Comparison Results for a Class of Variational Inequalities

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ABSTRACT. In this paper we study a variational inequality related to a linear differential operator of elliptic type. We give a pointwise bound for the rearrangement of the solution u, and an estimate for the L^2-norm of the gradient of u.

1. INTRODUCTION

Let Ω be an open bounded set of $\mathbb{R}^n$ and let

$$Au = -(a_{ij}(x)u_{x_i})_{x_j} + (b_i(x)u)_{x_i} + c(x)u$$

be a differential operator whose coefficients satisfy:

(1.1) $\qquad \begin{cases} a_{ij}, \ b_i, \ c \in L^\infty(\Omega) \\ a_{ij}(x)\xi_i\xi_j \geq |\xi|^2, \qquad \text{for a.e. } x \in \Omega, \ \ \xi \in R^n \\ \Sigma b_i^2 \leq B^2. \end{cases}$

Let $u \in H_0^1(\Omega)$ be a solution of the related variational inequality

(1.2) $\quad a(u, U-u) \geq \displaystyle\int_\Omega f(x)(U-u)dx \qquad \forall \ U \in H_0^1(\Omega), \ U, \ u \geq 0$

where

$$a(\varphi, \psi) = \int_\Omega (a_{ij}(x)\varphi_{x_i}\psi_{x_j} - b_i(x)\varphi\psi_{x_i} + c(x)\varphi\psi)\,dx,$$

and $f \in L^2(\Omega)$.

Schwarz symmetrization allows us to obtain sharp estimates for a solution u of (1.2) comparing u with the solution $v \in H_0^1(\Omega^*)$ of the «symmetrized» variational inequality

(1.3)
$$a^*(v, V-v) = \int_{\Omega^*} v_{x_i}(V-v)_{x_i} + B\frac{x_i}{|x|}v(V-v)_{x_i} - c_-^{\#}(x)v(V-v)\,dx \ge$$

$$\ge \int_{\Omega^*} f^*(x)(V-v)\,dx \qquad \forall\, V \in H_0^1(\Omega^*), \qquad V, v \ge 0$$

where Ω^* is the n-dimensional ball centered at the origin with measure $|\Omega|$, $f^*(x)$, $c_-^{\#}(x)$ are the spherically symmetric decreasing rearrangements of $f(x)$ and $c_-(x) = \mathrm{Max}(-c(x), 0)$, respectively (see § 2).

More precisely in this paper we show (see theorem 2.2) that the pointwise estimate

(1.4) $u^*(x) \le v(x)$

holds for all $x \in \Omega^*$. Such an estimate gives an optimal upper bound for the L^p-norm of u and for the measure of the set $\{x : u(x) > 0\}$ or, that is the same, an optimal lower bound for the coincidence set of u.

As consequence of (1.4) we derive an estimate for $a(u, u)$ and for the L^2-norm of ∇u in terms of $a^*(v, v)$ and the L^2-norm of ∇v respectively. Finally the previous results are applied to the obstacle problem when the obstacle is in $H_0^1(\Omega)$.

The first results in this direction, in the case $b_i = 0$, $c \ge 0$, are due to Bandle-Mossino [BM] and Maderna-Salsa [MS]. In particular in [BM] the case of a non-linear operator is considered. Subsequently Alvino-Matarasso-Trombetti [AMT] considered a variational inequality of general form taking into account also the influence of the term $c_+(x) = \max(c(x), 0)$, but with constraints on the coefficients different from (1.1). As far as variational parabolic inequalities concern we recall [DM].

The method used was introduced by Talenti [Ta1] for an elliptic equation without lower-order terms and was extended to elliptic equations of more general form by several authors (see i.e. [AT], [ALT2], [Ba], [FP], [GT], [Ta2], [TV]).

2. MAIN RESULTS

If Ω is an open bounded set of $\mathbb{R}^n$, we will denote by $|\Omega|$ its measure and by Ω^* the ball of $\mathbb{R}^n$ centered at the origin whose measure is $|\Omega|$. Moreover if φ is a measurable function,

$$\mu(t) = |\{x \in \Omega : \varphi(x) > t\}|, \qquad t \in \mathbb{R},$$

is the distribution function of φ and

$$\varphi^*(s) = \sup\{t \in \mathbb{R} : \mu(t) > s\}, \qquad s \in [0, |\Omega|[,$$

is its decreasing rearrangement. If C_n is the measure of the n-dimensional unit ball,

$$\varphi^*(x) = \varphi^*(C_n|x|^n), \qquad x \in \Omega^*,$$

is the spherically symmetric decreasing rearrangement of $\varphi(x)$. For an exhaustive treatment of rearrangements we refer for example to [ALT1], [Ba], [CR], [HLP], [Ka], [Mo]. Here we just recall the well known Hardy inequality:

$$\int_\Omega u(x)v(x)dx \leq \int_0^{|\Omega|} u^*(s)v^*(s)\,\mathrm{d}s$$

where $u(x)$, $v(x)$ are measurable functions.

The firts step to get the comparison result is to obtain a differential inequality involving the decreasing rearrangement of u. In the following we will consider $f_+, f_- \not\equiv 0$. In the other cases proofs are simpler.

Theorem 2.1. *Let u be a solution of the variational inequality (1.2) with conditions (1.1), then*

(2.1)
$$-u^{*\prime}(s) \leq \frac{s^{-2+2/n}}{n^2 C_n^{2/n}} \int_0^s (f^*(\sigma) + c_{\pm}^*(\sigma)u^*(\sigma))d\sigma +$$

$$+ \frac{B s^{-1+1/n}}{n C_n^{1/n}} u^*(s) \text{ a.e. in } [0, |u>0|].$$

Proof.   Following [BM] we choose as test function U in (1.2) $u \pm \Phi_h$ where

$$\Phi_h(x) = \begin{cases} h & \text{if } u > t+h \\ (u(x) - t) & \text{if } t < u(x) \le t+h \\ 0 & \text{otherwise} \end{cases}$$

with $h > 0$ and $t \in [0, \sup u[$.

We have

$$\frac{1}{h} a(u, \Phi_h) = \frac{1}{h} \int_\Omega f \Phi_h dx$$

and by ellipticity condition, letting h go to 0, we obtain

$$-\frac{d}{dt} \int_{u>t} |\nabla u|^2 dx \le -\frac{d}{dt} \int_{u>t} b_i(x) u_{x_i} u \, dx + \int_{u>t} (f(x) - c(x)u) dx.$$

From now on we give a sketch of the proof because the tools are standard. We have (see [ALT2])

$$-\frac{d}{dt} \int_{u>t} b_i(x) u_{x_i} u \, dx \le Bt \left(-\frac{d}{dt} \int_{u>t} |\nabla u| dx \right)$$

and by Hardy inequality

$$\int_{u>t} (f(x) - c(x)u(x)) dx \le \int_0^{\mu(t)} (f^*(\sigma) + c_*^*(\sigma) u^*(\sigma)) d\sigma.$$

Then

(2.2)
$$-\frac{d}{dt} \int_{u>t} |\nabla u|^2 dx - Bt \left(-\frac{d}{dt} \int_{u>t} |\nabla u| dx \right) \le$$

$$\le \int_0^{\mu(t)} (f^*(\sigma) + c_*^*(\sigma) u^*(\sigma)) d\sigma.$$

Isoperimetric inequality [DG], Fleming-Rishel formula [FR] and Schwarz inequality give (see [ALT2], [Ta2])

$$n C_n^{1/n} \mu(t)^{1-1/n} \le -\frac{d}{dt} \int_{u>t} |\nabla u| \le$$

(2.3)
$$\leq [-\mu'(t)]^{1/2}\left(-\frac{d}{dt}\int_{u>t}|\nabla u|^2\right)^{1/2}.$$

Now we observe that the right hand side of (2.2) is non negative. As a matter of fact, letting t go to 0, from (2.2) we get

$$\int_0^{\mu(0)}(f^*(\sigma)+c_\pm^*(\sigma)u^*(\sigma))d\sigma \geq 0$$

and then

(2.4) $$\int_0^{\mu(t)}(f^*(\sigma)+c_\pm^*(\sigma)u^*(\sigma))d\sigma \geq 0 \qquad \forall\, t\in[0,\sup u]$$

because $f^*(\sigma)+c_\pm^*(\sigma)u^*(\sigma)$ is a decreasing function.

Hence by (2.3) and (2.4) we have

$$\int_0^{\mu(t)}(f^*(\sigma)+c_\pm^*(\sigma)u^*(\sigma))d\sigma \leq$$

$$\leq \frac{\mu(t)^{-1+1/n}}{nC_n^{1/n}}[-\mu'(t)]^{1/2}\left(-\frac{d}{dt}\int_{u>t}|\nabla u|^2\right)^{1/2}\int_0^{\mu(t)}(f^*(\sigma)+ \\ +c_\pm^*(\sigma)u^*(\sigma))d\sigma.$$

Then by (2.2) and (2.3)

$$\left(-\frac{d}{dt}\int_{u>t}|\nabla u|^2dx\right)^{1/2}\leq$$

(2.5)
$$\leq Bt[-\mu'(t)]^{1/2}+[-\mu'(t)]^{1/2}\frac{\mu(t)^{-1+1/n}}{nC_n^{1/n}}\int_0^{\mu(t)}(f^*(\sigma)+c_\pm^*(\sigma)u^*(\sigma))d\sigma.$$

Using again (2.3), we obtain

$$\frac{nC_n^{1/n}\mu(t)^{1-1/n}}{[-\mu'(t)]^{1/2}}\leq Bt[-\mu'(t)]^{1/2}+$$

$$+[-\mu'(t)]^{1/2}\frac{\mu(t)^{-1+1/n}}{nC_n^{1/n}}\int_0^{\mu(t)}(f^*(\sigma)+c_\pm^*(\sigma)u^*(\sigma))d\sigma.$$

Making a change of variables we get (2.1). $\square$

We consider the variational inequality (1.3) whose data are spherically symmetric. If its solution v is spherically symmetric and decreasing, then repeating the above proof, it is easy to verify that all the inequalities used to get (2.1) are equalities. As in [AMT] a way to establish the existence of such a solution is given by the following proposition:

Proposition 2.1.   *Let us consider the differential operator*

$$A^* v = -\Delta v - B\left(v\frac{x_i}{|x|}\right)_{x_i} - c_-^*(x)v.$$

Let us suppose $c_-^* = 0$ or let any one of the following conditions be satisfied:

(i)   *there exists a non negative function $H \not\equiv 0$ such that the Dirichlet problem*

$$\begin{cases} A^* Z = H & \text{in } \Omega^* \\ Z \in H_0^1(\Omega^*) \end{cases}$$

has a non negative solution Z;

(ii)   *the first eigenvalue of the problem*

$$(2.6) \qquad \begin{cases} A^* \varphi = \lambda p(x)\varphi & \text{in } \Omega^* \\ \varphi \in H_0^1(\Omega^*) \end{cases}$$

where $p \in L^{n/2}(\Omega^*)$, $p(x) \not\equiv 0$ is a non negative function, is positive;

(iii)   *there exists $\alpha > 0$ such that*

$$\int_{\Omega^*} e^{B|x|}\left(|\nabla \Phi|^2 - B\frac{(n-1)}{|x|}\Phi^2 - c_-^*\Phi^2\right)dx \geq$$

$$\geq \alpha \int_{\Omega^*} e^{B|x|}|\nabla \Phi|^2 dx \quad \forall\, \Phi \in H_0^1(\Omega^*).$$

Then (1.3) has a unique solution $v = v^*$. Moreover for A^* a maximum principle holds, that is

(2.7) $A^* V \geq 0$ *in* Ω^*, $V \geq 0$ *on* $\partial\Omega^* \Rightarrow V \geq 0$ *in* Ω^*.

The proof of this proposition will be given in the appendix.

The above arguments yield to state the following

Proposition 2.2. *If one of the conditions of proposition 2.1 is verified then the solution* $v = v^*$ *of (1.3) satisfies*

$$-v^{*\prime}(s) = \frac{s^{-2+2/n}}{n^2 C_n^{2/n}} \int_0^s (f^*(\sigma) + c_-^*(\sigma) v^*(\sigma)) d\sigma +$$

(2.8)

$$+ \frac{B s^{-1+1/n}}{n C_n^{1/n}} v^*(s) \qquad a.e. \ in \ [0, |v > 0|].$$

At this point we are able to prove the comparison result.

Theorem 2.2. *If one of the conditions of proposition 2.1 is verified and u, v are solutions respectively of (1.2), (1.3) with the assumptions (1.1), then*

$$u^*(s) \leq v^*(s) \qquad in \ [0, |\Omega|].$$

Proof. Starting from (2.1) and (2.8) we show that, setting $w = u^* - v^*$, we will show that

(2.9) $-w'(s) \leq B \dfrac{s^{-1+1/n}}{n C_n^{1/n}} w + \dfrac{s^{-2+2/n}}{n^2 C_n^{2/n}} \displaystyle\int_0^s c_-^*(\sigma) w(\sigma) d\sigma$ in $[0, |u > 0|]$.

Obviously (2.9) holds for all $s \in [0, \min\{|u > 0|, |v > 0|\}]$, then it is trivial if $|u > 0| \leq |v > 0|$. We suppose $|u > 0| > |v > 0|$. By regularity results (see [BS]) the solution of (1.3) is in $H^2(\Omega^*)$, then $v^* \in C^1(]0, |\Omega|])$ and $v^{*\prime}(|v > 0|) = 0$.

Hence by (2.8)

(2.10) $\displaystyle\int_0^{|v > 0|} f^*(\sigma) d\sigma + \int_0^{|v > 0|} c_-^*(\sigma) v^*(\sigma) d\sigma = 0.$

By (2.1), taking into account that $w(s) = u^*(s)$ for $|v > 0| \leq s \leq |u > 0|$ and (2.10), we get

$$-w'(s) \leq B \frac{s^{-1+1/n}}{nC_n^{1/n}} w + \frac{s^{-2+2/n}}{n^2 C_n^{2/n}} \int_0^s c_{\pm}^*(\sigma) w(\sigma) d\sigma + \frac{s^{-2+2/n}}{n^2 C_n^{2/n}} \int_{|v>0|}^s f^*(\sigma) d\sigma$$

for $|v > 0| \leq s \leq |u > 0|$.

Then we obtain (2.9) observing that by (2.10)

$$\int_0^{|v>0|} f^*(\sigma) d\sigma \leq 0,$$

which implies $f^*(s) < 0$ in $[|v > 0|, |\Omega|]$ because f^* is a decreasing function. To get the thesis we first suppose $c_{\pm}^* > 0$. Setting $V = \int_0^s c_{\pm}^*(\sigma) w(\sigma) d\sigma$, (2.9) can be written as

$$(2.11) \quad \begin{cases} -\left(\exp\left(\frac{Bs^{1/n}}{C_n^{1/n}} \right) \frac{V'}{c_{\pm}^*} \right)' - \frac{s^{-2+2/n}}{n^2 C_n^{2/n}} \exp\left(\frac{Bs^{1/n}}{C_n^{1/n}} \right) V \leq 0 \text{ in } [0, |u > 0|] \\ V(0) = 0 \\ V'(|u > 0|) \leq 0. \end{cases}$$

We will show that $V \leq 0$. If we suppose ab absurdo $V_+ \not\equiv 0$, then there exists $\bar{s}$ such that $V(\bar{s}) > 0$, $V'(\bar{s}) = 0$ and $V_+ \not\equiv 0$ in $[0, \bar{s}]$. We denote by $\tilde{B}$ the ball centered at the origin whose measure is $\bar{s}$ and we consider the eigenvalue problem

$$(2.12) \quad \begin{cases} A^* \varphi = \lambda c_-^*(x) \varphi & \text{in } \tilde{B} \\ \varphi \in H_0^1(\tilde{B}). \end{cases}$$

If we observe that

$$A^* \varphi = \lambda c_-^*(x) \varphi \Leftrightarrow -(e^{B|x|} \varphi_{x_i})_{x_i} - e^{B|x|} B \frac{(n-1)}{|x|} \varphi - e^{B|x|} c_-^*(x) \varphi = e^{B|x|} \lambda c_-^*(x) \varphi$$

then the first eigenvalue of (2.12) can be characterized by

$$\lambda_1 = \min_{\Phi \in H_0^1(\tilde{B})} \frac{\int_{\tilde{B}} e^{B|x|} (|\nabla \Phi|^2 - B \frac{(n-1)}{|x|} \Phi^2 - c_-^*(x) \Phi^2) dx}{\int_{\tilde{B}} e^{B|x|} c_-^*(x) \Phi^2 dx}.$$

Since $\bar{B} \subset \Omega^{*}$, by (ii) of proposition 2.1 we find $\lambda_1 > 0$. On the other hand if φ_1 is the eigenfunction corresponding to λ_1, we have

$$-\varphi_1'(s) - B \frac{s^{-1+1/n}}{nC_n^{1/n}} \varphi_1(s) - \frac{s^{-2+2/n}}{n^2 C_n^{2/n}} \int_0^s c_-^{*}(\sigma)\varphi_1(\sigma)d\sigma =$$

$$= \lambda_1 \frac{s^{-2+2/n}}{n^2 C_n^{2/n}} \int_0^s c_-^{*}(\sigma)\varphi_1(\sigma)d\sigma \qquad \text{a.e. in } [0,\bar{s}].$$

Hence setting $\Phi = \int_0^s c_-^{*}(\sigma)\varphi_1(\sigma)d\sigma$, it is easy to verify that λ_1 is the same as the first eigenvalue of the problem

(2.13)
$$\left| \begin{array}{l} -\left(\exp\left(\dfrac{Bs^{1/n}}{C_n^{1/n}}\right)\dfrac{\Phi'}{c_-^{*}}\right)' - \dfrac{s^{-2+2/n}}{n^2 C_n^{2/n}}\exp\left(\dfrac{Bs^{1/n}}{C_n^{1/n}}\right)\Phi = \\[3mm] = \lambda_1 \dfrac{s^{-2+2/n}}{n^2 C_n^{2/n}}\exp\left(\dfrac{Bs^{1/n}}{C_n^{1/n}}\right)\Phi \\[3mm] \Phi(0) = \Phi'(\bar{s}) = 0. \end{array} \right.$$

By (2.13), using the variational characterization of λ_1 and (2.11), we get

$$\lambda_1 \le \frac{\displaystyle\int_0^{\bar{s}} \exp\left(\frac{Bs^{1/n}}{C_n^{1/n}}\right)\frac{V_+'^2}{c_-^{*}} - \frac{s^{-2+2/n}}{n^2 C_n^{2/n}}\exp\left(\frac{Bs^{1/n}}{C_n^{1/n}}\right)V_+^2\,ds}{\displaystyle\int_0^{\bar{s}} \frac{s^{-2+2/n}}{n^2 C_n^{2/n}}\exp\left(\frac{Bs^{1/n}}{C_n^{1/n}}\right)V_+^2\,ds} \le 0$$

which gives the absurd.

Using again (2.11) we have

$$\left\{ \begin{array}{l} -\left(\exp\left(\dfrac{Bs^{1/n}}{C_n^{1/n}}\right)\dfrac{V'}{c_-^{*}}\right)' \le 0 \\[3mm] V(0) = 0 \\ V'(|u>0|) \le 0. \end{array} \right.$$

Integrating between s and $|u>0|$, we obtain $V'(s) \le 0$, that is $w(s) \le 0$. The result in the case $c_-^{*} \ge 0$ can be proved by approximation techniques. $\square$

Making use of the techniques of the proof of theorem 2.1 it is possible to estimate the L^2-norm of ∇u. We have the following

Theorem 2.3. *If one of the conditions of proposition 2.1 is verified and u, v are solution respectively of (1.2), (1.3) with the assumptions (1.1), then*

$$\int_\Omega |\nabla u|^2 dx \leq \int_{\Omega^*} |\nabla v|^2 dx.$$

Proof. Starting from (2.5) and squaring both sides we get

$$-\frac{d}{dt}\int_{u>t} |\nabla u|^2 dx \leq$$

$$\leq -\mu'(t)\left[Bt + \frac{\mu(t)^{-1+1/n}}{nC_n^{1/n}}\int_0^{\mu(t)} (f^*(\sigma) + c_\pm^*(\sigma)u^*(\sigma))d\sigma\right]^2.$$

Integrating between 0 and sup u and making a change of variables we obtain

$$\int_\Omega |\nabla u|^2 dx \leq$$

$$\leq \int_0^{|u>0|}\left[Bu^*(s) + \frac{s^{-1+1/n}}{nC_n^{1/n}}\int_0^s (f^*(\sigma) + c_\pm^*(\sigma)u^*(\sigma))d\sigma\right]^2 ds.$$

By Theorem 2.2 we have $u^*(s) \leq v^*(s)$, which implies also $|u>0| \leq |v>0|$, therefore we conclude observing that

$$\int_{\Omega^*} |\nabla v|^2 dx =$$

$$= \int_0^{|v>0|}\left[Bv^*(s) + \frac{s^{-1+1/n}}{nC_n^{1/n}}\int_0^s f^*(\sigma) + c_\pm^*(\sigma)v^*(\sigma)d\sigma\right]^2 ds. \qquad \square$$

Using the previous results we can get also an estimate for $a(u,u)$.

Proposition 2.3. *Under the same assumptions of theorem 2.2, the following estimates hold*

(α) $a(u,u)+\displaystyle\int_{\Omega}c_-(x)u^2dx\leq a^*(v,v)+\displaystyle\int_{\Omega^*}c_-^*(x)v^2dx;$

(β) $a(u,u)\leq a^*(v,v)$ $\text{if } b_i(x)=0.$

Proof. If in (1.2) we choose as test function $U=0$ and $U=2u$ we obtain

$$a(u,u)=\int_{\Omega}f(x)u\,dx.$$

By Hardy inequality we have

$$a(u,u)+\int_{\Omega}c_-(x)u^2dx\leq\int_0^{|\Omega|}(f^*(s)u^*+c_-^*(s)u^{*2})ds,$$

and integration by parts gives

$$\int_0^{|\Omega|}(f^*(s)+c_-^*(s)u^*)u^*ds=\int_0^{|\Omega|}\int_0^s(f^*(t)+c_-^*(t)u^*)dt(-du^*(s)).$$

By (2.4) we have

$$\int_0^s(f^*(t)+c_-^*(t)u^*)dt\geq 0,$$

moreover (2.1), (2.8) and theorem 2.1 imply

$$-\frac{du^*}{ds}\leq-\frac{dv^*}{dv}.$$

Then

$$\int_0^{|\Omega|}(f^*(s)u^*+c_-^*(s)u^{*2})ds\leq\int_0^{|\Omega|}\int_0^s(f^*(t)+c_-^*(t)v^*)dt(-dv^*(s))=$$

$$=\int_0^{|\Omega|}(f^*(s)+c_-^*(s)v^*)v^*ds=a^*(v,v)+\int_{\Omega^*}c_-^*(x)v^2dx,$$

that is (α) or, which is the same, (β) when $c_-(x)=0.$

If $b_i(x)=0$, then the bilinear form $a^*(u,v)$ is symmetric and hence we get

$$
(2.14) \qquad a(u,u) = \int_\Omega f(x)u\,dx \le \int_{\Omega^*} f^*(x)u^*dx = a^*(v,u^*) \le
$$

$$
\le \sqrt{a^*(v,v)}\ \sqrt{a^*(u^*,u^*)}.
$$

Ellipticity condition, Polya Szego and Hardy inequalities give

$$
(2.15) \qquad a^*(u^*,u^*) = \int_{\Omega^*} |\nabla u^*|^2 dx - \int_{\Omega^*} c_-^*(x)u^{*2}dx \le
$$

$$
\le \int_\Omega |\nabla u|^2 dx - \int_\Omega c_-(x)u^2 dx \le a(u,u).
$$

By (2.14) and (2.15) (β) follows. $\square$

Remark.   We can apply the previous results to the obstacle problem

$$
(2.16) \qquad a(u,\varphi-u) \ge \int_\Omega f(\varphi-u) \qquad \forall\, \varphi \in H_0^1(\Omega)\ \varphi, u \ge \psi
$$

where the obstacle $\psi \in H_0^1(\Omega)$ and

$$
A\psi = -(a_{ij}(x)\psi_{x_i})_{x_j} + (b_i(x)\psi)_{x_i} + c(x)\psi \qquad \text{in } L^2(\Omega).
$$

Setting $\bar{u} = u - \psi$ and $g = f - A\psi$ we have that $\bar{u}$ satisfies

$$
a(\bar{u},\varphi-\bar{u}) \ge \int_\Omega g(\varphi-\bar{u}) \qquad \varphi \in H_0^1(\Omega)\ \varphi,\ \bar{u} \ge 0.
$$

Hence we can compare $\bar{u}$ with the solution of the symmetrized problem

$$
a^*(v,V-v) \ge \int_{\Omega^*} g^*(x)(V-v)\,dx \qquad \forall\, V \in H_0^1(\Omega^*)\ V,v \ge 0.
$$

APPENDIX

Proof of Proposition 2.1.   Following [AMT] at first we prove that conditions (i), (ii), (iii) are equivalent. In order to do that let us observe that

(A.1) $A^* Z = H \Leftrightarrow - (e^{B|x|} Z_{x_i})_{x_i} - e^{B|x|} B \dfrac{(n-1)}{|x|} Z - e^{B|x|} c_-^*(x) Z = e^{B|x|} H$

then the first eigenvalue of (2.6) can be characterized by the Rayleigh principle

(A.2) $\lambda_1 = \min\limits_{\Phi \in H_0^1(\Omega^*)} \dfrac{\displaystyle\int_{\Omega^*} e^{B|x|} \left(|\nabla \Phi|^2 - B \dfrac{(n-1)}{|x|} \Phi^2 - c_-^*(x) \Phi^2 \right) dx}{\displaystyle\int_{\Omega^*} e^{B|x|} p(x) \Phi^2 dx}.$

(i) $\Rightarrow$ (ii)

By (A.1)

$$\lambda_1 \int_{\Omega^*} e^{B|x|} p(x) \Phi_1 Z \, dx = \int_{\Omega^*} e^{B|x|} H \Phi_1 dx,$$

where Φ_1 is the first eigenfunction of (2.6). Then, since Φ_1 has constant sign in Ω^*, we have $\lambda_1 > 0$.

(ii) $\Rightarrow$ (iii)

Choosing in (ii) $p(x) \geq B \dfrac{(n-1)}{|x|} + c_-^*(x)$, and using (A.2) we get

$$\int_{\Omega^*} e^{B|x|} \left(|\nabla \Phi|^2 - B \frac{(n-1)}{|x|} \Phi^2 - c_-^* \Phi^2 \right) dx - \alpha \int_{\Omega^*} e^{B|x|} |\nabla \Phi|^2 dx =$$

$$= (1-\alpha) \int_{\Omega^*} e^{B|x|} |\nabla \Phi|^2 dx - \int_{\Omega^*} e^{B|x|} \left(B \frac{(n-1)}{|x|} \Phi^2 + c_-^* \Phi^2 \right) dx \geq$$

$$\geq \lambda_1 (1-\alpha) \int_{\Omega^*} e^{B|x|} p(x) \Phi^2 - \alpha \int_{\Omega^*} e^{B|x|} \left(B \frac{(n-1)}{|x|} \Phi^2 + c_-^* \Phi^2 \right) dx \geq$$

$$\geq (\lambda_1 - \alpha \lambda_1 - \alpha) \int_{\Omega^*} e^{B|x|} \left(B \frac{(n-1)}{|x|} \Phi^2 + c_-^* \Phi^2 \right) dx.$$

Taking $0 < \alpha < \dfrac{\lambda_1}{\lambda_1 + 1}$, (iii) follows.

(iii) $\Rightarrow$ (i)

By (iii) the problem

$$
(A.3) \quad
\begin{cases}
-(e^{B|x|}Z_{x_i})_{x_i} - e^{B|x|} B \dfrac{(n-1)}{|x|} Z - e^{B|x|} c_-^*(x)Z = e^{B|x|}H & \text{in } \Omega^* \\[2mm]
Z \in H_0^1(\Omega^*)
\end{cases}
$$

has a unique solution. Moreover this solution is positive. As a matter of fact setting $Z_- = \max(-Z, 0)$ by (A.3)

$$
\alpha \int_{\Omega^*} e^{B|x|} |\nabla Z_-|^2 dx \le \int_{\Omega^*} e^{B|x|} \left(|\nabla Z_-|^2 - B \frac{(n-1)}{|x|} Z_-^2 - c_-^* Z_-^2 \right) dx =
$$

$$
= -\int_{\Omega^*} e^{B|x|} H Z_- dx \le 0
$$

and then $Z_- = 0$.

With similar arguments, by (iii), we obtain (2.7).

Now we observe that (1.3) is equivalent to the variational inequality

$$
\int_{\Omega^*} e^{B|x|} \left[v_{x_i}(\Phi - v)_{x_i} - B \frac{(n-1)}{|x|} v(\Phi - v) - c_-^*(x)v(\Phi - v) \right] dx \ge
$$

$$
\ge \int_{\Omega^*} e^{B|x|} f^*(x)(\Phi - v) dx \qquad \forall \, \Phi \in H_0^1(\Omega^*) \ \ \Phi, v \ge 0
$$

which, by (iii), has a unique solution v.

To prove that v is decreasing for the sake of simplicity we suppose v sufficiently smooth. Setting $r = |x|, R = \left(\dfrac{|v>0|}{C_n} \right)^{1/n}$, $V = v_r$ and deriving with respect to r the equation

$$
A^* v = f^* \qquad \text{in } \{x : v(x) > 0\}
$$

we obtain

$$
-V_{rr} - \frac{(n-1)}{r} V_r - BV_r + \left(\frac{(n-1)}{r^2} - B \frac{(n-1)}{r} - c_-^*(x) \right) V =
$$

$$
= f_r^* - B \frac{(n-1)}{r^2} v + (c_-^*)_r v \qquad \text{in } [0, R].
$$

Since $f_r^* - B \dfrac{(n-1)}{r^2} v \le 0$, we have

$$A^{\#}V + \frac{(n-1)}{|x|^2}V \leq 0$$

and $V(R) \leq 0$.

We conclude observing that the operator $A^{\#} + \dfrac{(n-1)}{|x|^2}$ has the property (2.7).   □

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