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Two-Fold Branched Coverings of S^3 Have Type Six (*)

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ABSTRACT. In this work, we prove that every closed, orientable 3-manifold M^3 which is a two-fold covering of S^3 branched over a link, has type six. This implies that M^3 is the quotient of the universal pseudocomplex $K(4,6)$ by the action of a finite index subgroup of a fuchsian group with presentation.

$$S(4,6) = < a_1,\ a_2,\ a_3,\ a_4 / a_1^3 = a_2^3 = a_3^3 = a_4^3 = a_1 a_2 a_3 a_4 = 1 >$$

Moreover, the same result is proved to be true in case M^3 being an unbranched covering of a two-fold branched covering of S^3.

1. INTRODUCTION

To every closed, orientable, P. L. n-manifold M^n, A. Costa associated an even integer $t(M^n)$, the so called «type» of M^n; the importance of this new invariant for manifolds lies in its relation with the existence of universal pseudocomplexes (whose geometrical structure is described in [C]).

Proposition 1. [C] – *Let M^n be a closed, orientable n-manifold. If $t(M^n) = 2h$, M^n is the quotient of the universal pseudocomplex $K(n+1, 2h)$, by the action of a finite index subgroup of a fuchsian group with presentation $S(n+1, 2h) = < a_1,\ a_2 \ldots,\ a_{n+1} / a_1^h = a_2^h = \ldots = a_{n+1}^h = a_1 a_2 \ldots a_{n+1} = 1 >$.*

Recently. A. Costa and L. Grasselli computed the type of every closed orientable n-manifold, with $n \neq 3$, and obtained the following results about the type of 3-manifolds.

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Proposition 2. $[CoG]$ – (a) *Let M_g^2 be the orientable surface of genus g. Then,*

$$t(M_g^2)=\begin{cases} 2 & \textit{iff} \;\; g=0 \\ 6 & \textit{iff} \;\; g=1 \\ 8 & \textit{otherwise} \end{cases}$$

(b) *Let M^3 be an orientable 3-manifold. Then,*

$$t(M^3)=\begin{cases} 2 & \textit{iff} \;\; M^3 \cong S^3 \\ 4 & \textit{iff} \;\; M^3 \textit{ is a lens space } L(p,q) \\ 6 \textit{ or } 8 & \textit{otherwise} \end{cases}$$

(c) *Let M^n be an orientable n-manifold, with $n \geq 4$. Then.*

$$t(M^n)=\begin{cases} 2 & \textit{iff} \;\; M^n \cong S^n \\ 4 & \textit{otherwise} \end{cases}$$

Thus, it is an open problem to find whether the type of a given 3-manifold M^3, different from S^3 and $L(p,q)$, is 6 or 8 (only $t(S^1 \times S^2)=6$ is directly computed).

In this paper, we give a partial answer, by proving that, *if M^3 is a two-fold covering of S^3 branched over a link, or if M^3 is an unbranched covering space of a two-fold branched covering of S^3, $M^3 \neq S^3$, $M^3 \neq L(p,q)$, then $t(M^3)=6$* (Propositions 6 and 8).

As a consequence, we obtain the possibility of «representing» every two-fold branched covering of S^3 by means of a finite index subgroup of the fuchsian group $S(4,6)=<a_1, a_2, a_3, a_4 / a_1{}^3=a_2{}^3=a_3{}^3=a_4{}^3=a_1 a_2 a_3 a_4=1>$ (Corollary 7).

Moreover, a well-known result originally proved by Viro ([*Vi*], [*BH*], [*T*], [CG_2]) allows to assert, as a particular case of Corollary 7, that the group $S(4,6)$ is «universal» with respect to all closed, orientable 3-manifolds of Heegaard genus two.

2. PRELIMINARIES AND NOTATIONS

This paper, like [*C*] and [*CoG*], that introduce and investigate the new invariant «type» for P. L.-manifolds, bases itself on the possibility of representing a large class of polyhedra, including P. L.-manifolds, by means of edge-coloured graphs (see [*BM*], [*FGG*], [*V*] and their bibliography).

An $(n+1)$-*coloured graph* is a pair (Γ, γ), $\Gamma = (V(\Gamma), E(\Gamma))$ being a multigraph (i. e. loops are forbidden, but multiple edges are allowed) regular of degree $n+1$, and $\gamma: E(\Gamma) \to \Delta_n = \{0, 1, ..., n\}$ being a proper edge-coloration of Γ (i.e. $\gamma(e) \neq \gamma(f)$ for every pair e, f of adjacent edges). For sake of conciseness, we shall often denote the $(n+1)$-coloured graph (Γ, γ) simply by the symbol Γ of its underlying multigraph.

For each $\Lambda \subseteq \Delta_n$, we set $\Gamma_\Lambda = (V(\Gamma), \gamma^{-1}(\Lambda))$; each connected component of Γ_Λ is said to be a Λ-*residue* of Γ. Note that every $\{i, j\}$-residue of Γ $(i, j \in \Delta_n)$ is a cycle whose edges are alternatively coloured by i and j; the (even) number of these edges is called the *valence* of the $\{i, j\}$-residue.

A 2-cell embedding [*W*] $f: |\Gamma| \to F$ of an $(n+1)$-coloured graph (Γ, γ) into a closed surface F, is said to be *regular* if there exists a cyclic permutation $\varepsilon = (\varepsilon_1, ..., \varepsilon_n)$ of Δ_n such that each region of f (i.e. each connected component of $F - f(|\Gamma|)$ is bounded by the image of an $\{\varepsilon_i, \varepsilon_{i+1}\}$-residue of Γ $(i \in Z_{n+1})$.

Actually, for every $(n+1)$-coloured graph (Γ, γ) and for every pair $(\varepsilon, \varepsilon^{-1})$ of cyclic permutations (ε^{-1} being the inverse of ε), there exists a unique regular embedding of (Γ, γ) into a closed surface F_ε; moreover, F_ε is orientable iff Γ is bipartite (see [*G*]).

Definition 1. *The type* $\tau_\varepsilon(\Gamma)$ *of an* $(n+1)$*-coloured graph* (Γ, λ) *with respect to the cyclic permutation* ε *of* Δ_n, *is the less common multiple of the valences of all* $\{\varepsilon_i, \varepsilon_{i+1}\}$*-residues of* (Γ, γ), $i \in Z_n$.

Definition 2. *The type* $\tau(\Gamma)$ *of an* $(n+1)$*-coloured graph* (Γ, γ) *is defined by:*

$$\tau(\Gamma) = \min \{\tau_\varepsilon(\Gamma) / \varepsilon \in \Sigma(\Delta_n)\},$$

$\Sigma(\Delta_n)$ *being the set of all cyclic permutations of* Δ_n.

Every $(n+1)$-coloured graph (Γ, γ) provides precise instructions for constructing an n-dimensional pseudocomplex [*HW*] $K(\Gamma)$, which is said to be *represented* by Γ: the n-simplexes of $K(\Gamma)$ are in bijection with the vertices of Γ, while the identifications between the $(n-1)$-dimensional faces are indicated by the coloured edges of Γ (see [*FGG*] for the detailed construction). By abuse of language, we will often say that (Γ, γ) represents $|K(\Gamma)|$ and every homeomorphic space, too.

A *crystallization* of a closed n-manifold M^n is an $(n+1)$-coloured graph (Γ, γ) representing M^n such that $\Gamma_{\hat{i}}$ is connected for each $i \in \Delta_n$ (where

$\hat{i}=\Delta_n-\{i\}$). A theorem of [*P*] ensures the existence, for every closed n-manifold M^n, of crystallizations of M^n (and hence of $(n+1)$-coloured graphs representing M^n); moreover, if (Γ,γ) represents M^n, then M^n is orientable if and only if Γ is bipartite.

Definition 3. *The type $t(M^n)$ of a closed n-manifold M^n is defined by:*

$$t(M^n)=\min\{\tau(\Gamma)\ /\ (\Gamma,\gamma) \text{ represents } M^n\}.$$

3. TWO-SYMMETRIC CRYSTALLIZATIONS

In [*F*], Ferri describes an algorithm for constructing a crystallization $F(L)$ of the (closed, orientable) 3-manifold which is the (cyclic) two-fold covering space of S^3 branched over a link $\mathscr{L}$, starting from a given bridge-presentation L of $\mathscr{L}$; the construction works as follows.

Let $L=(B_1,\dots,B_g;\ b_1,\dots,b_g)$ be the given g-bridge presentation of $\mathscr{L}$, B_i being the bridges and b_i being the arcs (for basic knot theory, see, for example, [*BZ*]). If π is the plane containing all arcs b_i, denote by a_i the projection of B_i on π; $P=(a_1,\dots,a_g;\ b_1,\dots,b_g)$ is said to be the *planar projection* of L. We can always assume that P is connected; otherwise, it can be made to be connected by isotoping arcs of P to pass «in and out» under bridges of different components. For every $i\in N_g=\{1,\dots,g\}$, draw an ellipse E_i on π having the bridge-projection a_i as principal axis and intersecting the arcs of P in exactly $2(h_i+1)$ points $P_i^1,\dots,P_i^{2(h_i+1)}$, where h_i is the number of undercrossings of B_i. Let V be the set of all the points P_i^j, $j=1,\dots,2(h_i+1)$, $i=1,\dots,g$. The elements of V subdivide the arcs of P into edges; let C (resp. D) be the set of these edges which are internal (resp. external) to the ellipses. The elements of V subdivide the ellipses into edges, too: let F be the set of these edges. Colour the edges in D by 2 and colour the edges of the ellipse E_1 alternatively by 0 and 1; then, complete the coloration on F by and 0 and 1 so that each region of the planar 2-cell embedding of $F\cup D$ is bounded by edges of only two colours. Let α be the involution on V which exchanges the end-points of the edges of C and fixes the end-points of the bridge-projections of P; let δ be the involution on V which exchanges the end-points of the edges of D. Draw a further set D' of edges, each connecting a pair of elements of V corresponding under the involution $\alpha\,\delta\,\alpha$, and finally colour all these edges by 3.

If Γ is the graph which has V as vertex-set and $D\cup D'\cup F$ as edge-set, and if γ is the described edge-coloration on Γ, then $(\Gamma,\gamma)=F(L)$ is proved to be a crystallization of the two-fold covering space of S^3 branched over the link $\mathscr{L}$. Note that the involution α, which may be thought of as an axial symmetry on the plane π, exchanges colour 0 (resp. 2) with colour 1 (resp. 3) in $F(L)$; for this reason, the crystallizations $F(L)$ resulting from Ferri's construction are said to be *2-symmetric*.

In [CG_2] every closed orientable 3-manifold M^{3} of Heegaard genus two is proved to admit a 2-symmetric crystallization; this leaded to an easy proof of the following well-known result.

Proposition 3. [*Vi*] [*BH*] [*T*] [CG_2] — *Every closed, orientable 3-manifold M^3 of Heegaard genus two is a two-fold covering space of S^3 branched over a link.*

4. COMPUTING THE TYPE OF TWO-FOLD BRANCHED COVERINGS OF S^3

Let $P=(a_1,..., a_g; b_1,..., b_g)$ be the planar projection of a g-bridge presentation L of a link $\mathscr{L}$, a_i being the bridge-projections and b_i being the arcs; let π be the plane containing P. The connected components of π-P are said to be the *regions* of P; note that every region of P is alternatively bounded by pieces of bridge-projections and pieces of arcs of L. We shall call *edge* to such pieces of bridge-projections and arcs.

Definition 4. *The valence of a region R of P is the (even) number of its boundary-edges.*

Definition 5. *The valence of the planar projection P is the less common multiple of the valences of all regions of P.*

Proposition 4. *Every link $\mathscr{L}$ admits a bridge-presentation $\bar{L}$ whose planar projection $\bar{P}$ has valence six.*

In order to prove Prop. 4, we need the following lemma.

Lemma 5. *Let P be the planar projection of a bridge-presentation of a link $\mathscr{L}$. Let $G(P)$ be the pseudograph which has a vertex v_R for every region R of P, and $n \geq 0$ edges between v_R and $v_{R'}$, if ∂R and $\partial R'$ contain n common pieces of bridge-projections.*

Then: a) *$G(P)$ is a multigraph (i.e. it contains no loop);*

b) *$G(P)$ is connected.*

Proof.

a) Let us suppose $G(P)$ to contain a loop based on the vertex v_R. This means that the region R of P contains a piece of bridge projection, $\bar{\alpha}$ say, twice in its boundary; thus, chosen an inner point A_0 of $\bar{\alpha}$, it is possible to draw in π a closed simple curve $\sigma (\cong S^1)$ whose points belong to $R \cup \{A_0\}$. On the other hand, the projection in P of the component of the link $\mathscr{L}$ containing $\bar{\alpha}$ is a closed curve τ in π whose double points, if any, are also double points of P. Then, σ intersects τ only in the regular point A_0, and this is an absurd.

b) Let us suppose $G(P)$ to be not connected. Let G' be a connected component of $G(P)$ not containing the vertex $v_{\bar{R}}$, $\bar{R}$ being the unlimited region of P; let v_{R_0} be an arbitrary vertex of G'. If $R_1, \dots, R_t$ are the regions of P such that, for $i \in \{1, \dots, t\}$, v_{R_i}, is adjacent to v_{R_0} in G', attach each R_i, one at a time, to R_0, by means of the common pieces of bridge-projections in their boundaries; then, repeat the same process for every attached region, and so on, until exhausting all regions R such that $v_R \in V(G')$. Since every region is a 2-ball and P is planar, at every stage a 2-ball (possibly with holes) is obtained; let D^2 be the 2-ball (with holes) which results at the end of the process. It is easy to check that $\partial \bar{D}^2$ is the projection in P of a component of the link $\mathscr{L}$, which contains no piece of bridge-projections; this contradicts the hypothesis that $\mathscr{L}$ is bridge-presented, since every component of the link must contain both bridges and arcs. ∎

Proof of Prop. 4.

The proof consists in the following two steps.

lst step: We will prove that $\mathscr{L}$ admits a bridge-presentation L^* such that the maximum among the valences of the regions of its planar projection P^* is ≤ 6;

2nd step: Starting from L^*, we will produce the required bridge-presentation $\bar{L}$ *of* $\mathscr{L}$.

lst step.

Let P be the (connected) planar projection of a given bridge-presentation L of $\mathscr{L}$; suppose that the maximum among the valences of the regions of P is $m > 6$ (otherwise, start with the 2nd step). Let R be a region of P having valence m, and let $\alpha_1, \beta_1, \dots, \alpha_{m/2}, \beta_{m/2}$ be the sequence of its boundary-edges, consistent with a fixed orientation of π, α_j being pieces of bridge-projections and β_j being pieces of arcs of L. (Fig. 1) First of all, isotope β_3 to pass «in and

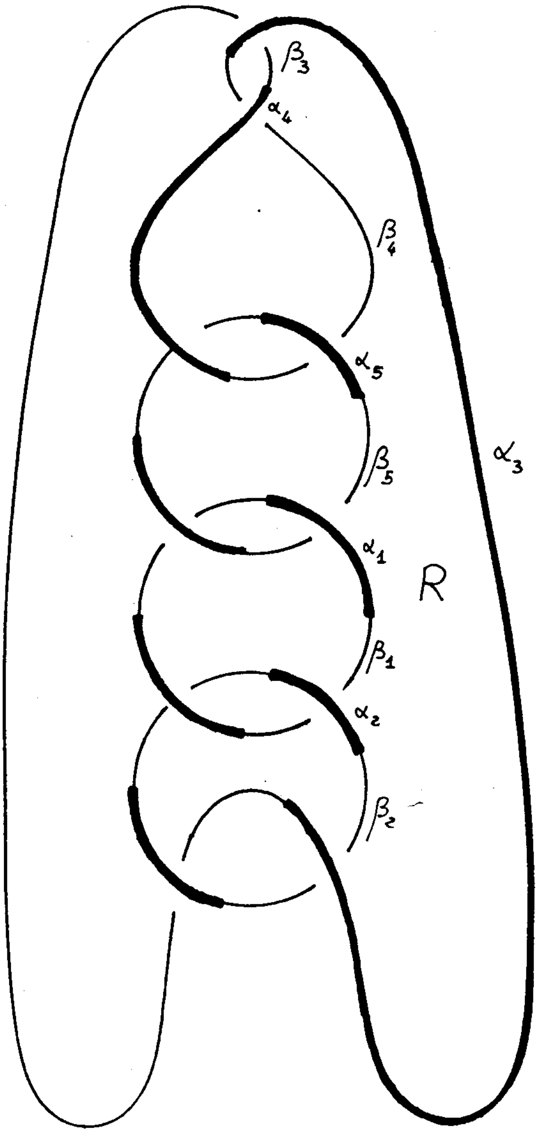


Fig. 1

out» under α_1, so that R gives rise to a region R' of valence six (bounded by $\alpha_1, \beta_1, \alpha_2, \beta_2, \alpha_3, \beta_3$) and a region R'' of valence $m-4$; note that the move adds a new piece of arc $\bar{\beta}$ to the boundary ∂Q of the region $Q(\neq R)$ of P containing α_1 and a new piece of bridge-projection $\bar{\alpha}$ to the boundary ∂Z of the region $Z(\neq R)$ of P containing β_3. Thus, at this stage, the regions Q and Z have their valence increased. (Fig. 2) However, lemma 5 (b) ensures the existence of a sequence $Q_1, Q_2, \ldots, Q_h$ of regions of P, such that $Q_1 \equiv Q$, $Q_h \equiv Z$, and ∂Q_i and ∂Q_{i+1} contain the same piece of bridge-projection $\bar{\alpha}_i$, for each $i \in \{1, \ldots, h-1)$; moreover, it can be assumed that the valence $v(Q_i)$ of the region Q_i is different from two, for each $i \in \{1, \ldots, h-1\}$, and, if $v(Z) > 6$, that the bridge-projection $\bar{\alpha}_{h-1}$ was not adjacent in P to the piece of arc β_3. Then, for each $i \in \{1, \ldots, h-1\}$, isotope the piece of arc $\bar{\beta}_i$ (with $\bar{\beta}_i \equiv \bar{\beta}$) in ∂Q_i to pass «in and out» under the piece of bridge-projection $\bar{\alpha}_i$, so that a new piece of arc $\bar{\beta}_{i+1}$ is added to ∂Q_{i+1} and Q_i gives rise to a «central» region $\tilde{Q}_i$ of valence four (containing $\bar{\alpha}_i$ in its boundary) and two regions Q_i', Q_i'' of valence not greater than $v(Q_i)$. Finally, isotope the piece of arc $\bar{\beta}_h$ in ∂Z to pass «in and out» under $\bar{\alpha}$. (Fig. 3) Note that the above sequence of moves, besides strictly lowering the valence of R, has increased the valence of no region of P. Hence, a (finite) iteration obviously leads to a planar projection P^* of $\mathscr{L}$ such that the maximum among the valences of its regions is ≤ 6.

2nd step.

Let L^* be a bridge-presentation of $\mathscr{L}$, such that the maximum among the valences of the regions of its planar projection P^* is ≤ 6. In order to obtain the required bridge-presentation $\bar{L}$ of $\mathscr{L}$, it is necessary to «adjust» all regions of P^* having valence four, in order to generate regions of valence two or six only.

First of all, note that two regions R, Q of P^* having valence four may obtain, together, valence six, if they are in one of the following situations:

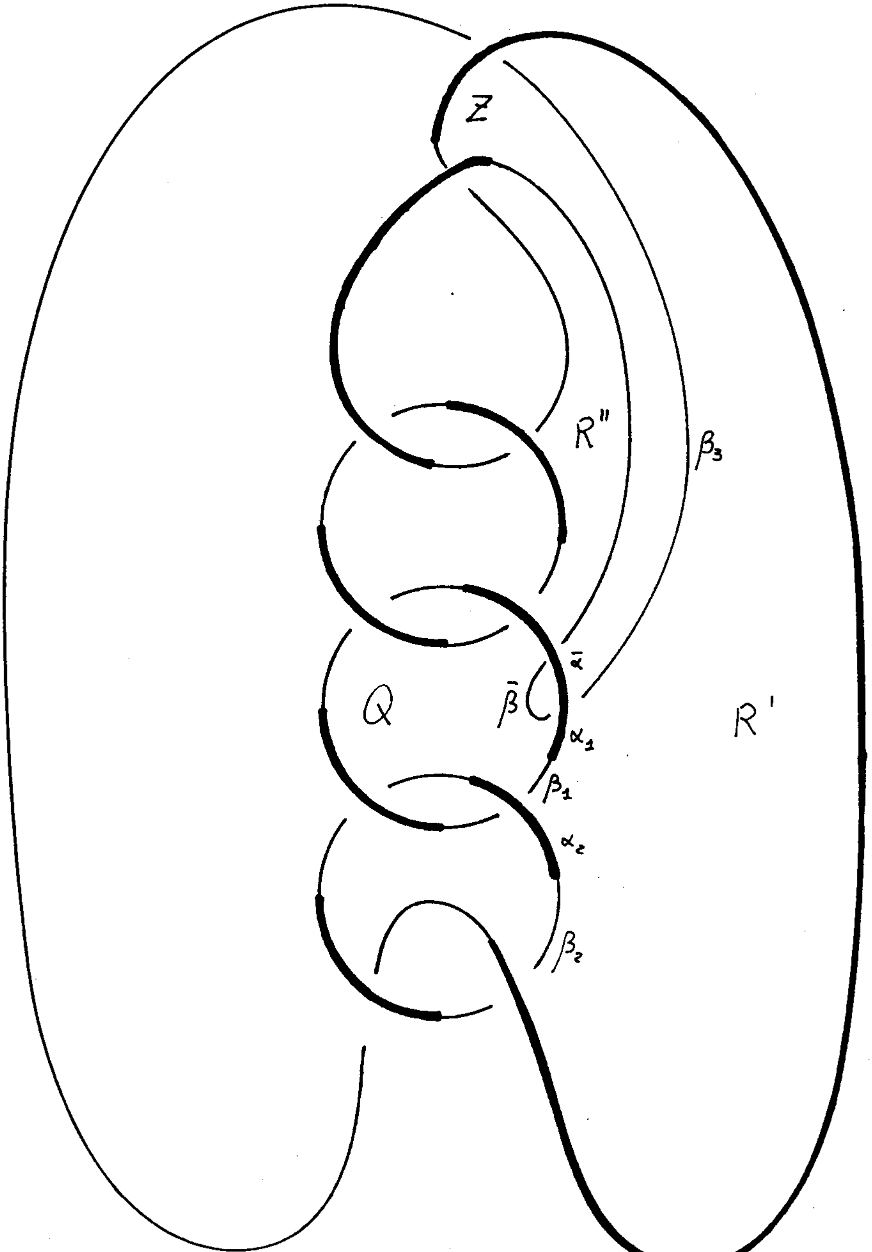
a) ∂R and ∂Q contain the same piece of bridge-projection $\bar{\alpha}$;

b) ∂R and ∂Q contain the same piece of arc $\bar{\beta}$;

c) ∂R and ∂Q contain the same vertex A (i.e. an edge β' of ∂R and an edge β'' of ∂Q are pieces of the same arc of L^*).

In fact: In case a), it is sufficient to introduce, within $\bar{\alpha}$, a new arc $\bar{\beta}$ without overcrossings; in case b), it is sufficient to introduce, within $\bar{\beta}$, a new arc $\bar{\alpha}$ without undercrossings; in case c), if α' is the piece of bridge-projection adjacent in A to β' and belonging to ∂R, it is sufficient to isotope the piece of arc β'' to pass «in and out» under α'. (Fig. 4 (a), (b), (c)).

On the other hand, note that a single region R of P^* having valence four may obtain valence six, if it is in the following situation:



Z
R''
β_3
α_3
$\bar{\alpha}$
$\bar{\beta}$
Q
R'
α_1
β_1
α_2
β_2

Fig. 2

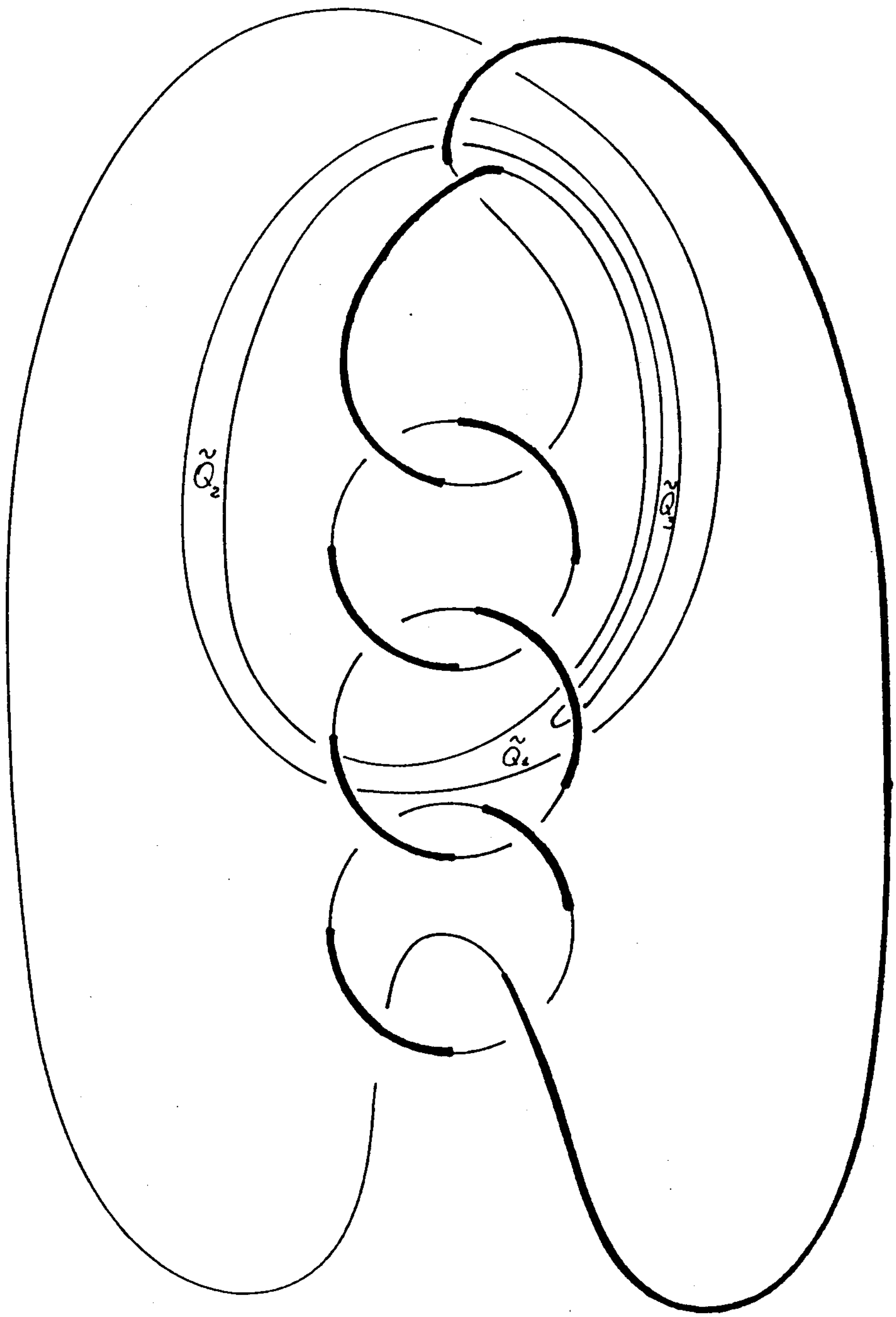


Fig. 3

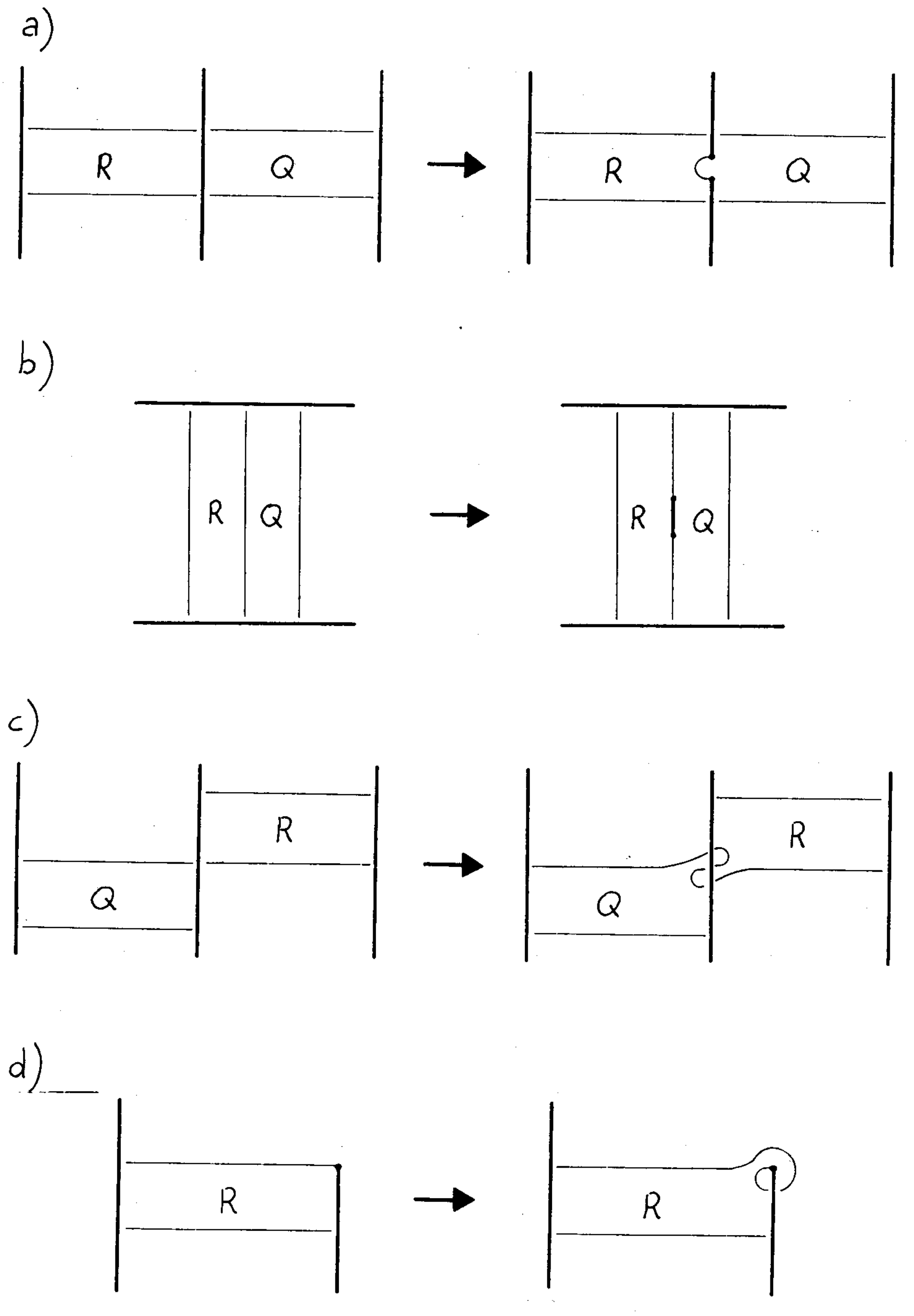


Fig. 4

d) ∂R contains a vertex A which is an end-point of a bridge-projection of L^*.

In fact: if α_1 and β_1 are respectively the piece of bridge-projection and the piece of arc adjacent in A and belonging to ∂R, it is sufficient to isotope β_1 to pass under α_1 from the side opposite to R, before arriving in A. (Fig. 4 (*d*)).

It is easy to check that the moves suggested in cases *a), b), c), d)* do not affect the valence of the other regions of P^*, and merely introduce (in cases c) and d)) new regions of valence two. Thus, it is always possible to obtain from P^* a new planar projection $P^{*\prime}$ of $\mathscr{L}$, such that the maximum among the valences of its regions is exactly six, and $P^{*\prime}$ does not contain regions of valence four belonging to the cases *a), b), c)* or *d)*.

If the valence of $P^{*\prime}$ is six, the thesis is proved; otherwise, let R be a region of $P^{*\prime}$ having valence four. As usual, denote by α_1, β_1, α_2, β_2 the sequence of its boundary-edges, consistent with a fixed orientation of π, α_1, α_2 being pieces of bridge-projections, β_1, β_2 being pieces of arcs of $L^{*\prime}$. The properties of $P^{*\prime}$ ensure that at least one between the edges β_1 and β_2, β_1 say, is such that the region $Q(\neq R)$ of $P^{*\prime}$ containing it has valence six; then, isotope β_1 to pass «in and out» under $\tilde{\alpha}$, $\tilde{\alpha}$ being the only piece of bridge-projection in ∂Q not adjacent to α_1 or α_2. In this way, R obtains valence six — as required —, while Q splits into two regions, Q', Q'' of valence four, and a new piece of arc $\tilde{\beta}$ is added to the boundary ∂S of the region $S(\neq Q)$ of $P^{*\prime}$ containing $\tilde{\alpha}$. (Fig. 5).

Note that $\partial Q'$ and ∂S contain two pieces (β' and β'', respectively, say) of the same arc $b_{\bar{i}}$ ($\bar{i} \in \{1,...,g\}$) of $P^{*\prime}$, which are both adjacent to $\tilde{\alpha}$. Let $\beta_{\bar{i}}^1$, $\beta_{\bar{i}}^2$,..., $\beta_{\bar{i}}^{\bar{j}}$, $\beta_{\bar{i}}^{\bar{j}+1}$,..., $\beta_{\bar{i}}^{m_{\bar{i}}}$ be the sequence of the pieces of the arc $b_{\bar{i}}$, consistent with a suitable orientation of the component of $\mathscr{L}$ which contains $b_{\bar{i}}$, so that $\beta_{\bar{i}}^{\bar{j}} \equiv \beta'$ and $\beta_{\bar{i}}^{\bar{j}+1} \equiv \beta''$, with $\bar{j} \in \{1,..., m_{\bar{i}}\}$. Let $S_1, S_2,..., S_{2m_{\bar{i}}}$ be the sequence of the (not necessarily distinct) regions of $P^{*\prime}$ such that: $S_1 \equiv S$, $S_{2m_{\bar{i}}} \equiv Q'$, $\beta_{\bar{i}}^{j}$ belongs both to $\partial S_{j-\bar{j}}$ and to $\partial S_{2m_{\bar{i}}-j-\bar{j}+1}$ (where the index i of $S_{\bar{i}}$ is written mod. $(2m_{\bar{i}})$), and, for each $i \in \{1, 2,..., 2m_{\bar{i}}-1\}$, ∂S_i and ∂S_{i+1} contain the same piece of bridge-projection $\tilde{\alpha}_i$. Note that $\tilde{\alpha}_{m_{\bar{i}}-\bar{j}}$ and $\tilde{\alpha}_{2m_{\bar{i}}-\bar{j}}$ are pieces of bridge-projections belonging to the same component of $\mathscr{L}$ than $b_{\bar{i}}$. Then, for each $i \in \{1,2,..., 2m_{\bar{i}}-1\}$, isotope the piece of arc $\tilde{\beta}_i$ with $\tilde{\beta}_1 \equiv \tilde{\beta}$) in ∂S_i to pass «in and out» under the piece of bridge-projection $\tilde{\alpha}_i$, so that a new piece of arc $\tilde{\beta}_{i+1}$ is added to ∂S_{i+1} and a new pair of adjacent regions S_i', S_i'' having valence four is placed near S_i, (Fig. 6) Note that, at the end of the above sequence of moves, every region S_i comes back to its original valence $v(S_i)$ in $P^{*\prime}$, while the region Q' obtains valence six. Let now a^* be the bridge-projection of $P^{*\prime}$ to which the adjacent pieces in ∂R and ∂Q (α_1 and α^*, respectively, say) belong, and let a^{*+} be the connected component of $a^* - \alpha^*$ not containing α_1; further, let K be the (possibly void) subset of $\{1, 2,...,$

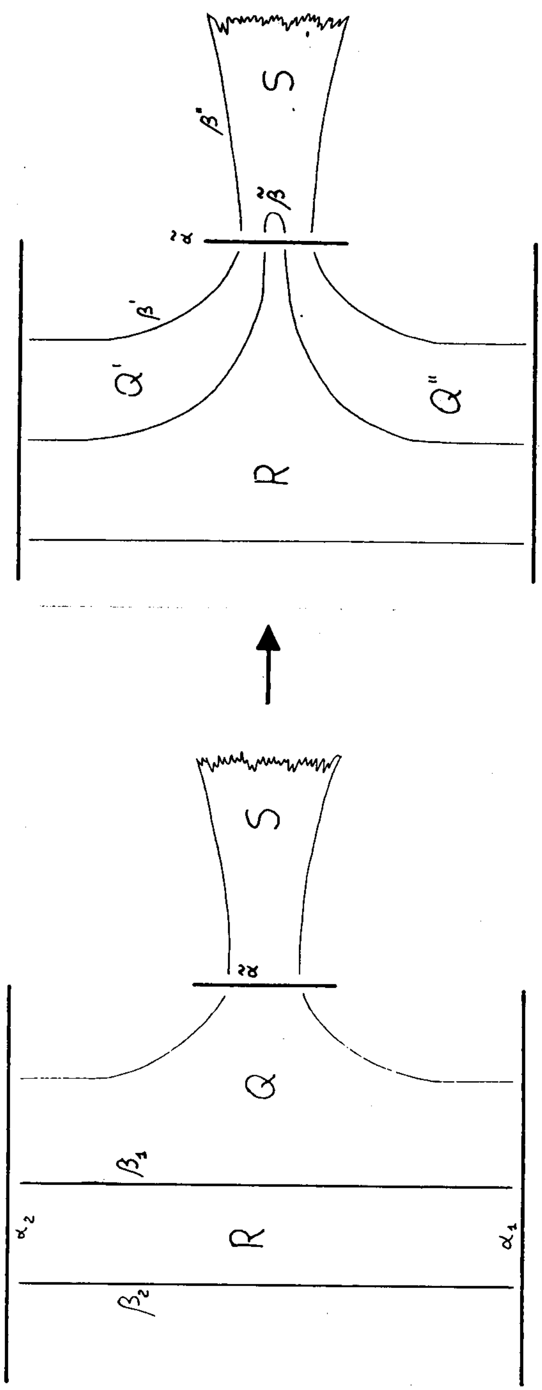


Fig. 5

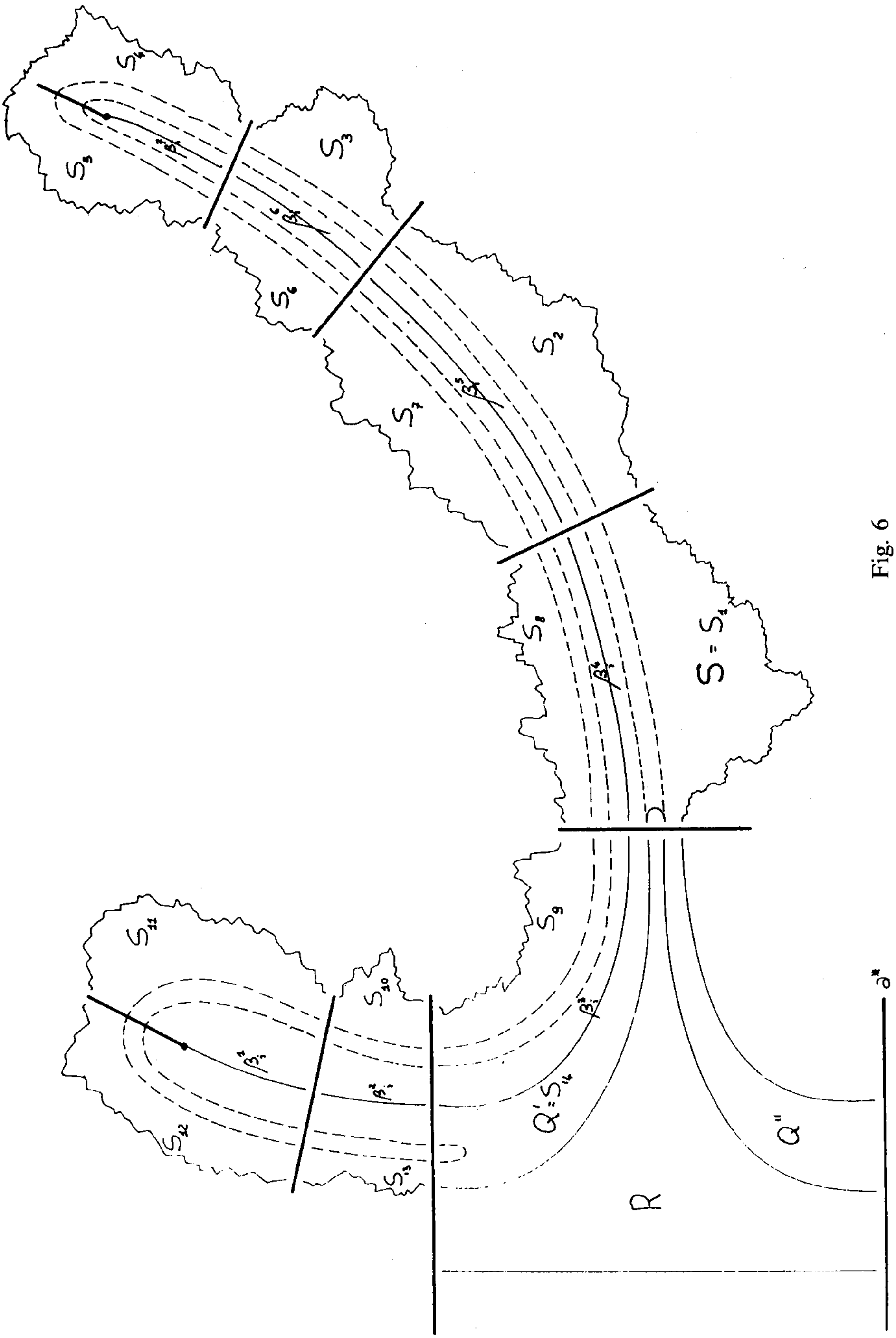


Fig. 6

$2m_{\bar{i}}-1\}$ such that, for every $k \in K$, α_k belongs to a^{*+}, and let $\bar{k}$ be the element of K such that $\alpha_{\bar{k}}$ is the closest to α^* among all α_k, $k \in K$. Then by applying the move suggested in case a) to the pairs $S_{\bar{k}}'$, $S_{\bar{k}+1}''$ and $S_{\bar{k}}''$, $S_{\bar{k}+1}'$, is any, and the move suggested in case b) to the pair S_i', S_i'', for each $i \in \{1, 2, \ldots, 2m_{\bar{i}}-1\} - \{\bar{k}\}$, the «adjustment» of the region R is obtained, with one only new region Q'' of valence four, However, it is easy to check that Q'', if not belonging to the cases a), b), c) or d), is strictly closer to an end-point of the bridge-projection a^* (either the one belonging to a^{*+}, or the new one, internal to $\alpha_{\bar{k}}$), than R was. Hence, the existence of a planar projection $\bar{P}$ of $\mathscr{L}$ having valence six, easily follows by (finite) iteration. ∎

Example: By applying the procedure of Prop. 4 to the Montesinos link $\mathscr{L} = M(-2; (2,1), (2,1), (2,1), (2,1))$ (see [$B\dot{Z}$]) represented in Fig. 1, one obtains the valence six planar projection of $\mathscr{L}$ represented in Fig. 7, passing through the ones depicted in Fig. 2 and Fig. 3.

We are now able to prove the main result of the paper.

Proposition 6. *Let M^3 be a (closed, orientable) 3-manifold, which is a two-fold covering space of S^3 branched over a link $\mathscr{L}$. Then,*

$$t(M^3) = \begin{cases} 2 & \text{iff } M^3 = S^3; \\ 4 & \text{iff } M^3 \text{ is a lens space } L(p,q); \\ 6 & \text{otherwise.} \end{cases}$$

Proof.

Prop. 4 ensures the existence of a bridge-presentation $\bar{L}$ of $\mathscr{L}$, such that the planar projection $\bar{P}$ of $\bar{L}$ has valence six. Let $F(\bar{L})$ be the 2-symmetric crystallization of M^3, obtained from $\bar{L}$ by Ferri's construction. It is easy to check that $F(\bar{L})$ contains $\{0,2\}-$, $\{1,2\}-$, $\{1,3\}-$ and $\{0,3\}-$ residues of valence two or six, only; thus, if ε is the cyclic permutation defined by $\varepsilon = (0,2,1,3)$, $\tau_\varepsilon(F(\bar{L})) = 6$. The result now easily follows from the characterization of the 3-manifolds of type two and four (see [CoG]). ∎

Remark. *If M^3 is a two-fold branched covering of S^3, the type of M^3 is obtained by the type of a crystallization of M^3.* It might be interesting to know whether this happens in the general case, or not.

The following result is a direct consequence of the above proposition and of the existence of a pseudocomplex $K(n+1, 2h)$, which is «universal» with respect to all closed orientable n-manifolds of type $2h$ (see [C]).

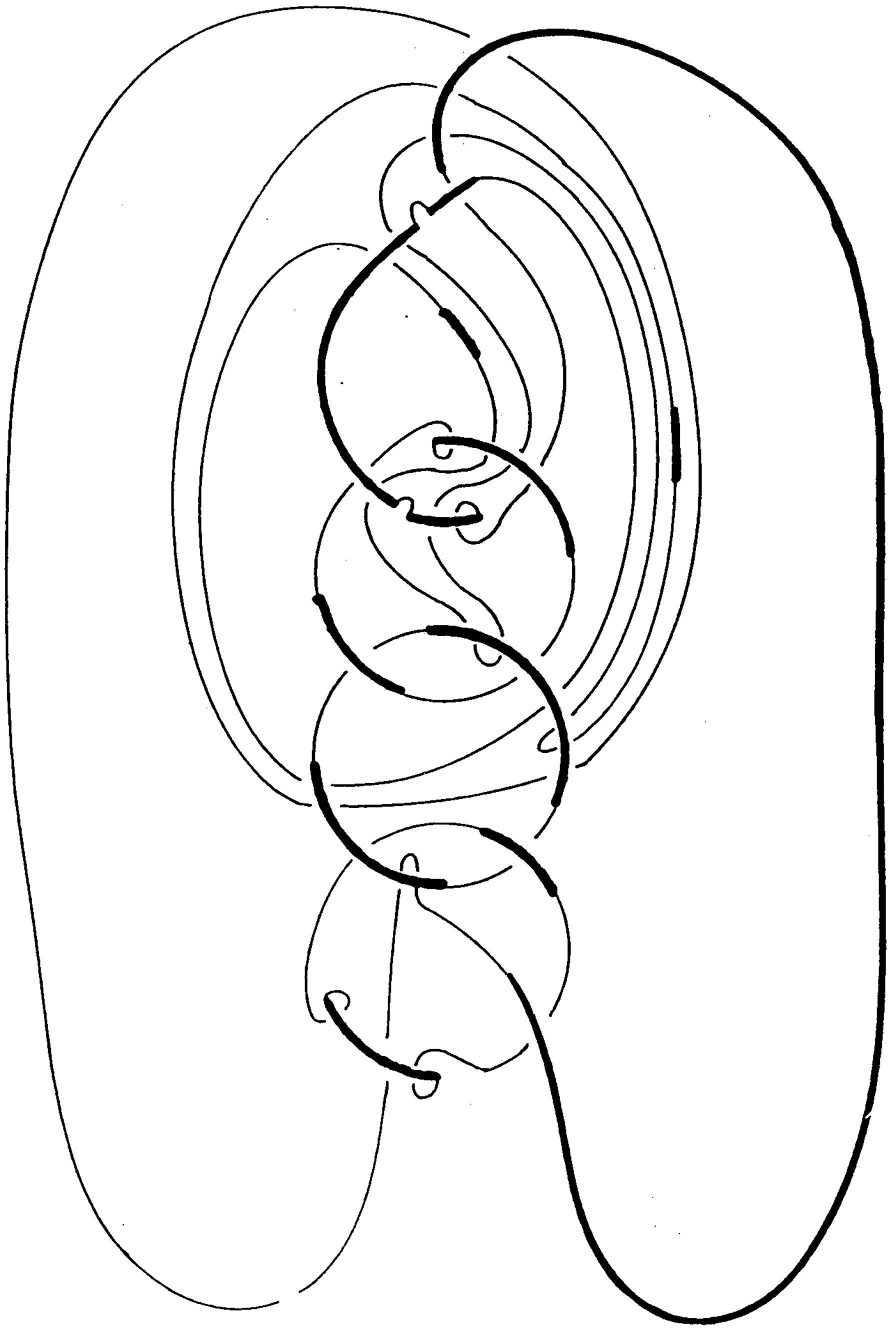


Fig. 7

Corollary 7. *Let M^3 be a two-fold branched covering space of S^3. Then, there exists a finite index subgroup N of a fuchsian group*

$$S(4,6) = < a_1, a_2, a_3, a_4 / a_1^3 = a_2^3 = a_3^3 = a_4^3 = a_1 a_2 a_3 a_4 = 1 >$$

such that

$$M^3 = \frac{K(4,6)}{N}$$ ∎

Remark that prop. 3 ensures that the property stated in Corollary 7 holds for every closed orientable 3-manifold of Heegaard genus two.

5. FURTHER TYPE-SIX 3-MANIFOLDS

The present last section is devoted to show that Prop. 6 actually implies the existence of a very large class of type-six 3-manifolds, properly comprehending two-fold branched coverings of S^3.

For, the notion of m-covering – originally due to [*V*] – is needed.

Definition 6. *Let (Γ, γ), (Γ', γ') be $(n+1)$-coloured graphs. A map $f: V(\Gamma') \to V(\Gamma)$ is said to be an m-covering, $1 \leq m \leq n$, if f preserves c-adjacency for all $c \in \Delta_n$ and is bijective when restricted to m-residues.*

The *branching $(m+1)$-residues* are the $(m+1)$-residues of (Γ, γ) covered by at least one $(m+1)$-residue of (Γ', γ') on which f is not injective.

The covering f naturally induces a topological map $|f| : K(\Gamma') \to K(\Gamma)$. An n-covering induces an (unbranched) topological covering between the underlying topological spaces, while a 1-covering induces a topological covering branched over the $(n-2)$-subcomplex of $K(\Gamma)$ whose $(n-2)$-simplexes are represented by the branching 2-residues of (Γ, γ).

We want now to illustrate a standard method for constructing m-coverings of graphs representing manifolds, which will be useful for our purposes.

Let (Γ, γ) be an $(n+1)$-coloured graph representing a closed orientable n-manifold $K(\Gamma) = M^n$. Suppose $\Gamma_{\hat{c}}$ connected, for some $c \in \Delta_n$, and let L be the $(n-2)$-subcomplex of $K(\Gamma)$ represented by a (possibly void) given set $\{C_1, C_2, ..., C_p\}$ of 2-residues containing colour c.

If $L = \phi$ (resp. $L \neq \phi$), then a presentation $< X: R >$ of $\Pi_1(M^n)$ (resp. $\Pi_1(M^n - L)$), called c-edge presentation, can be obtained in the following way:

*) the generators of X are the c-coloured edges, arbitrarily oriented;

**) the relators of R are obtained by walking along all the 2-residues of Γ containing colour c (resp. all the 2-residues of Γ containing colour c, but $C_1, C_2, \ldots, C_p$), giving the exponent $+1$ or -1 to each generator whether the orientation of the 2-residue is coherent or not with the orientation of the generator.

Note that, if $\Gamma_{\hat{c}}$ is not connected, the c-edge presentation can be obtained in a similar way: it is sufficient to complete the relators of R with a minimal set of generators such that the corresponding c-coloured edges connect $\Gamma_{\hat{c}}$. The existence of a one-to-one correspondence Φ between transitive d-representations ω of $\Pi_1(M^n)$ (resp. $\Pi_1(M^n - L)$) and d-fold unbranched covering spaces of M^n (resp. d-fold covering spaces of M^n branched over L), is well-known (see [F]). In [CG_1], the following method is described for constructing an $(n+1)$-coloured graph $(\tilde{\Gamma}, \tilde{\gamma})$ such that $K(\tilde{\Gamma}) = \Phi(\omega)$:

— set $V(\tilde{\Gamma}) = V(\Gamma) \times N_d$;

— for each $k \in \Delta_n - \{c\}$ and $i \in N_d$, join (v, i) with (w, i) by a k-coloured edge if v, w are k-adjacent in (Γ, γ);

— join (v, i) with (w, j) by a c-coloured edge if in (Γ, γ) there is an oriented c-coloured edge x_l from v to w and $\omega(x_l)(i) = j$.

It is easy to check that the projection map $f: V(\tilde{\Gamma}) \to V(\Gamma)$ defined by $f((v, i)) = v$ for every $v \in V(\Gamma)$ and $i \in N_d$, is a 2-covering (resp. a 1-covering having $C_1, C_2, \ldots, C_p$ as branching 2-residues).

As an application of the previous construction and of the results of section 4, we have the following existence theorem for type-six 3-manifolds.

Proposition 8. *If $\tilde{M}^3$ ($\tilde{M}^3 \neq S^3$, $L(p, q)$) is an unbranched covering of a two-fold branched covering of S^3, then $t(\tilde{M}^3) = 6$.*

Proof.

Let M^3 be a two-fold branched covering of S^3, and let $\omega: \Pi_1(M^3) \to S_d$ be the monodromy associated to the unbranched d-fold covering space $\tilde{M}^3$ of M^3.

Prop. 6 ensures the existence of a crystallization (Γ, γ) of M^3 such that, for $\varepsilon = (0, 2, 1, 3)$, $\tau_\varepsilon(\Gamma) = 6$. If $c \in \Delta_3$ is an arbitrarily chosen colour of (Γ, γ) and $\langle X; R \rangle$ is the *c-edge presentation of* $\Pi_1(M^3)$, then the construction above described yields a 4-coloured graph $(\tilde{\Gamma}, \tilde{\gamma})$ representing $\tilde{M}^3 = \Phi(\omega)$ and such that $\tau_\varepsilon(\tilde{\Gamma}) = 6$ (because of the 2-covering $f: V(\tilde{\Gamma}) \to V(\Gamma)$). Hence, the thesis follows. ∎

Actually, an even more general result holds.

Proposition 9. *Let (Γ, γ) be a 4-coloured graph representing a 3-manifold M^3, such that $\tau_\varepsilon(\Gamma) = 6$ (ε being a suitable cyclic permutation of Δ_3); let L be a subcomplex of $K(\Gamma)$ represented by a (possibly void) given set of $\{\varepsilon_c, \varepsilon_{c+2}\}$-residues, for some $c \in \Delta_3$. Then, every covering of $M^3 = K(\Gamma)$ branched over L is represented by a 4-coloured graph $(\tilde{\Gamma}, \tilde{\gamma})$, such that $\tau_\varepsilon(\tilde{\Gamma}) = 6$.*

The proof is an obvious adaptation of the one of Prop. 8. ∎

Remark. The fact that $T^3 = S^1 \times S^1 \times S^1$ is not a two-fold branched covering of S^3 is well-known ([Fox]). Nevertheless, Prop. 8 ensures $t(T^3) = 6$. In fact, T^3 is the (unbranched) two-fold covering of the Selfert manifold $ST(S_{2222}) = (OoO/-2; (2,1), (2,1), (2,1), (2,1))$, which is the two-fold covering space of S^3 branched over the Montesinos link $M(-2; (2,1), (2,1), (2,1), (2,1))$ of Fig. 1 (compare [*M*]).

Since Propositions 8 and 9 yield a very large class of type six 3-manifolds, the following two questions naturally arise:

— There exists a 3-manifold with type eight ?

— There exists a 3-manifold without any group action with type six?

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