REVISTA MATEMÁTICA de la
Universidad Complutense de Madrid
Volumen 6, número 1; 1993.

Dunford-Pettis-like Properties of Continuous Vector Function Spaces

JESUS M.F. CASTILLO AND FERNANDO SANCHEZ

ABSTRACT. In this paper, the structure of some operator ideals $\mathcal{A}$ defined on continuous function spaces is studied. Conditions are considered under which "$T\in\mathcal{A}$" and "the representing measure of T takes values in $\mathcal{A}$" are equivalent for the scales of p-converging (C_p) and weakly-p-compact (W_p) operators. The scale C_p is intermediate between the ideals $C_1=\mathcal{U}$ (unconditionally summing operators), and $C_\infty=\mathcal{B}$ (completely continuous operators), which have been studied by several authors (Bombal, Cembranos, Rodríguez-Salinas, Saab). The dual scale W_p is intermediate between the ideals $\mathcal{K}$(compact operators) and $W_\infty=W$ (weakly compact operators), and the results presented have a close connection with those of Diestel, Núñez and Seifert.

1. PRELIMINARIES

In this paper, $B(\Sigma,X)$ denotes the space of all bounded X-valued Σ-measurable functions; if $1\le p\le\infty$, p^* denotes the conjugate number of p; if $p=1$, l_{p^*} plays the role of c_0.

1.1. Definition. *A sequence (x_n) in a Banach space X is said to be weakly-p-summable ($1\le p\le\infty$) if $(x^*x_n)\in l_p$ for all $x^*\in X^*$, or equivalently, if there is a constant $C>0$ such that*

1991 Mathematics Subject Classification: 46E15, 46B28, 46B25.
Editorial Complutense. Madrid, 1993.

$$\sup_n \|\sum_{k=1}^{n} \xi_k x_k\| \leq C \cdot \|(\xi_n)\|_{l_{p^*}}$$

for any sequence $(\xi_n) \in l_{p^*}$. *We shall denote by* $w_p((x_n)_n)$ *the infimum of those constants C.*

We shall say that (x_n) *is weakly-p-convergent to* $x \in X$ *if* $(x_n - x)$ *is weakly-p-summable. Weakly-∞-convergent sequences are simply the weakly convergent sequences.*

1.2. Definition. *Let* $1 \leq p \leq \infty$. *An operator* $T \in \mathcal{L}(X,Y)$ *is said to be p-convergent if it transforms weakly-p-summable sequences into norm null sequences. We shall denote by* C_p *the class of p-convergent operators.*

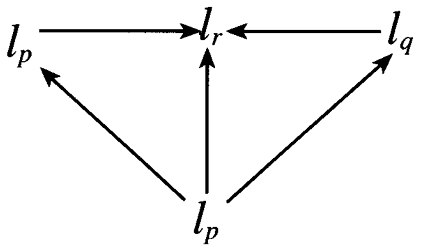
When $p=\infty$ this definition gives the ideal B of completely continuous operators, that is to say, those transforming weakly null sequences into norm null sequences. When $p=1$, it is easy to verify that $C_1=U$, the ideal of unconditionally summing operators, i.e., those transforming weakly-1-summable sequences into summable ones. Obviously $C_q \subset C_p$ when $p<q$.

The scale of C_p ideals are intermediate between the ideals B and U. It is clear (from the definition) that C_p are injective operator ideals, and, since any separable Banach space is a quotient of l_1, they are not surjective. On the other hand, it is easy to see that C_p is closed: let (T_n) be a sequence of p-converging operators with limit (in the operator norm) T. If (x_n) is a weakly p-summable sequence and $\varepsilon>0$, then $\|Tx_n\| \leq \varepsilon \|x_n\| + \|T_k x_n\| \leq 2\varepsilon$ and (Tx_n) is norm null.

1.3. Definition. *A bounded set K in a Banach space is said to be relatively weakly-p-compact* $(1 \leq p \leq \infty)$ *if every sequence in K has a weakly-p-convergent sub-sequence. An operator* $T \in \mathcal{L}(X,Y)$ *is said to be weakly-p-compact,* $1 \leq p \leq \infty$, *if* $T(B_X)$ *is relatively weakly-p-compact. We shall denote by* W_p *the ideal of weakly-p-compact operators.*

The W_p operators are meant to be a gradations of the class of weakly compact operators. It is clear that $W_\infty=W$ (weakly compact operators), and it is easy to see that $id(X)\in W_1$ if and only if X is finite dimensional. Obviously $W_p\subset W_q$ when $p<q$.

The ideals W_p are injective and surjective but not closed. The ideal W_1 is not closed since $W_1\neq W_1^2=K$, the ideal of compact operator (see [14]). To see W_p is not closed for $p>1$, we apply [14, Prop. 1.6] to the diagram:



for $1<p<r<q$. The left arrow is the identity and the right arrow is the inclusion, which belongs to W_{q^*}. If this operator ideal was closed, the middle inclusion should also be in W_{q^*}, which is not, since $C_p\circ W_p=K$ and .

1.4. Proposition. *Let* $1<p<\infty$, *then* $id\ (l_p)\in W_{p^*}$.

Proof. Let (x_n) be a bounded sequence in l_p. It admits a weakly convergent sub-sequence (x_k). Let x be its weak limit, and let us call $y_k=x_k-x$. If (y_k) is norm null, we have finished. If not, and we have $\|y_k\|\geq\varepsilon>0$ for some sub-sequence, applying the Bessaga-Pelczynski selection principle, we obtain a new sub-sequence, equivalent to the canonical basis (e_n) of l_p, which is weakly p^*-summable.

An easy consequence is:

1.5. Proposition. $\mathcal{L}(l_{p^*},X)=K(l_{p^*},X)$ *if and only if* $id(X)\in C_p$.

Moreover, an operator T belongs to $C_p(X,Y)$ if and only if for each $j\in\mathcal{L}(l_p,X)$ the composition $T\circ j$ is compact. From this and the proof of (2.5) we obtain

1.6. Proposition. *If* $T\in W_p(X,Y)$ *then* $T^*\in C_r(Y^*,X^*)$ *for all* $r<p^*$.

1.7. Corollary. *Let* $1<p<\infty$, $id(l_p)\in C_r$ *for all* $r<p^*$.

Remarks.

1. The progression expressed by (1.7) suddenly breaks down when $p<1$, due to [17], where it is shown that a weakly-1-summable sequence (x_n) exists in each l_p, $p<1$, for which $\|x_n\|_p \longrightarrow +\infty$.

2. Regarding Proposition 1.5, this result is equivalently to Pitt's lemma: $\mathfrak{L}(l_p,l_q) = K(l_p,l_q)$ if and only if $p>q$.

For L_p spaces the situation is:

1.8. Proposition.

a) *If* $2\leq p<\infty$ *then* $id(L_p)\in W_2$.
b) *If* $1<p<2$ *then* $id(L_p)\in W_{p^*}$.

Proof. Part a) can be obtained by using the Kadec-Pelczynski alternative: every normalized weakly null sequence in L_p has a sub-sequence equivalent either to the unit vector basis of l_p or the unit vector basis of l_2.

Part b) follows from a standard duality argument. If (x_n) is a normalized weakly null sequence in L_p and (x_k) is a basic sub-sequence of (x_n), consider a bounded sequence (y_k) of biorthogonal functionals in L_{p^*}, and (again) the Kadec-Pelczynski alternative.

1.9. Examples. (See [21] for details). We shall abbreviate $id(X)\in C_p$ (resp. $id(X)\in W_p$) by saying $X\in C_p$ (resp. $X\in W_p$).

a) If $1\leq p<\infty$, $l_p\in C_r$ for $1\leq r<p^*$, and $l_p\in W_{p^*}$ for $1<p<\infty$ (see (1.4) and (1.7)).

b) If $1 \le p < \infty$, $L_p(\mu) \in C_r$ for $r < \min(2, p^*)$. If $1 < p < \infty$, $L_p(\mu) \in W_r$ for $r = \max(2, p^*)$ (see (1.8) and (1.6)).

c) Tsirelson's space T is such that $T \in C_p$ for all $p \neq \infty$ (see [7]).

d) Tsirelson's dual space T^* is such that $T^* \in W_p$ for all $p > 1$ (see [7]).

e) Super-reflexive spaces belong to some class W_p and, consequently, to some class C_q (see [6]).

f) If $X, l_r \in W_p$ then so does $l_r(X)$ (see [8]).

It is well-known [12] that every operator T from $C(K,X)$ to Y has a finitely additive representing measure m of bounded semi-variation, defined on the Borel σ-field Σ of K and with values in $\mathcal{L}(X, Y^{**})$, in such a way that

$$T(f) = \int f dm, \quad (f \in C(K,X)).$$

If $m: Bo(K) \longrightarrow \mathcal{L}(X,Y)$ is a finitely additive measure, we shall denote by $|m|$ its semi-variation. One says that $|m|$ is continuous at $\varnothing$ if it has a control mesasure: a countably additive positive measure λ on $Bo(K)$ such that

$$\lim_{\lambda(A) \to 0} |m|(A) = 0.$$

1.10. Proposition. *When $T \in W(C(K,X),Y)$, its associated representing measure m is countably additive and verifies the following two conditions:*

a) $|m|$ *is continuous at* $\varnothing$, *and*
b) *for each* $A \in Bo(K)$, $m(A) \in W(X,Y)$.

Thus, it seems natural to ask which properties pass from T to m and viceversa.

2. OPERATORS AND MEASURES

By mimicry of the proofs made in [3], [4] and [20] for the cases $p=1,\infty$ one can easily obtain:

2.1. Proposition. *Let $T\in C_p(C(K,X),Y)$, and let m its representing measure. Then:*

a) *$|m|$ is continuous at $\varnothing$, and*
b) *for each $A\in Bo(K)$, $m(A)\in C_p(X,Y)$.*

Nevertheless, these two conditions a) and b) do not characterize C_p operators. In [1], there is shown an operator T from $C([0,1],c_0)$ to c_0 which is not in C_1 but is such that its representing measure m has continuous semi-variation at $\varnothing$, and $m(A)$ is a compact operator for any Borel set $A\subset[0,1]$.

2.2. Proposition. *Let $T\in \mathcal{L}(C(K,X),Y)$ have a representing measure m satisfying:*

a) *$|m|$ is continuous at $\varnothing$ and admits a discrete control measure, and*
b) *for each $A\in Bo(K)$, $m(A)\in C_p(X,Y)$.*

Then $T\in C_p(X,Y)$.

Since every Radon measure over a dispersed compact set is discrete (see [16, §2]), it follows that:

2.3. Corollary. *If K is dispersed and $T\in \mathcal{L}(C(K,X),Y)$ is such that its representing measure m satisfies:*

a) *$|m|$ is continuous at $\varnothing$, and*
b) *for each $A\in Bo(K)$, $m(A)\in C_p(X,Y)$,*
then $T\in C_p(X,Y)$.

Corollary (2.3) asserts that (2.1) is an equivalence when K is dispersed. We can also expect an equivalence when some condition is imposed on X.

2.4. Proposition. *Let $1 \leq p \leq \infty$. The following are equivalent:*

a) $id(X) \in C_p$.

b) *Given any compact space K and any Banach space Y, an operator $T \in C_p(C(K,X),Y)$ if and only if its representing measure satisfies*

i) $|m|$ *is continuous at* $\varnothing$, *and*

ii) *for each* $A \in Bo(K)$, $m(A) \in C_p$.

Concerning the dual scale of weakly-p-compact operators, we have:

2.5. Lemma. *Let $T \in \mathcal{L}(C(K,X),Y)$ and $p \geq 1$. The following are equivalent* ($\hat{T}$ is the restriction to $B(\Sigma,X)$ of the operator T^{**}):

a) $T \in W_p(C(K,X),Y)$, b) $\hat{T} \in W_p(B(\Sigma,X),Y)$, c) $T^{**} \in W_p(C(K,X)^{**},Y)$.

Proof. Since $T \in W(A,B)$ if and only if T^* (or any of its iterated duals) is weak*-to-weak continuous, and the unit ball of A is weak*-dense in the unit ball of A^{**}, we have:

$$T^{**}(B_{A^{**}}) = T^{**}(\overline{B_A}^{\sigma(A^{**},A^{*})}) \subset \overline{T(B_A)}$$

from which the result follows.

That immediately gives:

2.6. Proposition. *Let $T \in W_p\ (C(K,X),Y)$, $p \geq 1$. Its associated measure verifies:*

a) $|m|$ *is continuous at* $\varnothing$, *and*
b) *for each* $A \in Bo(K)$, $m(A) \in W_p(X,Y)$.

The converse is not true; see the comments after (2.1).

3. DUNFORD-PETTIS-LIKE PROPERTIES

A Banach space X is said to have the *Dunford-Pettis property* if any weakly compact operator $T:X \longrightarrow Y$ transforms weakly compact sets of X into norm compact sets of Y. This property can be described by means of the inclusion $W(X,Y) \subset B(X,Y) = C_\infty(X,Y)$. We can weaken this requirement in the following manner:

3.1. Definition. *Let* $1 \leq p \leq \infty$. *We shall say that a Banach space X has the Dunford-Pettis property of order p (in short* DPP_p*) if the inclusion* $W(X,Y) \subset C_p(X,Y)$ *holds for any Banach space Y.*

Obviously DPP_p implies DPP_q when $q<p$. Also, $DPP=DPP_\infty$ and every Banach space has DPP_1. It follows from the definition that if $id(X) \in C_p$ then X has DPP_p, and that if $id(X) \in W_p$ then X does not have DPP_p. The following result contains analytical and geometrical characterizations of the DPP_p.

3.2. Proposition. *For a given Banach space X, the following are equivalent:*

a) *X has* DPP_p $(1 \leq p \leq \infty)$.

b) *If* (x_n) *is a weakly-p-summable sequence of X and* (x_n^*) *is weakly null in* X^* *then* $(x_n^* x_n) \longrightarrow 0$.

c) *Every weakly compact operator* $T:X \longrightarrow Y$ *transforms weakly-p-compact sets of X into norm compact sets of Y.*

Proof. The proof of the equivalence between (a) and (b) is obtained as in [21]. We prove the equivalence of (a) and (c).

(c)⇒(a): Consider $T:X \longrightarrow Y$ a weakly compact operator, and (x_n) a weakly-p-summable sequence in X. We form the set:

$$conv_{p^*}((x_n)) = \{ \sum_{n=1}^{\infty} \lambda_n x_n : \Sigma_n |\lambda_n|^{p^*} \leq 1 \}$$

which we shall refer to as the *p^*-convex hull* of (x_n). Clearly, $conv_{p^*}(x_n)$, the continuous image by the natural operator associated to (x_n) of the unit ball of l_{p^*}, is a weakly-p-compact set. Since $T \in C_p$ and $l_p \in W_{p^*}$, $T(conv_{p^*}(x_n))$ is compact, and (Tx_n) is norm-null.

(a)⇒(c): If A is a weakly-p-compact set of X, then for each bounded sequence (z_m) of A there is a point $z \in A$, and a sub-sequence (z_n), such that $(z_n - z)$ is weakly-p-summable. We set $(x_n)=(z_n - z)$, and apply to this sequence the preceding argument, to conclude that (Tx_n) admits a norm null sub-sequence.

3.3. Examples. The following examples are immediate after (1.9). In fact, these results give the optimum values of p.

a) $C(K)$ and L_1 have the *DPP*, and therefore the DPP_p for all p.

b) If $1<r<\infty$, l_r has the DPP_p for $p<r^*$.

c) If $1<r<\infty$, $L_r(\mu)$ has the DPP_p for $p<\min(2,r^*)$.

d) Tsirelson's space T has DPP_p for all $p<\infty$. However, since T is reflexive, it does not have *DPP*.

e) Tsirelson's dual space T^* does not have DPP_p for any $p>1$.

Coming back to continuous vector function spaces, we have:

3.4. Proposition. *If $id(X) \in C_p$ then, for any compact K, $C(K,X)$ has DPP_p.*

Proof. Let $T \in W(C(K,X),Y)$. If (f_n) is a weakly-p-summable sequence in $C(K,X)$, then for each $t \in K$, the sequence $(f_n(t))$ is also weakly-p-summable in X, and thus it is norm null. The sequence (Tf_n) is also null by [5, Th. 2.1].

3.5. Corollary. *Given any compact space K and $1<p<\infty$, $C(K,l_p)$ has DPP_r for all $r<p^*$; it does not have DPP_{p^*}.*

A "limit case" is provided by Tsirelson's spaces (compare this result with (3.13)):

3.6. Corollary. *If T denotes Tsirelson's space then, given any compact space K and $1<p<\infty$, $C(K,T^*)$ has DPP_p but not DPP.*

Now, we see what happens if we replace the condition "$id(X) \in C_p$" by the weaker "*X has the DPP_p*".

3.7. Example. Talagrand's construction of a Banach space X having *DPP* but such that $C(K,X)$ does not have *DPP* (see [22]), can be modified in such a form that we obtain Banach spaces T_p ($p>1$) having *DPP*, and such that $C(K,T_p)$ does not have DPP_p. Talagrand's original example corresponds to T_2.

What can be said about $C(K,X)$ when X simply has DPP_p? The following theory was developed in [4] and [2] for *DPP*.

3.8. Definition. *An operator $T:C(K,X) \longrightarrow Y$, whose associated measure m has continuous semi-variation at $\varnothing$, is said to be almost-C_p if, for each weakly-p-summable sequence (x_n) of X and each bounded sequence (ϕ_n) of $C(K)$, the sequence $T(\phi_n x_n)$ converges to 0 in Y. Obviously, C_p-operators are almost-C_p.*

3.9. Theorem. *The following are equivalent:*

a) *X has DPP_p.*

b) *For each compact space K, every weakly compact operator $T:C(K,X) \longrightarrow Y$ is almost-C_p.*

c) *Every weakly compact operator $T:C([0,1],X) \longrightarrow Y$ is almost-C_p.*

d) *Every weakly compact operator $T:C([0,1],X) \longrightarrow c_0$ is almost-C_p.*

(The proof is exactly as [2, Th. 5]).

3.10. Corollary ([10, [13]). *Let $1 \leq p \leq \infty$. For a dispersed compact space K, the following are equivalent:*

a) *$C(K,X)$ has DPP_p.*
b) *X has DPP_p.*

Proof. Implication a)⇒b) follows from (3.9). Conversely, if $T \in W(C(K,X),Y)$ with representing measure *m*, for each Borel set $A \subset K$, $m(A) \in W(X,Y) \subset C_p(X,Y)$, since *X* has DPP_p. Applying (2.3), we obtain $T \in C_p$.

Concerning the scales W_p, Diestel and Seifert proved in [11] that weakly compact operators defined on *C(K)* spaces are *Banach-Saks* operators. Recall that an operator $T \in \mathfrak{L}(X,Y)$ is said to be Banach-Saks (in short $T \in BS$) if any bounded sequence (x_n) of *X* admits a sub-sequence (x_m) such that (Tx_m) has norm-convergent arithmetic means.

Núñez [18] extended this result to *C(K,X)* spaces showing that, when *X* is super-reflexive, then weakly compact operators defined on *C(K,X)* are Banach-Saks. In [9], it is shown a vector measure whose range is not a weakly-*p*-compact set for any *p*. That example provides a weakly compact operator *T*, defined on a certain *C(K)* space, which, for every *p*, does not belong to W_p, showing that, in general, $X \in W_p$ does not imply

$W(C(K,X),Y)\subset W_p(C(K,X),Y)$, and therefore, that in some sense, the result of Diestel and Seifert cannot be improved.

Despite that negative result, when K is a dispersed compact space, some positive results can be obtained:

3.11. Proposition. *If* $X\in W_p$ *then* $W(c_0(X),Y)\subset W_p(c_0(X),Y)$.

Proof. Let $T\in W(c_0(X),Y)$ and let (f_n) be a bounded sequence in $c_0(X)$. Let $\varepsilon>0$. For each $n\in\mathbb{N}$, a number p_n exists so that $\|f_n(k)\|\leq\varepsilon 2^{-n}$ for $k\geq p_n$.

We write $f_n=f_n^d+f_n^i$, where

$$f_n^i=(f_n(1),\dots,f_n(p_n-1),0,0,\dots)$$

and

$$f_n^d=(0,0,\dots,0,f_n(p_n),f_n+1),\dots).$$

Since $\|f_n^d\|\longrightarrow 0$, it is enough to see that $T(f_n^i))$ admits a weakly-p-convergent sub-sequence. For each $k\in\mathbb{N}$, there exists q_k such that $w_p((f_n^i(k)-x_k)_{n\geq q_k})\leq\lambda$ (the constant λ can be chosen uniformly [15]).

We determine inductively a sequence of indices $(q_{s(n)})$ as follows:

$$q_{s(0)}=q_1 \text{ and } q_{s(n+1)}\geq\max\{q_k : k\leq p(q_{s(n)})\}$$

so that $p(q_{s(n+1)})>p(q_{s(n)})$, and consider the sub-sequence $f_n^i=f_{q_{s(n)}}^i$.

We now write $f_n^i=s_n+t_n$ where

$$t_n=(0,0,0,\dots,f_n^i(p_{q_n}),\dots,f_n^i(p_{q_{n+1}}),0,0,\dots),$$

so that it is the continuous image of a block basic sequence constructed against the canonical basis of c_0. We see that, passing to a sub-sequence if necessary, (Tt_n) converges to 0.

The sequence

$$(z_n) = \begin{cases} z_n(k)=f_n^i(k) & \text{if } k\leq p(q_{s(n-1)}), \\ \\ z_n(k)=0 & \text{otherwise,} \end{cases}$$

however, is the continuous image of (a part of) the summing basis $(e_1+...+e_n)_n$ of c_0.

If we set $x=(x_1,x_2,x_3,...)\in l_\infty(X)$, we see, passing again to a sub-sequence if necessary, that $\|Tz_n-T^{**}x\|\leq 2^{-n}$.

Finally, if (ξ_n) is a finite sequence in the unit ball of l_{p^*}, then

$$\|\Sigma_n\xi_n(Ts_n-T^{**}x)\| \leq \|\Sigma_n\xi_n(Ts_n-Tz_n+Tz_n-T^{**}x)\|$$

$$\leq \|T\|\cdot\|\Sigma_n\xi_n(s_n-z_n)\| + 1 \leq \lambda\cdot\|T\| + 1,$$

thus finishing the proof.

Remark. If the choice of indices indicated in the proof is not possible because the sequence (p_n) does not go to infinity, then we would be working in a finite product space X^n; if it is because the sequence of q_n stops at q, then we shall follow the same reasoning as in the last part with the sub-sequence, f_q, f_{q+1}, ...

3.12. Theorem. *Let K be a dispersed compact space and $X\in W_p$. Then:*

$$W(C(K,X),Y)\subset W_p(C(K,X),Y).$$

Proof. Let $T\in W(C(K,X),Y)$ and let (f_n) be a bounded sequence in $C(K,X)$. By a standard argument we can assume K to be countable, $K=\{t_1,t_2,...\}$. Since m (the associated measure of T) has continuous semi-variation at $\varnothing$, a p_n exists for each $n\in\mathbb{N}$ such that, if we set $B_k=\{t_j\colon j\geq k\}$, then $|m|(B_{p_n})\leq 2^{-n}$.

Once more we write $f_n=f_n^d+f_n^i$ where f_n^d converges to 0 and f_n^i is eventually zero. Since f_n^i is a bounded sequence in a space isomorphic to some $c_0(\mathbb{N},X)$, the proof of (3.11) applies.

3.13. Corollary. *If K is a dispersed compact space and T^* denotes Tsirelson's dual space, then $W(C(K,T^*),Y)\subset W_p(C(K,T^*),Y)$ for all $p>1$.*

A sufficient condition on X which guarantees the inclusion $W(C(K,X),Y)\subset W_p(C(K,X),Y)$ is given by:

3.14. Theorem. *If X does not contain c_0 finitely represented, then*

$$W(C(K),X)\subset W_2(C(K),X).$$

Proof. If X does not contain c_0 finitely represented, then there is a $p>1$ such that $\mathfrak{L}(C(K),X)=W(C(K),X)\subset\Pi_p(C(K),X)$ by [19]. But each p-summing operator sub-factorizes through an L_p-space, which gives $\Pi_p\subset W_2$ when $p\geq 2$, and thus for all p.

The hypothesis is not necessary: just consider Tsirelson's space T^*.

4. FINAL REMARKS AND FURTHER QUESTIONS

Results (3.12) and (3.14) suggest the following problems:

Problem K. *Characterize the compacts K such that for any Banach space X*

$$W(C(K),X)\subset W_2(C(K),X).$$

Problem X. *Characterize those Banach spaces X such that for any compact K*

$$W(C(K),X) \subset W_2(C(K),X).$$

Notice that the hypothesis of (3.14) is not necessary: if K is dispersed, then $W(C(K,T^*),Y) \subset W_p(C(K,T^*),Y)$ for all $p>1$ and T^* is not, for any $p<\infty$, of cotype p.

An application could be the following conjecture, essentially due to Drewnowski: Is it true that $\mathcal{L}(l_2,X)=K(l_2,X) \Leftrightarrow \mathcal{L}(l_\infty,X)=K(l_\infty,X)$? One implication is clear. To see the other, notice that $X \in C_2$ and $\mathcal{L}(l_2,X)= K(l_2,X)$ are equivalent. Since $C_2 \circ W_2=K$, and since $X \in C_2$ implies $\mathcal{L}(l_\infty,X)= W(l_\infty,X)$, the question is whether **a**) Banach spaces $X \in C_2$ satisfy affirmatively Problem X, or **b**) the Stone-Čech compactification of $\mathbb{N}$, $\beta\mathbb{N}$, satisfies affirmatively Problem K.

Another unsolved question about the relationships between T and m is the following: Is it true that if K is a dispersed compact, and, for every Borel set A, the operator $m(A) \in W_p$, then $T \in W_p$?

The example in [9] mentioned before (3.11) shows that the hypothesis "*K dispersed*" cannot be removed.

Besides this, Núñez proved in [18] that if $T:C(K,X) \longrightarrow Y$, K is dispersed and, for every Borel set A, the operator $m(A) \in BS$, then $T \in BS$. The connection with Núñez's result is the following:

Obviously property W_p implies the Banach-Saks property. Moreover, for $p>1$, the *p-Banach-Saks* property is defined as follows: A Banach space X is said to have the *p-Banach-Saks* property when each bounded sequence (x_m) admits a sub-sequence (x_n) and a point x such that (x_n-x) is a p-Banach-Saks sequence, i.e., satisfies an estimate of the form

$$\|\sum_{k=1}^{n} x_k\| \leq C \cdot n^{1/p}$$

for some constant $C>0$ and all $n \in \mathbb{N}$. It is also clear that property W_p implies the p^*-Banach-Saks property. In [6] can be seen a proof that, conversely, the p^*-Banach-Saks property implies, for all $r>p$, the property W_r. Therefore, what this question is looking for is the extension of Núñez's result to the scale of p-Banach-Saks properties.

References

[1] BOMBAL, F. *Operators on spaces of vector-valued continuous functions.* Extracta Math. 1, n. 3, 103-114 (1986).

[2] BOMBAL, F. *On the Dunford-Pettis property.* Portugaliae Math. 45, Fasc. 3, 265-272 (1988).

[3] BOMBAL, F. and CEMBRANOS, P. *Characterization of some classes of operators on spaces of vector-valued continuous functions.* Math. Proc. Cambridge Phil. Soc. 97, 137-146 (1985).

[4] BOMBAL, F. and RODRIGUEZ-SALINAS, B. *Some classes of operators on C(K,E). Extension and applications.* Arch. Math. 47, 55-65 (1986).

[5] BROOKS, J.K. and LEWIS, P.W. *Operators on continuous function spaces and convergence in the spaces of operators.* Adv. in Math. 29, 157-177 (1978).

[6] CASTILLO, J.M.F. and SANCHEZ, F. *Weackly-p-compact, p-Banach-Saks and super-reflexive Banach spaces.* To appear in J. Math. Anal. and Appl.

[7] CASTILLO, J.M.F. and SANCHEZ, F. *Remarks on some basic properties of Tsirelson's space.* To appear in Note di Mat.

[8] CASTILLO, J.M.F. and SANCHEZ, F. *Upper-l_p-estimates in vector sequence spaces, with some applications.* Math. Proc. Cambridge Philos. Soc. 113 (1993) 329-334.

[9] CASTILLO, J.M.F. and SANCHEZ, F. *Remarks on the range of a vector measure.* To appear in Glasgow Math. J.

[10] CEMBRANOS, P. *On Banach spaces of vector valued continuous functions.* Bull. Austral. Math. Soc. 28, 175-186 (1983).

[11] DIESTEL, J. and SEIFERT, C.J. *The Banach-Saks ideal, I. Operators acting on $C(\Omega)$.* Commentationes Math. Tomus in honorem Ladislaw Orlicz I, 109-118 (1979).

[12] DIESTEL, J. and UHL, J.J. Jr. *Vector measures.* AMS Math. Surveys, 15 (1977).

[13] DOBRAKOV, I. *On representation of linear operators on $C_0(T,X)$.* Czechoslovak Math. J. 21, 13-30 (1971).

[14] HEINRICH, S. *Closed operators and interpolation.* J. Funct. Anal., vol. 35, n. 3, 397-411 (1980).

[15] KNAUST, H. and ODELL, E. *Weakly null sequences with upper l_p-estimates.* LNM 1470. Springer-Verlag.

[16] LACEY, H.E. *The isometric theory of classical Banach spaces.* Springer-Verlag (1974).

[17] MADDOX, I.J. *A non-absolutely summing operator*. J. Austral. Math. Soc. Ser A 43, n. 1, 70-73 (1987).

[18] NUÑEZ, C. *Banach-Saks operators on spaces of continuous functions*. Ann. Mat. Pura ed Appl.

[19] ROSENTHAL, H.P. *Some applications of p-summing operators to Banach space theory*. Studia Math. vol. 58, 21-43 (1976).

[20] SAAB, P. *Weakly compact, unconditionally converging and Dunford-Pettis operators on spaces of vector-valued continuous functions*. Math. Proc. Cambridge Phil. Soc. 95, 101-108 (1984).

[21] SANCHEZ, F. *Sucesiones débilmente p-sumables en espacios de Banach*. Pub. Dept. Mat. Univ. Extremadura vol. 28 (1991).

[22] TALAGRAND, M. *La propriété de Dunford-Pettis dans C(K,E) et $L_1(E)$*. Israel J. Math. 44, 317-321 (1983).

Departamento de Matemáticas
Universidad de Extremadura
Avda. de Elvas, s/n
06071 Badajoz
SPAIN

Recibido: 12 de diciembre de 1991
Revisado: 6 de marzo de 1992