))\right) \subset \sum_{n=0}^N W_n \subset W,$$

so

$$\begin{aligned} & A\left(\bigcup_{N=0}^{\infty} \left(\sum_{n=0}^N (B_{\varphi}(2^n) \cap B_{\mu}(\varepsilon_n))\right)\right) \subset \\ & \subset \bigcup_{N=0}^{\infty} A\left(\sum_{n=0}^N (B_{\varphi}(2^n) \cap B_{\mu}(\varepsilon_n))\right) \subset W. \end{aligned}$$

This means that  $A$  is  $(\mathcal{J}_I^{\varphi}, \eta)$ -continuous.

Now we are going to compare the topology  $\mathcal{J}_I^{\varphi}$  with the mixed topology  $\gamma[\mathcal{J}_{\varphi}, \mathcal{J}_{\mu|L^{\varphi}}]$  in the sense of Wiweger (see [24]). For this purpose we shall need the following

**Theorem 2.6.** *Assume that  $(\Omega, \Sigma, \mu)$  is an atomless measure space or that  $\mu$  is the counting measure on  $\mathcal{N}$ . If  $(L^{\varphi}, \mathcal{J}_{\varphi})$  is a locally bounded space then for a subset  $Z$  of  $L^{\varphi}$  the following statements are equivalent:*

- (i)  $Z$  is bounded for  $\mathcal{J}_I^{\varphi}$ .
- (ii)  $\sup \{\|x\|_{\varphi} : x \in Z\} < \infty$ .
- (iii)  $Z$  is bounded for  $\mathcal{J}_{\varphi}$ .

**Proof.** (i)  $\Leftrightarrow$  (ii) See Theorem 2.2.

(ii)  $\Rightarrow$  (iii) In view of [22, 0.3.10.2]  $\sup\{\|x\|_{\varphi} : x \in Z\} < \infty$  if and only if  $Z$  is additively bounded (see [22, 0.3.10.1]), so arguing as in the proof of [15, Lemma 2.5] we obtain that  $Z$  is bounded for  $\mathcal{J}_{\varphi}$ .

(iii)  $\Rightarrow$  (i) Obvious.

**Theorem 2.7.** *Assume that  $(\Omega, \Sigma, \mu)$  is an atomless measure space or that  $\mu$  is the counting measure on  $\mathcal{N}$ . If  $(L^{\varphi}, \mathcal{J}_{\varphi})$  is a locally bounded space, then the generalized mixed topology  $\mathcal{J}_I^{\varphi}$  coincides with the mixed topology  $\gamma[\mathcal{J}_{\varphi}, \mathcal{J}_{\mu|L^{\varphi}}]$ .*

**Proof.** In view of Theorem 2.6 it follows from [24, 2.2.1 and 2.2.2].

### 3. SOME PROJECTIVE TOPOLOGY ON ORLICZ SPACES

In [5], [6] H.W. Davis, F.J.Muray and J. Weber studied the spaces

$$L^P(S) = \bigcap_{p \in S} L^p(S \subset [1, \infty))$$

endowed with the appropriate projective topology.

There are some results concerning a representation of an Orlicz space  $L^\varphi$  as the intersection of some family of other Orlicz spaces (see [10], [17], [18]). In this section we examine the appropriate projective topology on  $L^\varphi$ . In section 4 we shall show that this projective topology coincides with the generalized mixed topology  $\mathcal{J}_I^\varphi$ .

We start with some equalities among Orlicz spaces, proved in [17] and [18], which are of key importance in this section. At the very beginning we distinguish some classes of Orlicz functions.

An Orlicz function  $\varphi$  continuous for all  $u \geq 0$ , taking only finite values, vanishing only at zero, and not bounded is usually called a  $\varphi$ -function. By  $\Phi$  we will denote the collection of all  $\varphi$ -functions.

A Young function  $\varphi$  vanishing only at zero and taking only finite values is called an  $N$ -function whenever  $\varphi(u)/u \rightarrow 0$  as  $u \rightarrow 0$  and  $\varphi(u)/u \rightarrow \infty$  as  $u \rightarrow \infty$ . By  $\Phi_N$  we will denote the collection of all  $N$ -functions.

Let  $\Phi_1$  be the set of all Orlicz functions  $\varphi$  vanishing only at zero and such that  $\varphi(u) \rightarrow \infty$  as  $u \rightarrow \infty$ . Denote by

$$\Phi_{11} = \{\varphi \in \Phi_1 : \varphi(u) < \infty \text{ for } u > 0\},$$

$$\Phi_{12} = \{\varphi \in \Phi_1 : \varphi \text{ jumps to } \infty\}.$$

Then  $\Phi_1 = \Phi_{11} \cup \Phi_{12}$ . In view of [17, Theorem 3.1, 3.2, 3.7 and 3.8] we get

**Theorem 3.1.** *Let  $\varphi \in \Phi_{1i}$  ( $i = 1, 2$ ). Then the following equalities hold:*

$$L^\varphi = \bigcap \{L^\psi : \psi \in \Psi_{1i}^\varphi\} = \bigcap \{E^\psi : \psi \in \Psi_{1i}^\varphi\}$$

where

$$\Psi_{11}^\varphi = \{\psi \in \Phi : \psi \overset{a}{\ll} \varphi\}, \quad \Psi_{12}^\varphi = \{\psi \in \Phi : \psi \overset{s}{\ll} \varphi\}.$$

Moreover, if  $\mu$  is an atomless measure or the counting measure on  $\mathcal{N}$ , then the strict inclusion  $L^\varphi \subsetneq E^\psi$  holds for each  $\psi \in \Psi_{1i}^\varphi$ .

Next let  $\Phi_1^c$  be the set of all Young functions  $\varphi$  vanishing only at zero and such that  $\varphi(u)/u \rightarrow \infty$  as  $u \rightarrow \infty$ .

Denote by

$$\Phi_{11}^c = \{\varphi \in \Phi_1^c : \varphi(u) < \infty \text{ for } u > 0 \text{ and } \varphi(u)/u \rightarrow 0 \text{ as } u \rightarrow 0\},$$

$$\Phi_{12}^c = \{\varphi \in \Phi_1^c : \varphi \text{ jumps to } \infty \text{ and } \varphi(u)/u \rightarrow 0 \text{ as } u \rightarrow 0\},$$

$$\Phi_{13}^c = \{\varphi \in \Phi_1^c : \varphi(u) < \infty \text{ for } u > 0 \text{ and } \varphi(u)/u \rightarrow a \text{ as } u \rightarrow 0, a > 0\},$$

$$\Phi_{14}^c = \{\varphi \in \Phi_1^c : \varphi \text{ jumps to } \infty \text{ and } \varphi(u)/u \rightarrow a \text{ as } u \rightarrow 0, a > 0\}.$$

Then  $\Phi_1^c = \bigcup_{i=1}^4 \Phi_{1i}^c$ , where the sets  $\Phi_{1i}^c$  are pairwise disjoint. It is seen that  $\Phi_{11}^c = \Phi_N$ . According to [18, Theorems 2.1-2.4] we get

**Theorem 3.2.** *Let  $\varphi \in \Phi_{1i}^c$  ( $i = 1, 2, 3, 4$ ). Then the following equalities hold:*

$$L^\varphi = \bigcap \{L^\psi : \psi \in \Psi_{1i}^\varphi(N)\} = \bigcap \{E^\psi : \psi \in \Psi_{1i}^\varphi(N)\}$$

where

$$\Psi_{11}^\varphi(N) = \{\psi \in \Phi_N : \psi \overset{a}{\ll} \varphi\}, \quad \Psi_{12}^\varphi(N) = \{\psi \in \Phi_N : \psi \overset{s}{\ll} \varphi\},$$

$$\Psi_{13}^\varphi(N) = \{\psi \in \Phi_N : \psi \overset{l}{\ll} \varphi\}, \quad \Psi_{14}^\varphi(N) = \Phi_N.$$

Next, let  $\Phi_2^c$  be the set of all Young functions  $\varphi$  taking only finite values and such that  $\varphi(u)/u \rightarrow 0$  as  $u \rightarrow 0$ .

Denote by

$$\Phi_{21}^c = \{\varphi \in \Phi_2^c : \varphi(u) > 0 \text{ for } u > 0 \text{ and } \varphi(u)/u \rightarrow \infty \text{ as } u \rightarrow \infty\},$$

$$\Phi_{22}^c = \{ \varphi \in \Phi_2^c : \varphi(u) > 0 \text{ for } u > 0 \text{ and } \varphi(u)/u \rightarrow a \text{ as } u \rightarrow \infty, a > 0 \},$$

$$\Phi_{23}^c = \{ \varphi \in \Phi_2^c : \varphi(u) = 0 \text{ near zero and } \varphi(u)/u \rightarrow \infty \text{ as } u \rightarrow \infty \},$$

$$\Phi_{24}^c = \{ \varphi \in \Phi_2^c : \varphi(u) = 0 \text{ near zero and } \varphi(u)/u \rightarrow a \text{ as } u \rightarrow \infty, a > 0 \}.$$

Then  $\Phi_2^c = \bigcup_{i=1}^4 \Phi_{2i}^c$ , where the sets  $\Phi_{2i}^c$  are pairwise disjoint. It is seen that  $\Phi_{21}^c = \Phi_N$ . According to [18, Theorems 1.1-1.4] we have

**Theorem 3.3.** *Let  $\varphi \in \Phi_{2i}^c$  ( $i = 1, 2, 3, 4$ ). Then the following equalities hold*

$$E^\varphi = \bigcup \{ E^\psi : \psi \in \Psi_{2i}^\varphi(N) \} = \bigcup \{ L^\psi : \psi \in \Psi_{2i}^\varphi(N) \}$$

where

$$\Psi_{21}^\varphi(N) = \{ \psi \in \Phi_N : \varphi \overset{a}{\prec} \psi \}, \quad \Psi_{22}^\varphi(N) = \{ \psi \in \Psi_N : \varphi \overset{s}{\prec} \psi \},$$

$$\Psi_{23}^\varphi(N) = \{ \psi \in \Phi_N : \varphi \overset{l}{\prec} \psi \}, \quad \Psi_{24}^\varphi(N) = \Phi_N.$$

At last, in view of [18, Lemma 3.1 and Theorem 3.3] we get

**Theorem 3.4** *Let  $\varphi_1$  and  $\varphi_2$  be a pair of complementary Young functions. Then  $\varphi_1 \in \Phi_{1i}^c$  if and only if  $\varphi_2 \in \Phi_{2i}^c$  ( $i = 1, 2, 3, 4$ ), and moreover, the sets  $\Psi_{1i}^\varphi(N)$  and  $\Psi_{2i}^\varphi(N)$  are mutually related in such a way that*

$$(\Psi_{1i}^{\varphi_1}(N))^* = \Psi_{2i}^{\varphi_2}(N) \text{ and } (\Psi_{2i}^{\varphi_2}(N))^* = \Psi_{1i}^{\varphi_1}(N).$$

We shall need the following

**Corollary 3.5.** *Let  $\varphi \in \Phi_{1i}^c$  ( $i = 1, 2, 3, 4$ ). Then*

$$E^{\varphi^*} = \bigcup \{ L^{\psi^*} : \psi \in \Psi_{1i}^\varphi(N) \}.$$

**Proof.** Since  $\varphi \in \Phi_{2i}^c$  and  $(\Psi_{1i}^\varphi(N))^* = \Psi_{2i}^{\varphi^*}(N)$  (see Theorem 3.4) by Theorem 3.3 we get

$$\begin{aligned} \bigcup \{L^{\psi^*} : \psi \in \Psi_{1i}^\varphi(N)\} &= \bigcup \{L^\psi : \psi \in (\Psi_{1i}^\varphi(N))^*\} \\ &= \bigcup \{L^\psi : \psi \in \Psi_{2i}^{\varphi^*}(N)\} = E^{\varphi^*}. \end{aligned}$$

We are now ready to define our projective topology on  $L^\varphi$ .

**Definition 3.1.** Let  $\varphi \in \Phi_{1i}$  ( $i = 1, 2$ ). By  $\mathcal{J}_P^\varphi$  we will denote the projective topology on  $L^\varphi$  with respect to the family  $\{(E^\psi, \mathcal{J}_{\psi|E^\psi}) : \psi \in \Psi_{1i}^\varphi\}$ , i.e.,  $\mathcal{J}_P^\varphi$  is defined to be the coarsest of all linear topologies  $\xi$  on  $L^\varphi$  for which  $\mathcal{J}_{\psi|L^\varphi} \subset \xi$  holds for every  $\psi \in \Psi_{1i}^\varphi$ . Thus

$$\mathcal{J}_P^\varphi = \sup \{\mathcal{J}_{\psi|L^\varphi} : \psi \in \Psi_{1i}^\varphi\}.$$

For  $\varphi$  being a  $\varphi$ -function the topology  $\mathcal{J}_P^\varphi$  has been examined in [14], [15], [16]. It is easy to verify that all properties of  $\mathcal{J}_P^\varphi$  which are obtained in [14], [15], [16] for  $\varphi$  being a  $\varphi$ -function remain valid for  $\varphi \in \Phi_{11}$ . In this section we extend results from [14], [15], [16] to the case of  $\varphi$  belonging to  $\Phi_1$ .

From the definition of  $\mathcal{J}_P^\varphi$  we have

**Theorem 3.6.** Let  $\varphi \in \Phi_1$ . Then  $\mathcal{J}_{\mu|L^\varphi} \subset \mathcal{J}_P^\varphi \subset \mathcal{J}_\varphi$ .

**Theorem 3.7.** Let  $\varphi \in \Phi_1$  and let  $\mu$  be an infinite atomless measure. Then there exists a sequence  $(x_n)$  in  $L^\varphi$  such that  $x_n \rightarrow 0$  for  $\mathcal{J}_P^\varphi$  and  $m_\varphi(x_n) = 1$  for  $n \in \mathcal{N}$ . Hence the strict inclusion  $\mathcal{J}_P^\varphi \subsetneq \mathcal{J}_\varphi$  holds.

**Proof.** For  $\varphi \in \Phi_{11}$  this fact is proved in [13, Theorem 2.5]. Now let  $\varphi \in \Phi_{12}$ , i.e.,  $\varphi(u) < \infty$  for  $u \leq a$  and  $\varphi(u) = \infty$  for  $u > a$ . Let  $(u_n)$  be a sequence of positive numbers such that  $u_n \downarrow 0$  and  $u_1 < a$ . Let  $(\Omega_n)$  be a sequence of measurable subsets of  $\Omega$  such that  $\mu(\Omega_n) = 1/\varphi(u_n)$ . Define

$$x_n(t) = \begin{cases} u_n & \text{for } t \in \Omega_n, \\ 0 & \text{for } t \notin \Omega_n. \end{cases}$$

We shall show that  $x_n \rightarrow 0$  for  $\mathcal{J}_P^\varphi$ , i.e.,  $\|x_n\|_\psi \rightarrow 0$  for each  $\psi \in \Psi_{12}^\varphi$ . Indeed, let  $\psi \overset{s}{\prec} \varphi$  and let  $\varepsilon > 0$  be given. Then there exists  $u_0 > 0$  such that  $\psi(u/\varepsilon) \leq \varepsilon\varphi(u)$  for  $u \leq u_0$ . Let  $n_0 \in \mathcal{N}$  be such that  $u_n \leq u_0$  for  $n \geq n_0$ . Then for  $n \geq n_0$  we have  $m_\psi(x_n/\varepsilon) = \psi(u_n/\varepsilon)/\varphi(u_n) \leq \varepsilon$  i.e.,  $\|x_n\|_\psi \leq \varepsilon$ . On the other hand,  $m_\varphi(x_n) = \varphi(u_n)/\varphi(u_n) = 1$ .

Arguing as in the proof of [13, Theorem 1.2] we get

**Theorem 3.8.** *Let  $\varphi \in \Phi_{1i}$ ; ( $i = 1, 2$ ). Then the topology  $\mathcal{J}_P^\varphi$  has a base of neighbourhoods of zero consisting of all sets of the form:*

$$B_\psi(r) \cap L^\varphi$$

where  $\psi \in \Psi_{1i}^\varphi$  and  $r > 0$ .

Repeating the arguments of the proof of [13, Theorem 5.1] and using the equalities from Theorem 3.1 we get

**Theorem 3.9.** *Let  $\phi \in \Phi_1$ . Then the space  $(L^\varphi, \mathcal{J}_P^\varphi)$  is complete.*

Since the space  $(L^\varphi, \mathcal{J}_\varphi)$  is complete, from Theorems 3.6 and 3.7, in view of the Open Mapping Theorem it follows

**Theorem 3.10.** *Let  $\varphi \in \Phi_1$  and let  $\mu$  be an infinite atomless measure. Then the space  $(L^\varphi, \mathcal{J}_P^\varphi)$  is not metrizable.*

To the end of this section we will assume that  $\varphi \in \Phi_1^c$ . We start with the following lemma.

**Lemma 3.11.** *Let  $\varphi \in \Phi_{1i}^c$  ( $i = 1, 2, 3, 4$ ) and let  $\psi$  be a  $\varphi$ -function such that  $\psi \overset{a}{\prec} \varphi$  if  $i = 1$  (resp.  $\psi \overset{s}{\prec} \varphi$  if  $i = 2$ ,  $\psi \overset{a}{\prec} \varphi$  if  $i = 3$ ,  $\psi \overset{s}{\prec} \varphi$  if  $i = 4$ ). Then there exists an  $N$ -function  $\psi_0$  such that  $\psi(u) \leq \psi_0(2u)$  for  $u \geq 0$  and  $\psi_0 \overset{a}{\prec} \varphi$  if  $i = 1$  (resp.  $\psi_0 \overset{s}{\prec} \varphi$  if  $i = 2$ ,  $\psi_0 \overset{l}{\prec} \varphi$  if  $i = 3$ ).*

**Proof.** Let  $\psi_1$  be an arbitrary  $N$ -function such that  $\psi_1 \overset{a}{\prec} \varphi$  if  $i = 1$  (resp.  $\psi_1 \overset{s}{\prec} \varphi$  if  $i = 2$ ,  $\psi_1 \overset{l}{\prec} \varphi$  if  $i = 3$ ,  $\psi_1 \overset{s}{\prec} \varphi$  if  $i = 4$ ). Let  $\psi_2 = \psi \vee \psi_1$ . Next, let us put  $p(0) = 0$  and  $p(s) = \sup_{0 < t \leq s} (\psi_2(t)/t)$  for  $s > 0$ . Let

$$\psi_0(u) = \int_0^u p(s) ds \text{ for } u \geq 0.$$

It is seen that  $\psi_0$  is an  $N$ -function. Arguing as in the proof of [13, Lemma 1.4] we can verify that  $\psi_0$  satisfies the desired properties.

**Theorem 3.12.** *Let  $\varphi \in \Phi_{1i}^c$  ( $i = 1, 2, 3, 4$ ). Then the topology  $\mathcal{J}_P^\varphi$  is generated by the family of  $B$ -norms  $\{|||\cdot|||_{\psi|L^\varphi} : \psi \in \Psi_{1i}^\varphi(N)\}$ .*

**Proof.** For example, let  $\varphi \in \Phi_{13}^c$ . Then  $\varphi \in \Phi_{11}$ . Given  $\psi \in \Psi_{11}^\varphi$  and  $r > 0$ , in view of Lemma 3.11 there exists  $\psi_0 \in \Psi_{13}^\varphi(N)$  and such that  $\psi(u) \leq \psi_0(2u)$  for  $u \geq 0$ . Hence

$$||x||_\psi \leq ||2x||_{\psi_0} \text{ for all } x \in L^{\psi_0}. \quad (1)$$

On the other hand, since the  $F$ -norms  $||\cdot||_{\psi_0}$  and  $|||\cdot|||_{\psi_0}$  are equivalent on  $L^{\psi_0}$ , there exists  $r_1 > 0$  such that

$$B_{(\psi_0)}(r_1) \subset B_{\psi_0}(r). \quad (2)$$

We shall show that  $B_{(\psi_0)}(r_1/2) \cap L^\varphi \subset B_\psi(r)$ . Indeed, let  $x \in B_{(\psi_0)}(r_1/2) \cap L^\varphi$ . Then  $|||2x|||_{\psi_0} \leq r_1$ , hence by (2) we get  $||2x||_{\psi_0} \leq r$  and next, by (1) we see that  $||x||_\psi \leq r$ .

For  $i=1,2,4$  the proof is similar.

Now we are ready to establish the general form of  $\mathcal{J}_P^\varphi$ -continuous linear functionals on  $L^\varphi$ .

**Theorem 3.13.** *Let  $\varphi \in \Phi_1^c$  and let  $\mu$  be a  $\sigma$ -finite measure. Then for a linear functional  $f$  on  $L^\varphi$  the following statements are equivalent:*

(i)  $f$  is continuous for  $\mathcal{J}_P^\varphi$ .



(ii) *There exists a unique  $y \in E^{\varphi^*}$  such that*

$$f(x) = f_y(x) = \int_{\Omega} x(t)y(t) d\mu \quad \text{for } x \in L^{\varphi}.$$

**Proof.** (i)  $\Rightarrow$  (ii). Let  $\varphi \in \Phi_{1i}^c$  ( $i = 1, 2, 3, 4$ ). In view of Theorem 3.12 there exist  $\psi \in \Psi_{1i}^{\varphi}(N)$  and  $r > 0$  such that  $f$  is bounded on  $B_{(\psi)}(r) \cap L^{\varphi}$ . This means that  $f$  is continuous on the linear subspace  $(L^{\varphi}, \mathcal{J}_{\psi|L^{\varphi}})$  of the normed space  $(E^{\psi}, \mathcal{J}_{\psi|E^{\psi}})$ . Hence, by the Hahn-Banach theorem there exists a  $\mathcal{J}_{\psi|E^{\psi}}$ -continuous linear functional  $\bar{f}$  on  $E^{\psi}$  such that  $\bar{f}(x) = f(x)$  for  $x \in L^{\varphi}$ . According to [11, p. 56] there exists  $y \in L^{\psi^*} \subset E^{\varphi^*}$  such that

$$\bar{f}(x) = \int_{\Omega} x(t)y(t) d\mu \quad \text{for } x \in E^{\psi}.$$

Hence

$$f(x) = f_y(x) = \int_{\Omega} x(t)y(t) d\mu \quad \text{for } x \in L^{\varphi}. \tag{1}$$

Now assume that there exists another  $y' \in E^{\varphi^*}$  such that

$$f(x) = f_{y'}(x) = \int_{\Omega} x(t)y'(t) d\mu \quad \text{for } x \in L^{\varphi}. \tag{2}$$

Then, for example, there exists a measurable set  $A \subset \{t \in \Omega : y'(t) > y(t)\}$  such that  $0 < \mu(A) < \infty$ . Hence by (1) and (2) we get

$$\int_{\Omega} \chi_A(t) (y'(t) - y(t)) d\mu = \int_A (y'(t) - y(t)) d\mu = 0.$$

This contradiction establishes that there exists a unique  $y \in E^{\varphi^*}$  such that (1) holds.

(ii)  $\Rightarrow$  (i) Let  $\varphi \in \Phi_{1i}^c$  ( $i = 1, 2, 3, 4$ ). According to Corollary 3.5 there exists  $\psi \in \Psi_{1i}^{\varphi}(N)$  such that  $y \in L^{\psi^*}$ . Then  $L^{\varphi} \subset E^{\psi} \subset L^{\psi}$  and using the Holder's inequality we get that  $|f_y(x)| \leq \|y\|_{\psi^*}^0 \|x\|_{\psi}$  for

$x \in L^\varphi$  (here  $\|\cdot\|_{\psi^*}^0$  denotes the Orlicz norm on  $L^{\psi^*}$ ). This means that  $f_y$  is  $\mathcal{J}_{\psi|L^\varphi}$ -continuous, so  $f_y$  is  $\mathcal{J}_P^\varphi$ -continuous, because  $\mathcal{J}_{\psi|L^\varphi} \subset \mathcal{J}_P^\varphi$ .

#### 4. THE IDENTITY OF THE TOPOLOGIES $\mathcal{J}_I^\varphi$ AND $\mathcal{J}_P^\varphi$ ON $L^\varphi$

In this section we shall prove that the topologies  $\mathcal{J}_I^\varphi$  and  $\mathcal{J}_P^\varphi$  coincide on  $L^\varphi$ . We start with the following lemma.

**Lemma 4.1.** *Let  $\varphi \in \Phi_{12}$  and  $\psi$  be a  $\varphi$ -function. Then the following statements hold:*

(i) *For every  $r > 0$  and  $\varepsilon > 0$  there exists  $\delta > 0$  such that*

$$\sup \{ \|x_A\|_\psi : x \in B_\varphi(r) \} < \varepsilon \quad \text{for } A \in \Sigma, \mu(A) < \delta.$$

(ii) *If  $\psi \overset{s}{\prec} \varphi$ , then for every  $r > 0$*

$$\mathcal{J}_{\mu|B_\varphi(r)} = \mathcal{J}_{\psi|B_\varphi(r)}.$$

**Proof.** (i) Assume that  $\varphi(u) < \infty$  for  $0 \leq u \leq a$  and  $\varphi(u) = \infty$  for  $u > a$ , where  $a > 0$ . Given  $x \in B_\varphi(r)$  we have  $\int_\Omega \varphi(|x(t)|/r) d\mu \leq r$ , so  $|x(t)|/r \leq a$  a.e. on  $\Omega$ . Given  $\varepsilon > 0$  let  $\delta = \varepsilon/\psi(ar/\varepsilon)$ . Then for  $A \in \Sigma$  with  $\mu(A) < \delta$

$$\int_\Omega \psi(|x_A(t)|/\varepsilon) d\mu \leq \psi(ar/\varepsilon)\mu(A) \leq \varepsilon$$

i.e.,  $\|x_A\|_\psi \leq \varepsilon$ .

(ii) Since the inclusion  $\mathcal{J}_{\mu|L^\varphi} \subset \mathcal{J}_{\psi|L^\varphi}$  holds it is enough to show that  $\mathcal{J}_{\psi|B_\varphi(r)} \subset \mathcal{J}_{\mu|B_\varphi(r)}$  holds for every  $r > 0$ . To this end we shall show that for any  $x \in B_\varphi(r)$  and  $\xi > 0$  there exists  $\eta_0 > 0$  such that

$$B_\mu(x, \eta_0) \cap B_\varphi(r) \subset B_\psi(x, \xi)$$

where for  $\eta > 0$

$$\begin{aligned} B_\mu(x, \eta) &= \{y \in L^{\varphi_0} : \|y - x\|_\mu \leq \eta\} \\ &= \{y \in L^{\varphi_0} : \mu(\{t \in \Omega : |y(t) - x(t)| > \eta\}) \leq \eta\} \end{aligned}$$

and

$$B_\psi(x, \eta) = \{y \in L^\psi : \|y - x\|_\psi \leq \eta\}.$$

Indeed, let  $x \in B_\varphi(r)$  and  $\xi > 0$  be given. For  $\eta > 0$  and  $y \in B_\varphi(r)$  let put

$$E(\eta, y) = \{t \in \Omega : |y(t) - x(t)| > \eta\}, \quad G(\eta, y) = \Omega \setminus E(\eta, y).$$

It is seen that

$$m_\varphi((y - x)/2r) \leq 2r. \tag{1}$$

Since  $\psi \prec \varphi$ , there exists  $u_0 > 0$  such that

$$\psi\left(\frac{2u}{\xi}\right) \leq \frac{\xi}{4r} \varphi\left(\frac{u}{2r}\right) \quad \text{for } 0 \leq u \leq u_0. \tag{2}$$

Moreover, in view of (i) there exists  $\delta > 0$  such that

$$\|(y - x)_A\|_\psi \leq \frac{1}{2}\xi \quad \text{for } A \in \Sigma \text{ with } \mu(A) < \delta. \tag{3}$$

Now let  $\eta_0 = \min(u_0, \delta)$  and let  $y \in B_\mu(x, \eta_0) \cap B_\varphi(r)$ . Then  $\mu(E(\eta_0, y)) \leq \eta_0 \leq \delta$ , and hence from (3) we get

$$\|(y - x)_{E(\eta_0, y)}\|_\psi \leq \frac{1}{2}\xi. \tag{4}$$

On the other hand, since  $\eta_0 \leq u_0$ , from (2) and (1) we get

$$\begin{aligned} m_\psi\left(\frac{2}{\xi}(y-x)_{G(\eta_0,y)}\right) &= \int_{G(\eta_0,y)} \psi\left(\frac{2|y(t)-x(t)|}{\xi}\right) d\mu \\ &\leq \frac{\xi}{4r} \int_{\Omega} \varphi\left(\frac{|y(t)-x(t)|}{2r}\right) d\mu = \frac{\xi}{2}. \end{aligned}$$

Hence

$$\|(y-x)_{G(\eta_0,y)}\|_\psi \leq \frac{\xi}{2}. \quad (5)$$

Thus from (1), (4) and (5) we get

$$\|y-x\|_\psi \leq \|(y-x)_{E(\eta_0,y)}\|_\psi + \|(y-x)_{G(\eta_0,y)}\|_\psi \leq \xi$$

and this means that  $y \in B_\psi(x, \xi)$ .

As an application of Lemma 4.1 we get

**Corollary 4.2.** *Let  $\varphi \in \Phi_1$ . Then for every  $r > 0$*

$$\mathcal{J}_{P|B_\varphi(r)}^\varphi = \mathcal{J}_{\mu|B_\varphi(r)}.$$

**Proof.** This equality is proved in [14, Theorem 1.4] for  $\varphi$  being a  $\varphi$ -function, but the proof can be applied for  $\varphi \in \Phi_{11}$ . For  $\varphi \in \Phi_{12}$  our equality follows Lemma 4.1, because

$$\mathcal{J}_{P|B_\varphi(r)}^\varphi = \sup\{\mathcal{J}_{\psi|B_\varphi(r)} : \psi \in \Psi_{12}^\varphi\} = \mathcal{J}_{\mu|B_\varphi(r)}.$$

In view of Corollary 4.2 and (2.2) we have:  $\mathcal{J}_P^\varphi \subset \mathcal{J}_I^\varphi$ . Repeating the arguments of the proof of [14, Theorem 2.2] we get

**Theorem 4.3.** *Let  $\varphi \in \Phi_1$ . If a sequence  $(x_n)$  in  $L^\varphi$  is modular convergent to  $x \in L^\varphi$  (i.e.,  $m_\varphi(\lambda(x_n - x)) \rightarrow 0$  for some  $\lambda > 0$ ) then  $x_n \rightarrow 0$  for  $\mathcal{J}_I^\varphi$*

It is well known that the set of all simple integrable functions  $\mathfrak{S}$  is dense in  $L^\varphi$  in the sense of modular convergence. Therefore, in view of the previous theorem and the inclusion  $\mathcal{J}_P^\varphi \subset \mathcal{J}_I^\varphi$  we get:

**Theorem 4.4.** *Let  $\varphi \in \Phi_1$ . The set of all simple integrable functions  $\mathfrak{S}$  is dense in  $L^\varphi$  with respect to  $\mathcal{J}_P^\varphi$  and  $\mathcal{J}_I^\varphi$ .*

Now we are in position to prove our main theorem.

**Theorem 4.5.** *Let  $\varphi \in \Phi_1$ . Then the equality*

$$\mathcal{J}_I^\varphi = \mathcal{J}_P^\varphi$$

*holds, i.e., for  $\varphi \in \Phi_{1i}$  ( $i = 1, 2$ ) the generalized mixed topology  $\mathcal{J}_I^\varphi$  is generated by the family  $\{ \|\cdot\|_\psi |_{L^\varphi} : \psi \in \Psi_{1i}^\varphi \}$ .*

**Proof.** For  $\varphi \in \Phi_{11}$  this equality is proved in [14, Theorem 2.4].

Next let  $\varphi \in \Phi_{12}$ . It is enough to show that the inclusion  $\mathcal{J}_I^\varphi \subset \mathcal{J}_P^\varphi$  holds. Since the spaces  $(L^\varphi, \mathcal{J}_P^\varphi)$  and  $(L^\varphi, \mathcal{J}_I^\varphi)$  are complete (see Theorems 2.1 and 3.9), in view of Theorem 4.4 and [4, Corollary of Lemma 4, p. 34] it suffices to show that

$$\mathcal{J}_{I|\mathfrak{S}}^\varphi \subset \mathcal{J}_{P|\mathfrak{S}}^\varphi.$$

To this end, in view of Definition 2.1, given a sequence of positive numbers  $(\varepsilon_n : n \geq 0)$  we shall find  $\psi_0 \in \Psi_{12}^\varphi$  (i.e.,  $\psi_0 \overset{s}{\prec} \varphi$ ) and  $r_0 > 0$  such that

$$B_{\psi_0}(r_0) \cap \mathfrak{S} \subset \bigcup_{N=0}^{\infty} \left( \sum_{n=0}^N (B_\varphi(2^n) \cap B_\mu(\varepsilon_n)) \right). \tag{1}$$

Without loss of generality we can assume that  $\varepsilon_n \downarrow 0$ ,  $\varepsilon_0 < 1$  and  $\varepsilon_0 \varphi(1) < 1$ . Moreover, for the reasons of convenience we can assume that  $\varphi(u) < \infty$  for  $u \leq 1$  and  $\varphi(u) = \infty$  for  $u > 1$ .

Let us choose subsequence  $(\varepsilon_{k_n})$  of  $(\varepsilon_n)$  in such a way that:

(a)  $k_0 = 0$ .

(b)  $k_1$  is the smallest natural number such that

$$\varphi(\varepsilon_{k_1}) < \frac{1}{\varepsilon_{k_1}}.$$

(c) Given  $k_n$  ( $n \geq 1$ ) we take  $k_{n+1} > k_n$  such that

$$\varphi(\varepsilon_{k_n})/2 > \varphi(\varepsilon_{k_{n+1}}) \quad \text{and} \quad \varepsilon_{k_n} < \varepsilon_{n+1}/2.$$

Let

$$N(t) = \sup\{n \in \mathcal{N} : \varepsilon_{k_n} \geq t\} \quad \text{for } t \in \varepsilon_{k_1}.$$

We shall now define a  $\varphi$ -function  $\psi_0$  such that  $\psi_0 \overset{s}{\ll} \varphi$  and

$$\psi_0(u) \geq \frac{1}{N(u)} \varphi\left(\frac{u}{N(u)}\right) \quad \text{for } 0 \leq u \leq \varepsilon_{k_2}, \quad (2)$$

$$\psi_0(n) \geq \frac{1}{\varepsilon_{n+1}} \quad \text{for } n \geq 1. \quad (3)$$

Let us denote by:

$$A'_n = \{t > 0 : \varepsilon_{k_{n+1}} < t \leq \varepsilon_{k_n}\}, \quad n = 1, 2, \dots,$$

$$A = \{t > 0 : \varepsilon_{k_1} < t < 1\},$$

$$A''_n = \{t > 0 : n \leq t < n + 1\}, \quad n = 1, 2, \dots,$$

and

$$B'_n = \{s > 0 : \varphi(\varepsilon_{k_{n+1}}) < s \leq \varphi(\varepsilon_{k_n})\}, \quad n = 1, 2, \dots,$$

$$B = \{s > 0 : \varphi(\varepsilon_{k_1}) < s < \varphi(1)/2\},$$

$$B''_n = \{s > 0 : \varphi(1) n/2 \leq s < \varphi(1)(n + 1)/2\}, \quad n = 1, 2, \dots.$$

Let us put

$$p(t) = \begin{cases} 0 & \text{for } t = 0, \\ \frac{2}{n-1} & \text{for } t \in A'_n, n \geq 2, \\ 2 & \text{for } t \in A'_1 \cup A \cup A''_1, \\ n & \text{for } t \in A''_n, n \geq 2, \end{cases}$$

and

$$q(s) = \begin{cases} 0 & \text{for } s = 0, \\ \frac{2}{n-1} & \text{for } s \in B'_n, n \geq 3 \\ 1 & \text{for } s \in B'_2 \cup B'_1 \cup B, \\ \frac{2}{\varphi(1)\varepsilon_{n+1}} & \text{for } s \in B''_n, n \geq 1. \end{cases}$$

Next, define for  $u \geq 0$  and  $v \geq 0$

$$\xi(u) = \int_0^u p(t) dt \quad \text{and} \quad \eta(v) = \int_0^v q(s) ds.$$

Let us put

$$\varphi_0(u) = \begin{cases} \varphi(u) & \text{for } u \leq 1 \\ \varphi(1)u & \text{for } u > 1 \end{cases}$$

At last let

$$\psi_0(u) = \eta(\varphi_0(\xi(u))) = \int_0^{\varphi_0(\xi(u))} q(s) ds \quad \text{for } u \geq 0.$$

Similarly as in the proof of [14, Theorem 2.4] we can show that  $\psi_0 \overset{s}{\prec} \varphi$  and that the conditions (2) and (3) hold.

Now let us put

$$r_0 = \min\left(\frac{1}{2}, d_0 \varepsilon_0 \varphi(\varepsilon_0)\right) \quad (4)$$

Where

$$\frac{1}{d_0} = \sup\left\{\frac{\varphi(u)}{\psi_0(u)} : \varepsilon_{k_2} < u \leq 1\right\}.$$

We shall now show that the inclusion (1) holds. Indeed, let

$$x = \sum_{i \in I} \lambda_i \chi_{H_i} \in B_{\psi_0}(r_0)$$

where  $I$  is a finite subset of  $\mathcal{N}$ , and  $\mu(H_i) < \infty$ . Denote by

$$K = \{i \in I : \varepsilon_{k_2} < |\lambda_i| \leq 1\},$$

$$L = \{i \in I : |\lambda_i| \leq \varepsilon_{k_2}\}, J = \{i \in I : |\lambda_i| > 1\}.$$

Let

$$x_1 = \sum_{i \in K} \lambda_i \chi_{H_i}, \quad x_2 = \sum_{i \in L} \lambda_i \chi_{H_i}, \quad x_3 = \sum_{i \in J} \lambda_i \chi_{H_i}.$$

Since  $x \in B_{\psi_0}(r_0)$  and  $r_0 < 1$  we have

$$m_{\psi_0}(x) = \sum_{i \in I} \psi_0(|\lambda_i|) \mu(H_i) = c \leq r_0.$$

Write

$$c_i = \psi_0(|\lambda_i|) \mu(H_i).$$

Arguing as in the proof of [14, Theorem 2.4] we get

$$x_1 \in B_{\varphi}(1) \cap B_{\mu}(\varepsilon_0) \quad (5)$$

and moreover, using (2) we can find  $N_1 \in \mathcal{N}$  such that

$$x_2 \in \sum_{n=1}^{N_1} \left( B_{\varphi}\left(\frac{1}{2} 2^n\right) \cap B_{\mu}\left(\frac{1}{2} \varepsilon_n\right) \right). \quad (6)$$



Now we shall find  $N_2 \in \mathcal{N}$  such that

$$x_3 \in \sum_{n=1}^{N_2} \left( B_\varphi\left(\frac{1}{2} 2^n\right) \cap B_\mu\left(\frac{1}{2} \varepsilon_n\right) \right). \quad (7)$$

Let

$$n_i = \sup\{n \in \mathcal{N} : n \leq |\lambda_i|\} \text{ for } i \in J.$$

Then  $n_i \geq 1$  and  $n_i \leq |\lambda_i| < n_i + 1$ . Let  $j_1, \dots, j_{m_0}$  be the different numbers in the set  $\{n_i : i \in J\}$  and let us assume that  $j_1 < \dots < j_{m_0}$ . Write

$$J_l = \{i \in J : n_i = j_l\} \quad \text{for } 1 \leq l \leq m_0.$$

Then

$$x_3 = \sum_{i \in J} \lambda_i \chi_{H_i} = \sum_{l=1}^{m_0} \left( \sum_{i \in J_l} \lambda_i \chi_{H_i} \right).$$

and let

$$y_l = \sum_{i \in J_l} \lambda_i \chi_{H_i} \text{ for } 1 \leq l \leq m_0.$$

Then  $j_l \leq |\lambda_i| < j_l + 1$  for  $i \in J_l$  and using (4) we get

$$\begin{aligned} m_\varphi(y_l/(j_l + 1)) &= \sum_{i \in J_l} \varphi(|\lambda_i|/(j_l + 1)) \mu(H_i) \\ &\leq d_0^{-1} \sum_{i \in J_l} \psi_0(|\lambda_i|/(j_l + 1)) \mu(H_i) \leq d_0^{-1} r_0 < j_l + 1. \end{aligned}$$

Thus

$$y_l \in B_\varphi(j_l + 1) \subset B_\varphi\left(\frac{1}{2} 2^{j_l + 1}\right). \quad (8)$$

Let

$$E_{y_l}(\varepsilon) = \{t \in \Omega : |y_l(t)| > \varepsilon\} \text{ for any } \varepsilon > 0.$$

Then

$$E_{y_l}\left(\frac{1}{2} \varepsilon_{j_l + 1}\right) = \bigcup_{i \in J_l} H_i.$$

Hence, using (3) we get

$$\begin{aligned} \mu(E_{y_i}(\frac{1}{2} \varepsilon_{j_i+1})) &\leq \sum_{i \in J_i} \mu(H_i) \leq \sum_{i \in J_i} c_i / \psi_0(|\lambda_i|) \\ &\leq \sum_{i \in J_i} c_i / \psi_0(j_i) \leq (\sum_{i \in J_i} c_i) / \psi_0(j_i) \leq \frac{1}{2} \varepsilon_{j_i+1} \end{aligned}$$

and this means that

$$y_i \in B_\mu(\frac{1}{2} \varepsilon_{j_i+1}). \quad (9)$$

Thus from (8) and (9) we have

$$y_i \in B_\varphi(\frac{1}{2} 2^{j_i+1}) \cap B_\mu(\frac{1}{2} \varepsilon_{j_i+1}).$$

Hence for  $N_2 = j_{m_0} + 1$  we obtain

$$x_3 \in \sum_{l=1}^{m_0} (B_\varphi(\frac{1}{2} 2^{j_l+1}) \cap B_\mu(\frac{1}{2} \varepsilon_{j_l+1})) \subset \sum_{n=1}^{N_2} (B_\varphi(\frac{1}{2} 2^n) \cap B_\mu(\frac{1}{2} \varepsilon_n)).$$

At last, using (5), (6) and (7), for  $N_0 = \max(N_1, N_2)$  we get

$$\begin{aligned} x = x_1 + x_2 + x_3 &\in B_\varphi(1) \cap B_\mu(\varepsilon_0) + \sum_{n=1}^{N_0} (B_\varphi(2^n) \cap B_\mu(\varepsilon_n)) \\ &\subset \bigcup_{N=0}^{\infty} \left( \sum_{n=0}^N (B_\varphi(2^n) \cap B_\mu(\varepsilon_n)) \right). \end{aligned}$$

Thus the proof is completed.

**5. A TOPOLOGICAL CHARACTERIZATION OF THE  $\gamma_\varphi$ -CONVERGENCE IN  $L^\varphi$**

In this section by applying of Theorem 4.5 we obtain a topological characterization of the  $\gamma_\varphi$ -convergence in  $L^\varphi$ .

**Theorem 5.1.** *Let  $\varphi \in \Phi_{1i}$  ( $i = 1, 2$ ). Then for a sequence  $(x_n)$  in  $L^\varphi$  and  $x \in L^\varphi$  the following statements are equivalent:*

- (i)  $x_n \rightarrow x$  for  $\mathcal{J}_1^\varphi$ .
- (ii)  $\|x_n - x\|_\psi \rightarrow 0$  for every  $\psi \in \Psi_{1i}^\varphi$ .
- (iii)  $x_n \rightarrow x$  ( $\mu - \Omega$ ) and  $\sup_n \|x_n\|_\varphi < \infty$ .

Moreover, for  $\varphi \in \Phi_{1i}^c$  ( $i = 1, 2, 3, 4$ ) the above statements are equivalent to

- (iv)  $\| \|x_n - x\| \|_\psi \rightarrow 0$  for every  $\psi \in \Psi_{1i}^\varphi(N)$ .

**Proof.** (i)  $\Leftrightarrow$  (ii)  $\Leftrightarrow$  (iii). It follows from Theorem 2.4. and Theorem 4.5.

(i)  $\Leftrightarrow$  (iv). It follows from Theorem 4.5 and Theorem 3.12.

Now we apply the above theorem to some classes of Orlicz spaces. Let

$$\chi_p(u) = u^p \text{ for } u \geq 0 \text{ and } \chi_\infty(u) = \begin{cases} 0 & \text{for } 0 \leq u \leq 1, \\ \infty & \text{for } u > 1. \end{cases}$$

Let  $\|\cdot\|_p$  and  $\|\cdot\|_\infty$  denote the usual norms in  $L^p$  ( $p \geq 1$ ) and  $L^\infty$  respectively.

**Examples**

A. For  $p \geq 1$  let

$$\varphi(u) = \begin{cases} u^p & \text{for } 0 \leq u \leq 1, \\ \infty & \text{for } u > 1. \end{cases}$$

Hence  $\varphi(u) = \chi_p(u) \vee \chi_\infty(u)$  for  $u \geq 0$ , so  $L^\varphi = L^p \cap L^\infty$  by Lemma 1.1. We see that  $\varphi \in \Phi_{14}^c$  for  $p = 1$  and  $\varphi \in \Phi_{12}^c$  for  $p > 1$ . Hence by applying of Theorem 5.1 and Lemma 1.1 we get the following two theorems:

**Theorem 5.2.** *For a sequence  $(x_n)$  in  $L^1 \cap L^\infty$  and  $x \in L^1 \cap L^\infty$  the following statements are equivalent:*

- (i)  $x_n \rightarrow x$  ( $\mu - \Omega$ ) and  $\sup_n \|x_n\|_1 < \infty$ ,  $\sup_n \|x_n\|_\infty < \infty$ .
- (ii)  $\| \|x_n - x\| \|_\psi \rightarrow 0$  for every  $N$ -function  $\psi$ .

**Theorem 5.3.** *Let  $p > 1$ . For a sequence  $(x_n)$  in  $L^p \cap L^\infty$  and  $x \in L^p \cap L^\infty$  the following statements are equivalent:*

- (i)  $x_n \rightarrow x$  ( $\mu - \Omega$ ) and  $\sup_n \|x_n\|_p < \infty$ ,  $\sup_n \|x_n\|_\infty < \infty$ .
- (ii)  $\| \|x_n - x\| \|_\psi \rightarrow 0$  for every  $N$ -function  $\psi$  such that  $\psi(u)/u^p \rightarrow 0$  as  $u \rightarrow 0$ .

B. For  $p > 1$  let

$$\varphi(u) = \begin{cases} u & \text{for } 0 \leq u \leq 1, \\ u^p & \text{for } u > 1. \end{cases}$$

Then  $\varphi(u) = \chi_1(u) \vee \chi_p(u)$  for  $u \geq 0$ , so  $L^\varphi = L^1 \cap L^p$ . Then  $\varphi \in \Phi_{13}^c$  and using Theorem 5.1 and Lemma 1.1 we get:

**Theorem 5.4.** *Let  $p > 1$ . For a sequence  $(x_n)$  in  $L^1 \cap L^p$  and  $x \in L^1 \cap L^p$  the following statements are equivalent:*

- (i)  $x_n \rightarrow x$  ( $\mu - \Omega$ ) and  $\sup_n \|x_n\|_1 < \infty$ ,  $\sup_n \|x_n\|_p < \infty$ .
- (ii)  $\| \|x_n - x\| \|_\psi \rightarrow 0$  for every  $N$ -function  $\psi$  such that  $\psi(u)/u^p \rightarrow 0$  as  $u \rightarrow \infty$ .

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Institute of Mathematics,  
Adam Mickiewicz University  
Matejki 48/49,  
60-769 Poznan,  
Poland.

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