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Existence and Nonexistence of Nontrivial Solutions for Some Nonlinear Elliptic Systems

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ABSTRACT. In this paper we give some existence and nonexistence results of non trivial solutions of nonlinear elliptic systems involving the p-Laplacian.

0. INTRODUCTION

In this paper, we give some existence and nonexistence results concerning nonlinear elliptic systems. The case of one equation has been studied by many authors.

Let Ω be a bounded regular open set in $\mathbb{R}^n$ and consider the problem

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$$(P_\lambda)\qquad \begin{cases} \text{Find } u \in C^2(\Omega) \cap C^0(\bar{\Omega}) \text{ such that} & \\ -\Delta u = \lambda f(u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

where $f(u) \in C^{0,\alpha}(\mathbf{R})$, $0 < \alpha < 1$, is such that: $f(0) = 0$ and $|f(u)| \leq A + B|u|^m$.

Any solution u^* of (P_λ) satisfies the Pohožaev's identity [21]:

$$n\int_\Omega \lambda\left[\frac{n-2}{2n}u^* f(u^*) - \int_0^{u^*} f(s)ds\right]dx = -\frac{1}{2}\int_{\partial\Omega} |\nabla u^*|^2 (x\cdot\nu)d\sigma,$$

whence $u^* = 0$ if Ω is starshaped and

$$\lambda\left[\frac{n-2}{2n}u^* f(u^*) - \int_0^{u^*} f(s)ds\right] > 0.$$

On the other hand, if

$$0 < m+1 < \frac{2n}{n-2},$$

Pohožaev [21] has shown that (P_λ) admits an eigenfunction $u^* \neq 0$ corresponding to λ.

Always in the scalar case, Ôtani [19], [20] and de Thélin [25] generalize these results for the *p-Laplacian* $\Delta_p u = div(|\nabla u|^{p-2}\nabla u)$.

For example, they give the following results concerning the equation

$$(E_\lambda)\qquad -\Delta_p u = \lambda |u|^{m-1}u$$

- If Ω is a strictly starshaped open set and $(m+1)(n-p) \geq np$ the only solution $u^* \in W_0^{1,p}(\Omega)$ of (E_λ) is $u^* \equiv 0$.

- If $(m+1)(n-p) < np$ and $m+1 \neq p$, then for any $\lambda > 0$, (E_λ) admits a positive solution $u^* \in W_0^{1,p}(\Omega)$.

- If $m+1 = p$, we have an eigenvalue problem [3].

More recently, in [32], we have given some results concerning the existence and nonexistence of a nontrivial solution $(u^*,v^*) \in W_0^{1,p}(\Omega) \times W_0^{1,q}(\Omega)$ of the following system

$$\begin{cases} -\Delta_p u = u\,|\,u\,|^{\alpha-1}\,|\,v\,|^{\beta+1} & \text{in } \Omega \\ -\Delta_q v = \,|\,u\,|^{\alpha+1} v\,|\,v\,|^{\beta-1}. \end{cases}$$

We prove

1) nonexistence results when

$$(\alpha+1)\frac{n-p}{np}+(\beta+1)\frac{n-q}{nq}\geq 1$$

when Ω is a strictly starshaped open set;

2) existence results when

$$(\alpha+1)\frac{n-p}{np}+(\beta+1)\frac{n-q}{nq}<1$$

and when

$$\frac{\alpha+1}{p}+\frac{\beta+1}{q}\neq 1.$$

Now, in this paper, we extend the study of existence and nonexistence of positive solutions of the nonlinear elliptic problem

$$(P) \qquad \begin{cases} -\Delta_p u = f(x;u,v) & \text{in } \Omega \\ -\Delta_q v = g(x;u,v) & \text{in } \Omega \\ u = 0,\ v = 0 & \text{on } \partial\Omega. \end{cases}$$

We say that (P) is a potential system if there is a C^1 function H such that

$$f(x;s,t)=\frac{\partial H}{\partial s}(x;s,t),\ g(x;s,t)=\frac{\partial H}{\partial t}(x;s,t).$$

In a first part, following Egnell [10] and Pucci-Serrin [22], we obtain a Pohožaev type identity for potential systems. In the case when Ω is a starshaped bounded open set, this identity gives nonexistence results.

In a second part, we give some existence results for non potential systems. Following Deuel and Hess [7], we construct appropriate sub-supersolutions for (P) and use a suitable comparison principle.

In a third part, we give some existence results for potential systems. Following Nirenberg [18], we apply Mountain-Pass Lemma to find a nontrivial solution; after that, we extend an iterative method previously used by Ôtani [20] for the equation (E_λ) to prove that the solution is bounded.

Concerning the systems, we can notice the existence results obtained in [4], [6], [11], [12], [28]. Independently, [13], [22] give nonexistence results.

1. NONEXISTENCE RESULT

In this first section, we propose to extend the non-existence study, made by de Thélin [26] and Egnell [10] in the scalar case, to the following problem (P)

$$(P)\qquad \begin{cases} \text{Find } (u,v)\in X\cap [L^\infty(\Omega)]^2 \text{ such that} & \\ (1)\ \ -\Delta_p u = \dfrac{\partial \mathrm{H}}{\partial u}(x;u,v) & \text{in } \Omega \\ (2)\ \ -\Delta_q v = \dfrac{\partial \mathrm{H}}{\partial v}(x;u,v) & \text{in } \Omega \\ \qquad u>0 & \text{in } \Omega \\ \qquad v>0 & \text{in } \Omega \end{cases}$$

Hereafter, X denotes the space $W_0^{1,p}(\Omega) \times W_0^{1,q}(\Omega)$.

1.1. Properties and Results.

Theorem 1.1. *Assume the following hypotheses*

i) $H(x;0,0) = 0$ and $\dfrac{\partial H}{\partial s}(x;0,0) = \dfrac{\partial H}{\partial t}(x;0,0) = 0$

ii) $\dfrac{\partial H}{\partial s}(x;s,t)$, $\dfrac{\partial H}{\partial t}(x;s,t)$ are in $C(\Omega\times\mathbb{R}\times\mathbb{R})$ and $\dfrac{\partial H}{\partial s}(x;s,t) \geq 0$

$$\frac{\partial H}{\partial t}(x;s,t) \geq 0 \text{ for any } s,t \geq 0 \text{ and } x \in \Omega$$

iii) $\forall (s,t) \in \mathbb{R}^2$

$$H(x;s,t) \leq \frac{n-p}{np}\left\{s\frac{\partial H}{\partial s}(x;s,t)\right\} + \frac{n-q}{nq}\left\{t\frac{\partial H}{\partial t}(x;s,t)\right\} - \frac{x}{n}\cdot\nabla_x H(x;s,t)$$

iv) Ω *is a bounded strictly starshaped domain in* $\mathbb{R}^n$ *containing 0.*

Then, $(u^*,v^*) \equiv 0$ *is the only solution of (P) in* $X \cap [L^\infty(\Omega)]^2$.

Corollary 1.1. *Let* Ω *be a bounded strictly starshaped domain in* $\mathbb{R}^n$ *and* $H(x;s,t) = |s|^{\alpha+1}|t|^{\beta+1}$.

If

$$(\alpha+1)\frac{n-p}{np} + (\beta+1)\frac{n-q}{nq} \geq 1,$$

(P) has only the trivial solution (0,0) *in* $X \cap [L^\infty(\Omega)]^2$.

Proof of the Corollary 1.1. Since

$$(\alpha+1)\frac{n-p}{np} + (\beta+1)\frac{n-q}{nq} \geq 1,$$

we have

$$H(x;s,t)\leq\left[(\alpha+1)\frac{n-p}{np}+(\beta+1)\frac{n-q}{nq}\right]H(x;s,t)$$

(1.1)

$$\leq\frac{n-p}{np}\left\{s\frac{\partial H}{\partial s}(x;s,t)\right\}+\frac{n-q}{nq}\left\{t\frac{\partial H}{\partial t}(x;s,t)\right\}$$

and all the hypotheses of Theorem 1.1 are satisfied.

The proof of Theorem 1.1 needs the following lemma which extends Egnell's one [10].

Lemma 1.1.1. *Let (u^*,v^*) be a solution of the problem (P); then for all x on the boundary of Ω, we have: $|\nabla u^*(x)|\neq 0$ and $|\nabla v^*(x)|\neq 0$.*

Proof. Let $x_0\in\partial\Omega$; there is a ball $B_{r_0}\subset\Omega$.

By translation we assume that $B_{r_0}=\{x\in\Omega;\ |x|<r_0\}$ and, proceeding as in [10], we introduce the function

$$g(x)=k(e^{-\alpha|x|^2}-e^{-\alpha r_0^2}).$$

For $p>1$, a suitable choose of α gives g_p such that

$$-div(|\nabla g_p|^{m-2}\nabla g_p)\leq a g_p^{m-1}\ \text{in}\ B_{r_0}\backslash B_{r_0/2} \tag{1.2}$$

Multiplying (1) and $(1.2)_p$ [resp. (2) and $(1.2)_q$] by the test function $\varphi_p=(g_p-u^*)_+$ [resp $\varphi_q=(g_q-v^*)_+$] and integrating on the set $B_p^+=\{x\in B_{r_0}\backslash B_{r_0/2};\ \varphi_p>0\}$ [resp. B_q^+] where u^* and v^* are regular, we obtain

$$0\leq\int_{B_p^+}(|\nabla g_p|^{p-2}\nabla g_p-|\nabla u^*|^{p-2}\nabla u^*)\cdot\nabla\varphi_p dx\leq-\int_{B_p^+}\frac{\partial H}{\partial u}(x;u^*,v^*)\varphi_p dx$$

whence, $g_p\leq u^*$ in $B_{r_0}\backslash B_{r_0/2}$.

By construction $g_p(x_0) = u^*(x_0) = 0$, therefore

$$|\nabla u^*(x_0)|>2k_p\alpha_p e^{-\alpha_p}>0 \tag{1.3}$$

Proof of Theorem 1.1. Let (u^*, v^*) be a nontrivial solution of (P). For $i = 1,...,n$; $l = 1,...,n$ let

$$P_i=\sum_{l=1}^{n} |\nabla u^*|^{p-2}\frac{\partial u^*}{\partial x_i}x_l\frac{\partial u^*}{\partial x_l} \quad and \quad Q_i=\sum_{l=1}^{n} |\nabla v^*|^{q-2}\frac{\partial v^*}{\partial x_i}x_l\frac{\partial v^*}{\partial x_l}$$

Let $K_p = \{x \in \Omega;\ |\nabla u^*(x)| = 0\}$, $K_q = \{x \in \Omega;\ |\nabla v^*(x)| = 0\}$.

Lemma 1.1. allows us to consider as in [10], the sets $\tilde{\Omega}_k$ and $\tilde{\Omega}_k'$ such that $K_p \subset \tilde{\Omega}_k \subset\subset \Omega$, $K_q \subset \tilde{\Omega}_k' \subset\subset \Omega$, with $dist(K_p;\partial\tilde{\Omega}_k) \to 0$, $dist(K_q;\partial\tilde{\Omega}_k') \to 0$, as $k \to +\infty$ and we define $\Omega_k = \Omega\backslash\tilde{\Omega}_k, \Omega_k' = \Omega\backslash\tilde{\Omega}_k'$.

$$\sum_{i=1}^{n}\int_{\Omega_k}\frac{\partial P_i}{\partial x_i}dx=\sum_{i=1}^{n}\int_{\Omega_k}\sum_{l=1}^{n}x_l\frac{\partial u^*}{\partial x_l}\frac{\partial}{\partial x_i}\left(|\nabla u^*|^{p-2}\frac{\partial u^*}{\partial x_i}\right)dx+\int_{\Omega_k}|\nabla u^*|^p dx$$

$$+\sum_{i=1}^{n}\int_{\Omega_k}\sum_{l=1}^{n}\frac{\partial u^*}{\partial x_i}x_l|\nabla u^*|^{p-2}\frac{\partial}{\partial x_i}\left(\frac{\partial u^*}{\partial x_l}\right)dx$$

$$=-\int_{\Omega_k}\sum_{l=1}^{n}x_l\frac{\partial u^*}{\partial x_l}\frac{\partial H}{\partial u}(x;u^*,v^*)dx+\int_{\Omega_k}|\nabla u^*|^p dx \tag{1.4}$$

$$+\int_{\Omega_k}\sum_{l=1}^{n}\frac{\partial}{\partial x_l}\left(x_l\frac{1}{p}|\nabla u^*|^p\right)dx-\frac{n}{p}\int_{\Omega_k}|\nabla u^*|^p dx$$

∇u^* do not vanish in Ω_k and therefore u^* is of class C^2 in Ω_k, so we can use the Gauss's formula to obtain

$$(1.5)\quad \int_{\Omega_k}\sum_{i=1}^{n}\frac{\partial P}{\partial x_i}dx=\int_{\partial\Omega_k}\sum_{i=1}^{n}P_i\nu_i d\sigma=\int_{\partial\Omega_k}|\nabla u*|^{p-2}(x\cdot\nabla u*)(\nu\cdot\nabla u*)d\sigma$$

and

$$(1.6)\quad \int_{\Omega_k}\sum_{l=1}^{n}\frac{\partial}{\partial x_l}\left(x_l\frac{1}{p}|\nabla u*|^p\right)=\int_{\partial\Omega_k}\frac{1}{p}|\nabla u*|^p(x\cdot\nu)d\sigma$$

Whence, by (1.4), (1.5) and (1.6)

$$\int_{\partial\Omega_k}|\nabla u*|^{p-2}(x\cdot\nabla u*)(\nu\cdot\nabla u*)d\sigma-\frac{1}{p}\int_{\partial\Omega_k}|\nabla u*|^p(x\cdot\nu)d\sigma$$

(1.7)

$$=-\int_{\Omega_k}\sum_{l=1}^{n}x_l\frac{\partial u*}{\partial x_l}\frac{\partial H}{\partial u}(x;u*,v*)dx+\frac{p-n}{p}\int_{\Omega_k}u*\frac{\partial H}{\partial u}(x;u*,v*)dx$$

In the same way, an analogous relation is also obtained relatively to v^*. Summing up these relations, we have

$$\int_{\partial\Omega_k}|\nabla u*|^{p-2}(x\cdot\nabla u*)(\nu\cdot\nabla u*)d\sigma+\int_{\partial\Omega_k'}|\nabla v*|^{q-2}(x\cdot\nabla v*)(\nu\cdot\nabla v*)d\sigma$$

$$-\frac{1}{p}\int_{\partial\Omega_k}|\nabla u*|^p(x\cdot\nu)d\sigma-\frac{1}{q}\int_{\partial\Omega_k'}|\nabla v*|^q(x\cdot\nu)d\sigma$$

(1.8)

$$=\frac{p-n}{p}\int_{\Omega_k}u*\frac{\partial H}{\partial u}(x;u*,v*)dx+\frac{q-n}{q}\int_{\Omega'_k}v*\frac{\partial H}{\partial v}(x;u*,v*)dx$$

$$-\int_{\Omega_k}\sum_{l=1}^{n}x_l\left\{\frac{\partial u*}{\partial x_l}\frac{\partial H}{\partial u}(x;u*,v*)\right\}dx-\int_{\Omega'_k}\sum_{l=1}^{n}x_l\left\{\frac{\partial v*}{\partial x_l}\frac{\partial H}{\partial v}(x;u*,v*)\right\}dx.$$

Passing to the limit on k in this equality, as u^* and $v^* \equiv 0$ on $\partial\Omega$ and using the results of Egnell (2.1 [10, p. 64]).

$$\frac{p-1}{p}\int_{\partial\Omega}|\nabla u*|^p(x\cdot\mathrm{v})d\sigma+\frac{q-1}{q}|\nabla v*|^q(x\cdot\mathrm{v})d\sigma$$

$$=-\frac{n-p}{p}\int_{\Omega}u*\frac{\partial H}{\partial u}(x;u*,v*)dx-\frac{n-q}{q}\int_{\Omega}v*\frac{\partial H}{\partial v}(x;u*,v*)dx$$

(1.9)

$$-\int_{\Omega}\sum_{l=1}^{n}x_l\left\{\frac{\partial u*}{\partial x_l}\frac{\partial H}{\partial u}(x;u*,v*)+\frac{\partial v*}{\partial x_l}\frac{\partial H}{\partial v}(x;u*,v*)\right\}dx.$$

We have the following relation

$$\sum_{l=1}^{n}\frac{\partial}{\partial x_l}\{x_lH(x;s,t)\}=nH(x;s,t)+x\cdot\nabla_xH(x;s,t)$$

(1.10)

$$+\sum_{l=1}^{n} x_l\left\{\frac{\partial s}{\partial x_l}\cdot\frac{\partial H}{\partial s}(x;s,t)+\frac{\partial t}{\partial x_l}\cdot\frac{\partial H}{\partial t}(x;s,t)\right\}.$$

Moreover, since the application $x \longrightarrow H(x;u^*(x),v^*(x))$ is of class $C^1(\bar{\Omega})$, using again the Gauss's formula then we have from hypothesis *i)* $\int_{\partial\Omega} H(x;u^*(x),v^*(x))\ (x{\cdot}\nu d\sigma) = 0$. Hence, we obtain
(1.11)

$$-\left(\frac{p-1}{p}\int_{\partial\Omega}|\nabla u*|^p(x{\cdot}\nu)d\sigma+\frac{q-1}{q}\int_{\partial\Omega}|\nabla v*|^q(x{\cdot}\nu)d\sigma\right)$$

$$=\int_\Omega\left[-x{\cdot}\nabla_x H(x;u*,v*)-nH(x;u*,v*)+\frac{n-p}{p}\left\{u*\frac{\partial H}{\partial u}(x;u*,v*)\right\}\right.$$

$$\left.+\frac{n-q}{q}\left\{v*\frac{\partial H}{\partial v}(x;u*,v*)\right\}\right]dx$$

According to the hypothesis *iii)* the integral on Ω is nonnegative, whence a contradiction.

2. EXISTENCE RESULTS VIA COMPARISON ARGUMENTS

Ω denotes a bounded regular open set in $\mathbb{R}^n$ and $X = W_0^{1,p}(\Omega)\times W_0^{1,q}(\Omega)$.

Throughout this second section, we shall prove some existence results for the following problem.

$$(P)\qquad\begin{cases}\text{Find } (u,v)\in X) \text{ such that} & \\ -\Delta_p u = f(x;u,v) & \text{on } \Omega\\ -\Delta_q v = g(x;u,v) & \text{on } \Omega.\end{cases}$$

We make the following assumptions

(H1) f and g belong to $C(\Omega \times \mathbf{R} \times \mathbf{R})$
moreover, for any $s \geq 0$. $t \geq 0$; $f(x;s,t) \geq 0$ and $g(x;s,t) \geq 0$

(H2) There are nonnegative constants:

$\alpha > 0$, $\beta > 0$, p_i, q_i (i = 1,2)
a_j, b_j ($j = 1,...,6$) where $a_1>0$, $a_3>0$, $b_1>0$, $b_3>0$ satisfying $(H2)_a$ and $(H2)_b$:

$$(H2)_a: \begin{cases} \dfrac{\alpha+1}{p} + \dfrac{\beta+1}{p} < 1 \\ 1 < p_1 < p; \quad 0 < q_1 - 1 < \dfrac{q}{p^*} \\ 0 < p_2 - 1 < \dfrac{p}{q^*}; \quad 1 < q_2 < q \end{cases}$$

$$(H2)_b: \begin{cases} a_1 s|s|^{\alpha-1}|t|^{\beta+1} - a_2|s|^{p-1} \leq f(x;s,t) \leq a_3 s|s|^{\alpha-1}|t|^{\beta+1} + a_4|s|^{p_1-1} \\ \qquad\qquad + a_5|t|^{q_1-1} + a_6 \\ b_1|s|^{\alpha+1} t|t|^{\beta-1} - b_2|t|^{q-1} \leq g(x;s,t) \leq b_3 s^{\alpha+1} t|t|^{\beta-1} + b_4|s|^{p_2-1} \\ \qquad\qquad + b_5|t|^{q_2-1} + b_6 \end{cases}$$

We have the following existence theorem:

Theorem 2.1. *Under hypotheses (H1) and (H2), (P) has a nontrivial solution (u^*,v^*) in $X \cap [L^\infty(\Omega)]^2$.*

Example: existence result for $f(x;s,t) = a(x)s|s|^{\alpha-1}|t|^{\beta+1}$ and $g(x;s,t) = b(x)|s|^{\alpha+1}t|t|^{\beta-1}$.

Corollary 2.1. *Let f and g be as above where a and b are* *nonnegative continuous functions and assume that $\alpha > 0$ and $\beta > 0$ are such that*

$$(\alpha+1)\frac{n-p}{np}+(\beta+1)\frac{n-q}{nq}<1;\quad \frac{\alpha+1}{p}+\frac{\beta+1}{q}<1.$$

Then, the corresponding problem (P) has a nontrivial solution in $X \cap [L^\infty(\Omega)]^2$.

The proof of Theorem 2.1 is in three steps.

1st step: Construction of sub-supersolutions of (*P*).

Definition 2.1. *A pair $[(u_0,v_0),(u^0,v^0)]$ is said a weak sub-super solution for the Dirichlet problem (P) if the following conditions are satisfied:*

$$(1):\quad \begin{cases} (u_0,v_0) \in (W^{1,p}(\Omega) \times W^{1,q}(\Omega)) \cap [L^\infty(\Omega)]^2 \\ (u^0,v^0) \in (W^{1,p}(\Omega) \times W^{1,q}(\Omega)) \cap [L^\infty(\Omega)]^2 \end{cases}$$

$$(2.1)\qquad (2):\quad \begin{cases} -\Delta_p u_0 - f(x;u_0,v) \le 0 \le -\Delta_p u^0 - f(x;u^0,v) & \text{in } \Omega \quad \forall v \in [v_0,v^0] \\ -\Delta_q v_0 - g(x;u,v_0) \le 0 \le -\Delta_q v^0 - g(x;u,v^0) & \text{in } \Omega \quad \forall u \in [u_0,u^0] \\ u_0 \le u^0 & \text{in } \Omega \\ v_0 \le v^0 & \text{in } \Omega \\ u_0 \le 0 \le u^0 & \text{on } \partial\Omega \\ v_0 \le 0 \le v^0 & \text{on } \partial\Omega \end{cases}$$

Similar definitions can be found in Díaz-Hernández [8], Díaz-Herrero [9], Hernández [16].

Proposition 2.1. *Assume (H2) and*

$$\frac{\alpha+1}{p}+\frac{\beta+1}{q}<1;$$

then, for any $M > 0$, the problem (P) admits a pair $[(u_0,v_0),(u^0,v^0)]$ of sub-super solution satisfying $u_0(x) \leq M \leq u^0(x)$, $v_0(x) \leq M \leq v^0(x)$ in Ω.

Proof. a) Construction of (u^0,v^0)

Consider $R > 0$ such that $\Omega \subset B(0;R)$. We seek for u^0,v^0 in the following forms:

$$(2.2)\qquad \begin{aligned} u^0(x) &= \varphi^0(r) = ar^{p^*} + b \\ v^0(x) &= \psi^0(r) = cr^{q^*} + d \end{aligned}$$

with: $a < 0$ *and* $c < 0$, $b > 0$ *and* $d > 0$, $\|x\| = r$.

We fix a real $M > 0$ and choose

$$(2.3)\qquad a=-\frac{b-M}{R^{p^*}} \text{ and } c=-\frac{d-M}{R^{q^*}},$$

we have, for b and d greater than M

$$(2.4)\qquad M\leq u^0(x);\ M\leq v^0(x)\quad \forall x\in\Omega.$$

and for each point x in Ω, we have:

(2.5)

$$\Delta_p u^0(x)=(p-1)|\varphi'(r)|^{p-2}\varphi''(r)+\frac{n-1}{r}|\varphi'(r)|^{p-2}\varphi'(r)=-np*|a|^{p-1}=np*\left(\frac{b-M}{R^{p^*}}\right)^{(p-1)}$$

For $u \leq u^0$, $v \leq v^0$ and $a < 0$; $c < 0$ we have

$$(2.6)\qquad \begin{cases} \Delta_p u^0 + f(x;u^0,v) \le -np*\left(\dfrac{b-M}{R^{p*}}\right)^{(p-1)} + a_3 b^{\alpha} d^{\beta+1} \\ \qquad + a_4 b^{p_1-1} + a_5 d^{q_1-1} + a_6 \quad , \quad \forall v_0 \le v \le v^0 \\ \Delta_q v^0 + g(x;u,v^0) \le -nq*\left(\dfrac{d-M}{R^{q*}}\right)^{(q-1)} + b_3 b^{\alpha+1} d^{\beta} \\ \qquad + b_4 b^{p_2-1} + b_5 d^{q_2-1} + b_6 \quad , \quad \forall u_0 \le u \le u^0. \end{cases}$$

Let $k > 0$, $b = k^{1/p}$ and $d = k^{1/q}$. Comparing, the growth of the different terms in (2.6) for large k, we obtain

$$(2.7)\qquad \begin{cases} \Delta_p u^0 + f(x;u^0,v) \le 0 \quad \forall v^0 \le v \le v^0 \\ \Delta_q v^0 + g(x;u,v^0) \le 0 \quad \forall u_0 \le u \le u^0. \end{cases}$$

b) Construction of (u_0,v_0). Consider $x_0 \in \Omega$, and $R > 0$ such that $B(x_0;R) \subset \Omega$; we can assume $0 \in \Omega$.

As in [11], [26], we seek (u_0,v_0) in the following form

$$(2.8)\qquad u_0(x)=\varphi_0(r)= \begin{cases} Ar^{p*}+B & \text{for } 0 \le r \le \dfrac{nR}{n+1}, \\ C(R-r)^{p*} & \text{for } \dfrac{nR}{n+1} \le r \le R, \\ 0 & \text{for } R<r, \end{cases}$$

$$(2.9)\qquad v_0(x)=\psi_0(r)= \begin{cases} \tilde{A}r^{q*}+\tilde{B} & \text{for } 0 \le r \le \dfrac{nR}{n+1}, \\ \tilde{C}(R-r)^{q*} & \text{for } \dfrac{nR}{n+1} \le r \le R, \\ 0 & \text{for } R<r \end{cases}$$

Take

$$A=-B\left(\frac{n+1}{n}\right)^{p*-1}\frac{1}{R^{p*}},\quad \tilde{A}=-\tilde{B}\left(\frac{n+1}{n}\right)^{q*-1}\frac{1}{R^{q*}}$$

(2.10)
$$C=-An^{p*-1},\quad \tilde{C}=-\tilde{A}n^{q*-1}$$

$$B>0,\ \tilde{B}>0.$$

By (2.10) u_0 and v_0 are in $C^1(\bar{\Omega})$ and moreover they vanish on $\partial\Omega$.

First consider x such that

$$\frac{nR}{n+1}\leq r=\|x\|\leq R;$$

we have

(2.11)
$$\begin{cases} 0\leq u_0(x)\leq C\left(R-\dfrac{nR}{n+1}\right)^{p*} \\ 0\leq v_0(x)\leq \tilde{C}\left(R-\dfrac{nR}{n+1}\right)^{q*} \end{cases}$$

Consequently

$$\Delta_p u_0(x)=p*^{p-1}C^{p-1}\left\{1-(n-1)\frac{R-r}{r}\right\}$$

(2.12)

$$\geq\frac{p*^{p-1}C^{p-1}}{n}$$

Whence for any $(u,v)\in[u_0,u^0]\times[v_0,v^0]$ and for sufficiently small R:

$$(2.13)\quad \begin{cases} \Delta_p u_0 + f(x;u_0,v) \geq C^{p-1} \left\{ \dfrac{p*^{p-1}}{n} - a_2\left(\dfrac{R}{n+1}\right)^p \right\} \geq 0 \\ \Delta_q v_0 + g(x;u,v_0) \geq \tilde{C}^{q-1} \left\{ \dfrac{q*^{q-1}}{n} - b_2\left(\dfrac{R}{n+1}\right)^q \right\} \geq 0 \end{cases}$$

Now consider $x \in \Omega$ such that:

$$0 \leq \|x\| \leq \frac{nR}{n+1}$$

We have in this case

$$(2.14)\quad 0 \leq u_0(x) \leq B \text{ and } 0 \leq v_0(x) \leq \tilde{B}.$$

Moreover

$$(2.15)\quad \Delta_p u_0(x) = -B^{(p-1)} \frac{n+1}{R^p} p*^{(p-1)}$$

Using the hypothesis ($H2$), for any $(u,v) \in [u_0,v_0] \times [v_0,v^0]$, we obtain

$$(2.16)\quad \begin{cases} -B^{p-1}\dfrac{n+1}{R^{p*}}(p*)^{p-1} + a_1 B^{\alpha}\tilde{B}^{\beta+1}\dfrac{1}{(n+1)^{\alpha+\beta+1}} - a_2 B^{p-1} \leq \Delta_p u_0 + f(x;u_0,v) \\ -\tilde{B}^{q-1}\dfrac{n+1}{R^{q*}}(q*)^{q-1} + b_1 B^{\alpha+1}\tilde{B}^{\beta}\dfrac{1}{(n+1)^{\alpha\beta+1}} - b_2\tilde{B}^{q-1} \leq \Delta_q v_0 + g(x;u,v_0) \end{cases}$$

Hence the conclusion follows for $B = D^{1/p}$, $\tilde{B} = D^{1/q}$, $D > 0$ sufficiently small.

2nd Step: The troncated problem ($\tilde{P}$) associed to (P).

Following [7], we define a troncated problem ($\tilde{P}$), associated to (P).

$$(\tilde{P}) \quad \begin{cases} \text{Find } (u,v) \in X \text{ such that} \\ (\tilde{1}) \ -\Delta_p u = \tilde{f}(x;u,v) - \gamma_1(x,u) & \text{in } \Omega \\ (\tilde{2}) \ -\Delta_q v = \tilde{g}(x;u,v) - \gamma_2(x,v) & \text{in } \Omega \end{cases}$$

Where

$$(2.17) \quad \begin{aligned} \gamma_1(x,u(x)) &= -(u_0(x) - u(x))_+^{p-1} + (u(x) - u^0(x))_+^{p-1} \\ \gamma_2(x,v(x)) &= -(v_0(x) - v(x))_+^{q-1} + (v(x) - v^0(x))_+^{q-1} \\ \tilde{f}(x;u(x),v(x)) &= f(x;U(x),V(x)) \\ \tilde{g}(x;u(x),v(x)) &= g(x;U(x),V(x)) \end{aligned}$$

With

$$(2.18) \quad \begin{aligned} U(x) &= u(x) + (u_0(x) - u(x))_+ - (u(x) - u^0(x))_+ \\ V(x) &= v(x) + (v_0(x) - v(x))_+ - (v(x) - v^0(x))_+ \end{aligned}$$

For any $(u,v) \in X$, $(\hat{u},\hat{v}) \in X$, we define:

$$A(u,v) = -\begin{pmatrix} \Delta_p & 0 \\ 0 & \Delta_q \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} \gamma_1(x;u) - \tilde{f}(x;u,v) \\ \gamma_2(x;v) - \tilde{g}(x;u,v) \end{pmatrix}$$

$$(2.19) \quad = -\begin{pmatrix} \displaystyle\sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left(|\nabla u|^{p-2} \frac{\partial u}{\partial x_i} \right) \\ \displaystyle\sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left(|\nabla v|^{q-2} \frac{\partial v}{\partial x_i} \right) \end{pmatrix} + \begin{pmatrix} \gamma_1(.;u) - \tilde{f}(x;u,v) \\ \gamma_2(.;v) - \tilde{g}(x;u,v) \end{pmatrix}$$

$$a[(u,v);(\hat{u},\hat{v})] = \int_\Omega A(u,v) \cdot W dx$$

with $W = \begin{pmatrix} \hat{u} \\ \hat{v} \end{pmatrix}$

We have

$$a[(u,v);(\hat{u},\hat{v})] = \int_\Omega |\nabla u|^{p-2} \nabla u \nabla \hat{u} dx + \int_\Omega |\nabla v|^{q-2} \nabla v \nabla \hat{v} dx$$

$$(2.20) \qquad - \int_\Omega \tilde{f}(x;u,v) \hat{u} dx - \int_\Omega \tilde{g}(x;u,v) \hat{v} dx$$

$$+ \int_\Omega \gamma_1(x,u) \hat{u} dx + \int_\Omega \gamma_2(x,v) \hat{v} dx.$$

Lemma 2.1. *A is a bounded operator from X to X*.*

Proof [31].

Definition 2.2 (C.f [17]). *An operator $A : X \longrightarrow X^*$ is called a calculus of variations operator, if it is bounded and if it can be represented in the form*

(1) $$A(u,v) = \mathcal{A}[(u,v);(u,v)]$$

where $((u,v),(\hat{u},\hat{v})) \longrightarrow \mathcal{A}[(u,v);(\hat{u},\hat{v})]$ is an operator $X \times X \longrightarrow X^*$ which satisfies

$$\begin{cases} \forall (u,v)\in X;\ (\hat{u},\hat{v})\longrightarrow \mathcal{A}[(u,v);(\hat{u},\hat{v})] \text{ is a hemicontinuous bounded} \\ \text{operator } X \longrightarrow X^* \text{ and} \\ <\mathcal{A}[(u,v);(u,v)]-\mathcal{A}[(u,v);(\hat{u},\hat{v})],(u,v)-(\hat{u},\hat{v})>\geq 0;\ \forall (u,v),(\hat{u},\hat{v})\in X \end{cases} \tag{2}$$

For any $(\hat{u},\hat{v}) \in X$, $(u,v) \longrightarrow \mathcal{A}[(u,v);(\hat{u},\hat{v})]$
is a bounded hemicontinuous operator $X \longrightarrow X^*$. (3)

If $(u_\mu,v_\mu) \longrightarrow (u,v)$ *weakly in* X
and
if $<\mathcal{A}[(u_\mu,v_\mu),(u_\mu,v_\mu)] - \mathcal{A}[(u_\mu,v_\mu),(u,v)],(u_\mu-u,v_\mu-v)> \longrightarrow 0$ (4)
then, for any $(\hat{u},\hat{v})$ *in* X
the sequence $\mathcal{A}[(u_\mu,v_\mu),(\hat{u},\hat{v})]$ *converges weakly to* $\mathcal{A}[(u,v),(\hat{u},\hat{v})]$
in X^*.

If $(u_\mu,v_\mu) \rightharpoonup (u,v)$ *in* X
and if $\mathcal{A}[(u_\mu,v_\mu),(\hat{u},\hat{v})] \longrightarrow (\phi,\psi)$ *weakly in* X^*
(5)
then
$<\mathcal{A}[(u_\mu,v_\mu),(\hat{u},\hat{v})];(u_\mu,v_\mu)>_{X^*,X} \longrightarrow <(\phi,\psi),(u,v)>_{X^*,X}$.

In our problem, we define $\mathcal{A}$ by the following relation; for any (u_1,v_1), $(u_2,v_2),(\hat{u},\hat{v})$:

$$\begin{aligned} <\mathcal{A}\ [(u_1,v_1),(u_2,v_2)];(\hat{u},\hat{v})> &= \int_\Omega |\nabla u_2|^{p-2}\nabla u_2 \nabla \hat{u}dx + \int_\Omega |\nabla v_2|^{q-2}\nabla v_2\nabla \hat{v}dx \\ &\quad - \int_\Omega \tilde{f}(x;u_1,v_1)\hat{u}dx - \int_\Omega \tilde{g}(x;u_1,v_1)\hat{v}dx \\ &\quad + \int_\Omega \gamma_1(x,u_1)\hat{u}dx + \int_\Omega \gamma_2(x,v_1)\hat{v}dx \end{aligned} \tag{2.21}$$

Lemma 2.2. *$\mathcal{A}$ is a calculus of variations operator.*

Proof. (c.f [31])

Lemma 2.3. *Let V be a Banach space and let A be a coercive calculus of variations operator.*

Then, for any f in V^*, the equation $A(u) = f$ has a solution u in V.

Proof (c.f [17], proposition 2.6, theorem 2.7, p. 180-181).

Lemma 2.4. *If the application $\tilde{f}$, $\tilde{g}$, γ_1 and γ_2 are defined as above, then the problem $(\tilde{P})$ has a solution $(\bar{u},\bar{v})$ in X.*

3st Step: Existence of a non-trivial solution for (*P*).

Now, we prove that $u_0 \leq \bar{u} \leq u^0$ $\ v_0 \leq \bar{v} \leq v^0$, in Ω.

W show for example $\bar{u} \leq u^0$.

Consider $\hat{u} = (\bar{u}-u^0)_+$ and $\hat{v} = (\bar{v} - v^0)_+$.

Multiplying $(\tilde{1})$ by $\hat{u}$ and $(\tilde{2})$ by $\hat{v}$, we have

$$\int_\Omega |\nabla\bar{u}|^{p-2}\nabla\bar{u}\nabla\hat{u}dx-\int_\Omega\tilde{f}(x;\bar{u},\bar{v})\cdot\hat{u}dx+\|(\bar{u}-u^0)_+\|^p_{L^p(\Omega)}=0 \tag{2.22}$$

but, according to the definition of u^0, $\forall v \in [v_0,v^0]$, we have

$$\int_\Omega |\nabla u^0|^{p-2}\nabla u^0\nabla\hat{u}dx-\int_\Omega f(x;u^0,v)\hat{u}dx\geq 0 \tag{2.23}$$

Thus, combining (2.22) and (2.23), we obtain

$$0\geq\int_\Omega\{|\nabla\bar{u}|^{p-2}\nabla\bar{u}-|\nabla u^0|^{p-2}\nabla u^0\}\nabla(\bar{u}-u^0)^+dx$$

(2.24)

$$+\int_\Omega\{f(x;u^0,v)-\tilde{f}(x;\bar{u},\bar{v})\}(\bar{u}-u^0)_+dx+\|(\bar{u}-u^0)_+\|^p_{L^p(\Omega)}$$

Take $v = \bar{V}$ where $\bar{V}$ is associated to $\bar{v}$ as in (2.18). On the set $\{x\in\Omega;\ \bar{u}(x)-u^0(x)>0\}$, we have $\bar{U}(x) = u^0(x)$,

(2.27)

$$\int_{\Omega}\{f(x;u^0,\bar{V})-\bar{f}(x;\bar{u},\bar{v})\}(\bar{u}-u^0)_+(x)dx=\int_{\Omega}\{f(x;u^0,\bar{V})-f(x;\bar{U},\bar{V})\}(\bar{u}-u^0)_+(x)dx=0$$

By monotonicity of $-\Delta_p$, we get that $0 \geq \|(\bar{u}-u^0)_+\|^p_{L^p(\Omega)} \geq 0$.

Thus $\bar{u} \leq u^0$ on Ω and similarly $\bar{v} \leq v^0$ on Ω.

3. EXISTENCE RESULTS VIA VARIATIONAL METHODS

3.0. Introduction. We present in this final section an existence result for the following problem (P)

$$\begin{cases} \text{Find } (u,v) \in X \text{ such that} \\ (1^*) \quad -\Delta_p u = \dfrac{\partial H}{\partial u}(x;u,v) \qquad \text{in } \Omega \\ (2^*) \quad -\Delta_q v = \dfrac{\partial H}{\partial v}(x;u,v) \qquad \text{in } \Omega \end{cases}$$

This result extends to a potential system those obtained by L. Nirenberg [18] and F. de Thélin [26], in the scalar case. Our existence result follows from an appropriate adaptation of the variational method given by Ambrosetti-Rabinowitz [2].

Recal that $X = W_0^{1,p}(\Omega) \times W_0^{1,q}(\Omega)$.

In the next section, we shall prove that in fact $(u,v) \in X \cap [L^\infty(\Omega)]^2$.

We make the following assumptions

(H1) $\quad H \in C^1(\Omega \times \mathbf{R} \times \mathbf{R})$

(H2) There exist two positive real numbers δ, A, with $\delta < A$ such that, for a partition of $\mathbf{R}^2$ in D_1, D_2, D_3 respectively defined by

$$D_1 = \{(s,t) \in \mathbb{R}^2;\ |s| \geq A \ \ or \ \ |t| \geq A\}$$

$$D_2 = \{(s,t) \in \mathbb{R}^2 \setminus D_1;\ |s| > \delta \ \ and \ \ |t| > \delta\}$$

$$D_3 = \mathbb{R}^2 \setminus (D_1 \cup D_2)$$

We have:

$(H2)_a$ there exists a nonnegative constant C and

$$p' \in \left]p, \frac{np}{n-p}\right[,\ q' \in \left]q, \frac{nq}{n-q}\right[,$$

such that $0 \leq H(x;s,t) \leq C(|s|^{p'} + |t|^{q'})$, for any $x \in \Omega$ and for any pair $(s,t) \in D_3$.

$(H2)_b$ There exists a positive function $a \in L^\infty(\Omega)$ such that $H(x;s,t) = a(x)|s|^{\alpha+1}|t|^{\beta+1}$ for any $x \in \Omega$ and $(s,t) \in D_1$.

Remark. We are interested by the nonnegative solutions for the problem (P), so we can add the following hypothesis

(H3) For any $x \in \Omega$, $s \leq 0$ or $t \leq 0$;

$$\frac{\partial H}{\partial s}(x;s,t)=0 \text{ and } \frac{\partial H}{\partial t}(x;s,t)=0.$$

For any (u,v) in X, we define:

(3.0) $$J(u,v)=\frac{1}{p}\int_\Omega |\nabla u|^p dx+\frac{1}{p}\int_\Omega |\nabla v|^q dx-\int_\Omega H(x;u,v)dx$$

We shall use the Mountain-Pass Lemma to obtain an existence theorem for (P). The nontrivial solution is obtained as a critical point of J.

Theorem 3.1. *We suppose that the hypotheses (H1) and (H2) are satisfied and that the real numbers* α *and* β *in* $(H2)_b$ *are such that*

$$\begin{cases} 1) \quad (\alpha+1)\dfrac{n-p}{np}+(\beta+1)\dfrac{n-q}{nq}<1 \\ 2) \quad \dfrac{\alpha+1}{p}+\dfrac{\beta+1}{q} \qquad >1, \end{cases}$$

then, the problem (P) possesses a nontrivial solution (u^*,v^*) *in* $X \cap [L^\infty(\Omega)]^2$.

Corollary 3.1. *All the hypotheses of Theorem 3.1. are satisfied for* $H(x;s,t) = a(x)|s|^{\alpha+1}|t|^{\beta+1}$.

If

$$(\alpha+1)\frac{n-p}{np}+(\beta+1)\frac{n-q}{nq}<1, \quad \frac{\alpha+1}{p}+\frac{\beta+1}{q}>1$$

then, the corresponding problem possesses a nontrivial solution (u^*,v^*) in $X \cap [L^\infty(\Omega)]^2$.

Proof of Corollary 3.1. Consider a truncature $\tilde{H}$ of the application H

$$\tilde{H}(x;s,t)=\begin{cases} 0 & \text{if } s\leq 0 \text{ or } t\leq 0 \\ H(x;s,t) & \text{otherwise} \end{cases}$$

$\tilde{H}$ satisfies the hypotheses $(H1)$, $(H2)$. For proving $(H2)_a$, we write for any real s and t

$$(*) \quad |s|^{\alpha+1}|t|^{\beta+1}\leq C(|s|^{\lambda p}+|t|^{\mu q})$$

Where λ and μ are such that

$$\frac{\alpha+1}{\lambda p}+\frac{\beta+1}{\mu q}=1,\ 1<\lambda<\frac{n}{n-p}\ \textit{and}\ 1<\mu<\frac{n}{n-q}.$$

3.1. Existence of a solution in X.

Lemma 3.1.1. *If*

$$(\alpha+1)\frac{n-p}{np}+(\beta+1)\frac{n-q}{nq}<1,$$

there exist γ_1 *and* γ_2 *such that*

$$\begin{cases}\dfrac{\alpha+1}{\gamma_1}+\dfrac{\beta+1}{\gamma_2}=1\\ \gamma_1\in\left]1,\dfrac{np}{n-p}\right[,\gamma_2\in\left]1,\dfrac{nq}{n-q}\right[\end{cases}$$

Moreover, if (u_k,v_k) *is bounded in* X*, the applications*

$$x\to u_k(x)|u_k(x)|^{\alpha-1}|v_k(x)|^{\beta+1}\ \textit{and}\ x\to v_k(x)|v_k(x)|^{\beta-1}|u_k(x)|^{\alpha+1}$$

are bounded in $L^{\gamma_1'}(\Omega)$ *and* $L^{\gamma_2'}$ *respectively.*

Lemma 3.1.2. *If*

$$\frac{\alpha+1}{p}+\frac{\beta+1}{q}>1,$$

J satisfies the Palais-Smale (P.S) condition.

Proof. Let $\{(u_k,v_k);\ k\in\mathbb{N}\}$ be a sequence in X such that

there exist $M>0$, $|J(u_k,v_k)|\leq M$ $(P.S)_1$

$J'(u_k,v_k) \longrightarrow 0$ strongly in X^* *as* k goes to $+\infty$ $(P.S)_2$.

We claim that this sequence is bounded in X.

By contradiction, suppose that we can extract from (u_k,v_k) a subsequence denoted again by (u_k,v_k) such that $\|(u_k,v_k)\|_X \longrightarrow +\infty$.

Hereafter, we set

$$e_k=\frac{1}{p}\int_\Omega |\nabla u_k|^p dx+\frac{1}{q}\int_\Omega |\nabla v_k|^q dx.$$

The $(P.S)_1$ condition implies that

$$-\frac{M}{e_k}\leq 1-\frac{1}{e_k}\int_\Omega H(x;u_k,v_k)dx\leq\frac{M}{e_k}. \tag{3.1.1}$$

Let $\Omega_{i,k} = \{x \in \Omega\colon (u_k(x),v_k(x)) \in D_i\}$, for $i = 1,2,3$; we obtain

$$-\frac{M}{e_k}\leq 1-\frac{1}{e_k}\left\{\int_{\Omega_{1,k}} a(x)u_k^{\alpha+1}v_k^{(\beta+1)}dx+\int_{\Omega\setminus\Omega_{1,k}} H(x;u_k,v_k)dx\right\}\leq\frac{M}{e_k}. \tag{3.1.2}$$

On the other hand, by $(PS)_2$ we have:

$$-\varepsilon\|(u_k,v_k)\|_X \leq J'(u_k,v_k)\cdot\left(\frac{u_k}{p},\frac{v_k}{q}\right)\leq\varepsilon\|(u_k,v_k)\|_X.$$

That means

$$-\varepsilon\|(u_k,v_k)\|_X\leq e_k-\frac{1}{p}\int_{\Omega_{1,k}} u_k\frac{\partial H}{\partial u}(x;u_k,v_k)dx-\frac{1}{q}\int_{\Omega_{1,k}} v_k\frac{\partial H}{\partial v}(x;u_k,v_k)dx$$

(3.1.3)
$$-\frac{1}{p}\int_{\Omega\backslash\Omega_{1,k}} u_k\frac{\partial H}{\partial u}(x;u_k,v_k)dx-\frac{1}{q}\int_{\Omega\backslash\Omega_{1,k}} v_k\frac{\partial H}{\partial v}(x;u_k,v_k)dx$$

$$\leq\varepsilon\,\|(u_k,v_k)\|_X$$

Then, taking the limit with respect to k in the inequalities (3.1.2) and (3.1.3), we obtain respectively

$$\lim_{k\to+\infty}\frac{1}{e_k}\int_{\Omega_{1,k}} a(x)u_k^{\alpha+1}v_k^{\beta+1}dx=1$$

(3.1.4)

$$\lim_{k\to+\infty}\frac{1}{e_k}\int_{\Omega_{1,k}} a(x)u_k^{\alpha+1}v_k^{\beta+1}dx=\frac{1}{\frac{\alpha+1}{p}+\frac{\beta+1}{q}}.$$

But, this contradicts the hypothesis

$$\frac{\alpha+1}{p}+\frac{\beta+1}{q}>1.$$

Thus, there exist positive contants C_1 et C_2 such that: $\|u_k\|_{1,p}\leq C_1$ and $\|v_k\|_{1,q}\leq C_2$.

Denoting again by $\{u_k;\ k\in\mathcal{N}\}$ and $\{v_k;\ k\in\mathcal{N}\}$ the extracted subsequences, they converge strongly in the spaces $L^{\gamma_1}(\Omega)$ and $L^{\gamma_2}(\Omega)$ respectively; we claim that the subsequence $\{(u_k,v_k);\ k\geq 0\}$ converges strongly in X.

In fact, for any integer pair (m,l)

(3.1.5)
$$\int_\Omega\{F_p(\nabla u_m)-F_p(\nabla u_l)\}\nabla(u_m-u_l)dx=A_{m,l}$$

where

$$A_{m,l}=\langle J'_{p,q}(u_m,v_m)-J'_{p,q}(u_l,v_l);(u_m-u_l,0)\rangle_{X,X*}$$
$$+\int_\Omega\left\{\frac{\partial H}{\partial u}(x;u_m,v_m)-\frac{\partial H}{\partial u}(x;u_l,v_l)\right\}(u_m-u_l)dx$$

and

$$\int_\Omega(F_q(\nabla v_m)-F_q(\nabla v_l))\nabla(v_m-v_l)dx=B_{m,l} \tag{3.1.6}$$

where

$$B_{m,l}=\langle J'(u_m,v_m)-J'(u_l,v_l);(0,v_m-v_l)\rangle_{X,X*}$$
$$+\int_\Omega\left\{\frac{\partial H}{\partial v}(x;u_m,v_m)-\frac{\partial H}{\partial v}(x;u_l,v_l)\right\}(v_m-v_l)dx$$

By $(P.S)_2$ it is easy to remark that $<J'_{p,q}(u_m,v_m)-J'_{p,q}(u_l,v_l);(u_m-u_l,0)>_{X,X*}$ converges to 0 as m and l tend to $+\infty$.

From the hypotheses $(H1)$ and $(H2)$, there exist two constants A_1 and A_2 such that for any (s,t) in $\mathbb{R}^2$ and x in Ω

$$\left|\frac{\partial H}{\partial s}(x;s,t)\right|\leq A_1+A_2|s|^\alpha|t|^{\beta+1}. \tag{3.1.7}$$

By use of Lemma 3.1.,

$$\int_\Omega\left\{\frac{\partial H}{\partial v}(x;u_m,v_m)-\frac{\partial H}{\partial v}(x;u_l,v_l)\right\}(v_m-v_l)dx$$

converges to 0 and therefore $A_{m,l}$ converges to 0.

We have the following algebraic relation [24]:

$$|\nabla u_m-\nabla u_l|^p\leq C\{[F_p(\nabla u_m)-F_p(\nabla u_l)](\nabla_m-\nabla u_l)\}^{s/2}(|\nabla u_m|^p+|\nabla u_l|^p)^{(1-s/2)}$$

(3.1.8) with $s = \begin{cases} p \text{ for } 1<p\leq 2 \\ 2 \text{ for } 2<p \end{cases}$

Integrating (3.1.8) on Ω and using Hölder's inequality in the right hand side, we obtain

(3.1.9) $$\|u_m - u_l\|_{1,p}^p \leq C|A_{m,l}|^{s/2}\{\|u_m\|_{1,p}^p + \|u_l\|_{1,p}^p\}^{(1-s/2)}$$

and

(3.1.10) $$\|v_m - v_l\|_{1,q}^q \leq C'|B_{m,l}|^{t/2}\{\|v_m\|_{1,q}^q + \|v_l\|_{1,q}^q\}^{(1-t/2)}$$

From the convergence results related above, these inequalities give strong convergence of $\{(u_k,v_k);\ k \in \mathcal{N}\}$.

Lemma 3.1.3. *Under the hypotheses of Theorem 3.1.*

1) There exist two positive real numbers ρ, υ_l *and a neighborhood* V_ρ *of the origin of X such that for any element (u,v) on the boundary of* V_ρ*;* $J(u,v) \geq \upsilon_l > 0$.

2) There exist (ϕ,ψ) *in X such that* $J(\phi,\psi) < 0$.

Proof. 1) By $(H1)$ and $(H2)$

$$\int_\Omega H(x;u,v)dx \leq C\int_{\Omega_1}(|u|^{p'} + |v|^{q'}dx + \int_{\Omega_2} Bdx + \int_{\Omega_3} a(x)|u|^{\alpha+1}|v|^{\beta+1}dx$$

(3.1.11)

$$\leq C(\|u\|_{1,p}^{p'} + \|v\|_{1,q}^{q'}) + b_\delta\int_\Omega |u|^{\alpha+1}|v|^{\beta+1}dx + \int_\Omega a(x)|u|^{\alpha+1}|v|^{\beta+1}dx$$

By lemma 3.1.1., we obtain

(3.1.12) $$\int_\Omega |u|^{\alpha+1}|v|^{\beta+1}dx \leq \|u\|^{\alpha+1}_{L^{\gamma_1}(\Omega)} \cdot \|v\|^{\beta+1}_{L^{\gamma_2}(\Omega)} \leq M\|u\|^{\alpha+1}_{1,p} \cdot \|v\|^{\beta+1}_{1,q}$$

Therefore, we get

(3.1.13) $$\int_\Omega H(x;u,v)dx \leq C(\|u\|^{p'}_{1,p} + \|v\|^{q'}_{1,q} + (b_\delta + \|a\|_\infty)\left\{\|u\|^{r(\alpha+1)}_{1,p} + \|v\|^{r*(\beta+1)}_{1,q}\right\}$$

where b_δ is a positive constant $B = b_\delta \delta^{\alpha+\beta+2}$, δ fixed,

$$r = 1 + \frac{p}{q}\frac{\beta+1}{\alpha+1} \quad and \quad r* = 1 + \frac{q}{p}\frac{\alpha+1}{\beta+1}.$$

Denoting by θ and η respectively $\|u\|_{1,p}$ and $\|v\|_{1,q}$, we therefore obtain the following minoration of J for any $(u,v) \in X$,

(3.1.14)

$$J(u,v) \geq \theta^p\left[1 - C\theta^{p'-p} - (b_\delta + \|a\|_\infty)\theta^{(r(\alpha+1)-p)}\right] + \eta^q\left[1 - C\eta^{q'-q} - (b_\delta + \|a\|_\infty)\eta^{(r*(\beta+1)-q)}\right]$$

Whence,

(3.1.15) $$J(u,v) \geq \upsilon_1 > 0$$

2) Let $\phi \in W^{1,p}_0(\Omega)$ and $\psi \in W^{1,q}_0(\Omega)$ be positive in Ω, for any $\sigma > 0$, we have

$$J(\sigma^{\frac{1}{p}}\phi;\sigma^{\frac{1}{q}}\psi) = \sigma\|\phi\|^p_{1,p} + \sigma\|\psi\|^q_{1,q} - \int_\Omega H(x;\sigma^{\frac{1}{p}}\phi, \sigma^{\frac{1}{q}}\psi)dx$$

(3.1.16)

$$= \sigma\|\phi\|^p_{1,p} + \sigma\|\psi\|^q_{1,q} - \int_{\Omega\setminus\Omega_1} H(x;\sigma^{\frac{1}{p}}\phi, \sigma^{\frac{1}{q}}\psi)dx - \sigma^{\frac{\alpha+1}{p}+\frac{\beta+1}{q}}\int_{\Omega_1}|\phi|^{\alpha+1}|\psi|^{\beta+1}dx$$

Taking σ sufficiently large to have $|\Omega_1| > 0$, we obtain

$$\lim_{\sigma\to+\infty} J(\sigma^{\frac{1}{p}}\tilde{\phi};\sigma^{\frac{1}{q}}\tilde{\psi})=-\infty, \text{ since } \frac{\alpha+1}{p}+\frac{\beta+1}{q}>1.$$

By the continuity for $J(.,.)$ on X, we find a pair (ϕ,ψ) in $X\backslash B_\rho(0)$ such that $J(\phi,\psi) < 0$.

Proof of the theorem 3.1. (1st part). By Mountain-Pass Lemma [2], there exist a pair (u^*,v^*) in X which is a critical point of J. This means that for any $(w_1,w_2) \in X$, $J'(u^*,v^*) \cdot (w_1,w_2) = 0$, i.e

$$\begin{cases} -\Delta_p u^* = \dfrac{\partial H}{\partial u}(x;u^*,v^*) & \text{in } \Omega \\ -\Delta_q v^* = \dfrac{\partial H}{\partial v}(x;u^*,v^*) & \text{in } \Omega. \end{cases}$$

So, we have proved that (P) posseses a nontrivial solution in X. The second part is devoted to prove that the solutions are bounded in Ω.

Moreover, [26] (c.f the definition for H) ensure $u^* \geq 0$ and $v^* \geq 0$ in Ω.

3.2. L^∞-Estimate of the solution

3.2.0. Introduction. In this part, we use an iterative method to estimate the solution (u^*,v^*) obtained in section 3.1. We prove here that in fact $(u^*,v^*) \in [L^\infty(\Omega)]^2$.

In this matter, the crucial point is the construction of two strictly increasing unbounded sequences $\{\lambda_k;\ k \geq 0\}$ and $\{\mu_k;\ k \geq 0\}$ such that u^* and v^* verify:

$$\text{If } \begin{cases} u^* \in L^{\lambda_k}(\Omega) \\ v^* \in L^{\mu_k}(\Omega) \end{cases} \quad \text{then } \begin{cases} u^* \in L^{\lambda_{k+1}}(\Omega) \\ v^* \in L^{\mu_{k+1}}(\Omega) \end{cases}$$

We shall present some properties deriving to the fact that u^* and v^* belong to $L^{\lambda_k}(\Omega)$ and $L^{\mu_k}(\Omega)$ respectively. In a second step, we shall proceed to the appropriate construction for these sequences.

It is very important to note that this iterative schema use some regularity properties of u^* and v^*, for example (u^*,v^*) belong to $[C^2(\Omega) \cap C^1(\bar{\Omega})]^2$. The study of regularized equations (cf. [20], [26]) allows us to suppose u^* and v^* smooth throughout all this part. Though we do not make extensive development about our iterative method, more detailed proofs are given in [31].

Proposition 3.2. *Suppose that all the hypotheses of Theorem 3.1. are satisfied. Then, there exist sequences $\{\lambda_k;\ k \geq 0\}$ and $\{\mu_k;\ k \geq 0\}$ such that*

1) For each k, u^* and v^* belong respectively to $L^{\lambda_k}(\Omega)$ and $L^{\mu_k}(\Omega)$.
2) There exist two real constants A_p and A_q be such that

$$\|u^*\|_\infty \leq \overline{\lim_{k \to +\infty}} \|u^*\|_{L^{\lambda_k}(\Omega)} \leq A_p$$

$$\|v^*\|_\infty \leq \overline{\lim_{k \to +\infty}} \|v^*\|_{L^{\mu_k}(\Omega)} \leq A_q$$

Lemma 3.2.1. *Let π_p (resp. π_q) be such that*

$$1 < \pi_p < \frac{np}{n-p} \quad (\text{resp. } 1 < \pi_q < \frac{nq}{n-q}),$$

and for any $k \geq 0$

$$a_k=\lambda_k\left(1-\frac{\alpha}{\lambda_k}-\frac{\beta+1}{\mu_k}\right)-1 \quad (1)_k$$

$$b_k=\mu_k\left(1-\frac{\alpha+1}{\lambda_k}-\frac{\beta}{\mu_k}\right)-1 \quad (2)_k$$

Then there are some constants c and c' such that for any $u^* \in L^{\lambda_k}(\Omega)$ and $v^* \in L^{\mu_k}(\Omega)$ we have

$$\int_\Omega |u*|^{\left(1+\frac{a_k}{p}\right)\pi_p} dx \le c\left(1+\frac{a_k}{p}\right)^{\pi_p} \theta_k^{(\pi_p/p)}, \quad \int_\Omega |v*|^{\left(1+\frac{b_k}{q}\right)\pi_q} dx \le c'\left(1+\frac{b_k}{q}\right)^{\pi_q} \Phi_k^{(\pi_q/p)}$$

where θ_k and Φ_k are defined as

$$\theta_k=\int_\Omega \frac{\partial H}{\partial u}(x;u*,v*)u*|u*|^{a_k}dx, \quad \Phi_k=\int_\Omega \frac{\partial H}{\partial v}(x;u*,v*)v*|v*|^{b_k}dx.$$

Proof of the Lemma 3.2.1. Multiplying (1*) by $u^*|u^*|^{a_k}$ and integrating on Ω, we obtain

(3.2.1) $$\int_\Omega |\nabla u*|^{p-2}\nabla u*\nabla[u*|u*|^{a_k}]=\int_\Omega \frac{\partial H}{\partial u}(x;u*,v*)u*|u*|^{a_k}dx$$

On the other hand, we have,

(3.2.2) $$\int_\Omega \left|\nabla\{u*\}^{1+\frac{a_k}{p}}\right|^p=\left(1+\frac{a_k}{p}\right)^p\int_\Omega |u*|^{a_k}|\nabla u*|^p dx$$

Since, u^* is in $C^1(\bar{\Omega})$, so is $\{u^*\}^{1+a_k/p}$ and consequently $\{u^*\}^{1+a_k/p}$ belongs to $W_0^{1,p}(\Omega)$. The continuous imbedding $W_0^{1,p}(\Omega) \hookrightarrow L^{\pi_p}(\Omega)$ implies the existence of a constant $c > 0$ such that

(3.2.3) $$\left(\int_\Omega |u*|^{\left(1+\frac{a_k}{p}\right)\pi_p}\right)^{1/\pi_p} dx \leq c\left(\int_\Omega \left|\nabla (u*)^{1+\frac{a_k}{p}}\right|^p\right)^{1/p}$$

Since a_k is nonnegative, (3.2.1), (3.2.2), (3.2.3) give,

$$\int_\Omega |u*|^{\left(1+\frac{a_k}{p}\right)\pi_p} \leq c\left(1+\frac{a_k}{p}\right)^{\pi_p}\left[\int_\Omega |\nabla u*|^p |u*|^{a_k} dx\right]^{\pi_p/p}$$

(3.2.4)

$$\leq c\left(1+\frac{a_k}{p}\right)^{\pi^p} \theta_k^{\pi_p/p}$$

Lemma 3.2.2. *Assume that*

$$\lambda_{k+1} \leq \left(1+\frac{a_k}{p}\right)\pi_p \quad (3)_k, \qquad \mu_{k+1} \leq \left(1+\frac{b_k}{q}\right)\pi_q \quad (4)_k.$$

Then, If $u^* \in L^{\lambda_k}(\Omega)$ and $v^* \in L^{\mu_k}(\Omega)$, we have

$\|u*\|_{L^{\lambda_{k+1}}(\Omega)}^{\lambda_{k+1}}$

(3.2.5)

$$\leq K_p^{\lambda_{k+1}}\left\{c^{\frac{1}{\pi_p}}\left(1+\frac{a_k}{p}\right)\left\{A_1\|u*\|_{L^{\lambda_k}(\Omega)}^{\lambda_k} + A_2\left(\|u*\|_{L^{\lambda_k}(\Omega)}^{\mu_k}\right)^{\frac{\alpha+a_k+1}{\mu_k}}\left(\|v*\|_{L^{\mu_k}(\Omega)}^{\lambda_k}\right)^{\frac{\beta+1}{\lambda_k}} + A_3\right\}^{\frac{1}{p}}\right\}^{\frac{\lambda_{k+1}}{1+\frac{\alpha_k}{p}}}$$

where $A_i (i=1;2;3)$ are positive constants.

Proof. We first call (c.f (3.1.7)) that the hypotheses on H imply the existence of positive constants A_i (i=1;2) such that for any real numbers s and t,

$$\frac{\partial H}{\partial s}(x;s,t)\leq A_1+A_2|s|^{\alpha}|t|^{\beta+1}$$

Thus, by Hölder's inequality we obtain

$$\int_\Omega \frac{\partial H}{\partial u}(x;u*,v*)u*|u*|^{a_k}dx\leq A_1\int_\Omega |u*|^{a_k+1}dx+A_2\int_\Omega |u*|^{\alpha+a_k+1}|v*|^{\beta+1}dx$$

(3.2.6)
$$\leq A_1\int_\Omega |u*|^{\lambda_k}dx+A_2\int_\Omega |u*|^{\alpha+a_k+1}|v*|^{(\beta+1)}dx+A_3$$

$$\leq A_1\int_\Omega |u*|^{\lambda_k}dx+A_2\left(\int_\Omega |u*|^{\lambda_k}dx\right)^{\frac{\alpha+a_k+1}{\lambda_k}}\left(\int_\Omega |v*|^{\mu_k}dx\right)^{\frac{(\beta+1)}{\mu_k}}+A_3$$

That implies with (3.2.4),

(3.2.7)

$$\int_\Omega |u*|^{\left(1+\frac{a_k}{p}\right)\pi_p}\leq c\left(1+\frac{a_k}{p}\right)^{\pi_p}\left[\int_\Omega |\nabla u*|^p|u*|^{a_k}dx\right]^{\pi_p/p}$$

$$\leq c\left(1+\frac{a_k}{p}\right)^{\pi_p}\left[A_1\int_\Omega |u*|^{\lambda_k}dx+A_2\left(\int_\Omega |u*|^{\lambda_k}dx\right)^{\frac{\alpha+a_k+1}{\lambda_k}}\left(\int_\Omega |v*|^{\mu_k}dx\right)^{\frac{(\beta+1)}{\mu_k}}+A_3\right]^{\pi_p/p}$$

Now, by (3_k), $L^{(1+a_k/p)\pi_p}(\Omega)$ is continuously imbedded into $L^{\lambda_{k+1}}(\Omega)$, so there exists a constant K_p such that

$$\left(\int_\Omega |u*|^{\lambda_{k+1}}dx\right)^{1/\lambda_{k+1}} \leq K_p\left(\int_\Omega |u*|^{\left(1+\frac{a_k}{p}\right)\pi_p}dx\right)^{1/\left(1+\frac{a_k}{p}\right)\pi_p}.$$

Combined with (3.2.7), we have

(3.2.8)

$$\int_\Omega |u*|^{\lambda_{k+1}}dx$$

$$\leq K_p^{\lambda_{k+1}}\left\{c^{\frac{1}{\pi_p}}\left(1+\frac{a_k}{p}\right)\left[A_1\int_\Omega |u*|^{\lambda_k}dx + A_2\left(\int_\Omega |u*|^{\lambda_k}dx\right)^{\frac{\alpha+a_k+1}{\lambda_k}}\left(\int_\Omega |v*|^{\mu_k}dx\right)^{\frac{(\beta+1)}{\mu_k}} + A_3\right]^{\frac{1}{p}}\right\}^{\frac{\lambda_{k+1}}{1+\frac{\alpha_k}{p}}}.$$

An analogous result is obtained for $v*$.

3.2.1. Definition and construction of sequences $\{\lambda_k;\ k\in\mathcal{N}\}$ and $\{\mu_k;\ k\in\mathcal{N}\}$. Here, we construct the sequences $\{\lambda_k;\ k\in N\}$ and $\{\mu_k;\ k\in N\}$. This construction requires similar tools as in [20], [26] or [27] use for the study of first eigenvalue, but here the problem is different from [27], because α and β do not verify

$$\frac{\alpha+1}{p}+\frac{\beta+1}{q}=1.$$

Here, the first terms of each sequence cannot be determined directly by using the Rellich-Kondrachov's continuous imbedding result. So, we first construct Lebesgue spaces of exponents $\hat{\lambda}_k$ and $\hat{\mu}_k$ containing respectively $u*$ and $v*$. By an appropriate choice for $k_0\in\mathcal{N}$, $\hat{\lambda}_{k_0}$ and $\hat{\mu}_{k_0}$ give the respective first terms of $\{\lambda_k;\ k\geq 0\}$ and $\{\mu_k;\ k\geq 0\}$. After that, we shall show that $u*$ and $v*$ are estimated independently to k by a same constant in every $L^{\lambda_k}(\Omega)$ and $L^{\mu_k}(\Omega)$ spaces respectively. This is not always the case when we are limiting us only to $L^{\hat{\lambda}_k}(\Omega)$ and $L^{\hat{\mu}_k}(\Omega)$ spaces.

a) Construction of $\{\hat{\lambda}_k;\ k>0\}$ and $\{\hat{\mu}_k;\ k>0\}$. We consider here α and β satisfying the relations

$$\frac{\alpha+1}{p}\left(\frac{n-p}{n}\right)+\frac{\beta+1}{q}\left(\frac{n-q}{n}\right)<1$$

(3.2.9)

$$\frac{\alpha+1}{p}(1)+\frac{\beta+1}{q}(1)>1$$

So, we can find $\hat{C}>1$ and (λ,μ) such that

$$\text{(3.2.10)}\qquad \begin{cases} 1<\lambda<\dfrac{n}{(n-p)\hat{C}} \\ 1<\mu<\dfrac{n}{(n-q)\hat{C}} \\ \dfrac{\alpha+1}{\lambda p}+\dfrac{\beta+1}{\mu q}=1 \end{cases}$$

Now, we take $\hat{\lambda}_k=\lambda p\hat{C}^k$, $\hat{\mu}_k=\mu q\hat{C}^k$.

From $(1)_k$ and $(2)_k$, we get

$$\text{(3.2.11)}\qquad \begin{cases} \hat{a}_k=\hat{\lambda}_k-\lambda p \\ \hat{b}_k=\hat{\mu}_k-\mu q \end{cases}$$

Lemma 3.2.3. *For each $k\in\mathcal{N}$, u^* and v^* belong respectively to $L^{\hat{\lambda}_k}(\Omega)$ and $L^{\hat{\mu}_k}(\Omega)$.*

Proof. We give a proof by induction.

By Sobolev imbedding Theorem, we have $u^*\in L^{\lambda p}(\Omega)$; $v^*\in L^{\mu q}(\Omega)$.

Then the Lemma is proved for $k = 0$. Suppose that it is true for all integer k' such that $0 \leq k' \leq k \in \mathcal{N}$.

Take $\pi_p = \lambda p\hat{C}$ and $\pi_q = \mu q\hat{C}$, and $u^* \in L^{\hat{\lambda}_k}(\Omega)$. The relation:

$$\left(1+\frac{a_k}{p}\right)\pi_p=\lambda^2 p\hat{C}^{k+1}+\lambda p\hat{C}-\lambda^2 p\hat{C}\geq\lambda p\hat{C}^{k+1}=\hat{\lambda}_{k+1},$$

and Lemma 3.2.1. give $u^* \in L^{\hat{\lambda}_{k+1}}(\Omega)$ and $v^* \in L^{\hat{\mu}_{k+1}}(\Omega)$.

b) Construction of sequences $\{\lambda_k;\ k \in N\}$ and $\{\mu_k;\ k \in N\}$. Let

$$C=\min\left(\frac{n}{n-p},\frac{n}{n-q}\right),\ \gamma=\frac{\alpha+1}{\lambda p}+\frac{\beta+1}{\mu q},\ \delta=\{M-(\gamma-1)\}C,$$

with $M > \gamma\text{-}1$; we define the sequences $\{\lambda_k;\ k \in \mathcal{N}\}$ and $\{\mu_k;\ k \in \mathcal{N}\}$ by

$$\lambda_k=pf_k,\quad \mu_k=qf_k$$

where f_k denotes the sequence

(3.2.12) $$f_k=\frac{C}{C-1}[\delta C^{k-1}+(\gamma-1)].$$

Remark the sequences $\{\lambda_k;\ k \in \mathcal{N}\}$ and $\{\mu_k;\ k \in \mathcal{N}\}$ are strictly increasing and tend to $+\infty$, futhermore, we have the iterative relation

$$f_{k+1}=C[f_k-(\gamma-1)]\quad (5)_k.$$

Proof of Proposition 3.2. We proceed again by induction.

First, we use the fact that the sequences $\hat{\lambda}_k$ and $\hat{\mu}_k$ are strictly increasing to establish the existence of an integer k_0 such that $\lambda_0\geq\hat{\lambda}_{k_0}$ and $\mu_0\geq\hat{\mu}_{k_0}$; we obtain from Lemma 3.2.3. that $u^* \in L^{\lambda_0}(\Omega)$ and $v^* \in L^{\mu_0}(\Omega)$.

Suppose that the proposition is true for $0 \leq k' \leq k$. Let $\pi_p = Cp$ and $\pi_q = Cq$, $(1)_k$ and $(2)_k$ give: $a_k = p(f_k\text{-}\gamma)$ and $b_k = q(f_k\text{-}\gamma)$.

So,

$$1+\frac{a_k}{p}=1+f_k-\gamma<\frac{C}{C-1}\left[\frac{\delta}{C}+(\gamma-1)\right]C^k.$$

Moreover by $(5)_k$ we obtain

$$\lambda_{k+1}=\left(1+\frac{a_k}{p}\right)\pi_p,$$

and similarily

$$\mu_{k+1}=\left(1+\frac{b_k}{q}\right)\pi_q.$$

Then, we conclude with Lemma 3.2.2. that $u^* \in L^{\lambda_{k+1}}(\Omega)$, according to (3.1.6) and taking

$$A=\frac{C}{C-1}\left[\frac{\delta}{C}+(\gamma-1)\right],$$

$$\|u*\|^{\lambda_{k+1}}_{L^{\lambda_{k+1}}(\Omega)}\leq c\left(1+\frac{a_k}{p}\right)^{C_p}\left\{A_1\|u*\|^{\lambda_k}_{L^{\lambda_k}(\Omega)}+A_2\left(\|u*\|^{\lambda_k}_{L^{\lambda_k}(\Omega)}\right)^{\frac{\alpha+a_k+1}{\lambda_k}}\left(\|v*\|^{\mu_k}_{L^{\mu_k}(\Omega)}\right)^{\frac{\beta+1}{\mu_k}}\right\}^C$$

(3.2.13)

$$\leq A^C C^{kCp}\max\left(1;\|u*\|^{\lambda_k}_{L^{\lambda_k}(\Omega)};\|v*\|^{\mu_k}_{L^{\mu_k}(\Omega)}\right)^C.$$

Considering the equality

$$-\Delta_q v^* = \frac{\partial H}{\partial v}(x; u*, v*),$$

we obtain an analogous inequality

$$\|v*\|^{\mu_{k+1}}_{L^{\mu_k}(\Omega)} \leq A^C C^{kCq} \max\left(1; \|u*\|^{\lambda_k}_{L^{\lambda_k}(\Omega)}; \|v*\|^{\mu_k}_{L^{\mu_k}(\Omega)}\right)^C \quad (3.2.14)$$

As in [20], [26], [27], we obtain the iterative relation $E_{k+1} \leq r_k + CE_k$, where

(3.2.15)
$$\begin{cases} E_k = \ln \max\left(\|u^*\|^{\lambda_k}_{L^{\lambda_k(\Omega)}}; \|v^*\|^{\mu_k}_{L^{\mu_k(\Omega)}}\right) \\ r_k = ak + b \quad a = \ln C^{C\max(p,q)}, \quad b = \ln(A)^C \end{cases}$$

So, we get the iterative relation $E_k \leq dC^{k-1}$, where d denotes a positive constant.

Thus,

(3.2.16)
$$\|u*\|_{L^{\lambda_k}(\Omega)} \leq \exp\left(\frac{E_k}{\lambda_k}\right) \leq \exp\left(\frac{d(C-1)}{pC\delta}\right)$$

$$\|v*\|_{L^{\mu_k}(\Omega)} \leq \exp\left(\frac{d(C-1)}{qC\delta}\right)$$

then, u^* and v^* are bounded in $L^{\lambda_k}(\Omega)$ and $L^{\mu_k}(\Omega)$ independently of $k \in \mathcal{N}$.

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