

# *On a Formula for the Jumps in the Semi-Fredholm Domain*

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**ABSTRACT.** In this paper we prove some properties of the lower  $s$ -numbers and derive asymptotic formulae for the jumps in the semi-Fredholm domain of a bounded linear operator on a Banach space.

## 1. INTRODUCTION AND PRELIMINARIES

In this note  $X$ ,  $Y$ ,  $Z$  and  $W$  are complex Banach spaces, and  $B(X, Y)$  ( $B(X)$ ) the set of all bounded linear operators from  $X$  into  $Y$  (on  $X$ ). Let  $K(X, Y)$  denote the set of compact linear operators from  $X$  into  $Y$ . Let  $U$  denote the closed unit ball of  $X$ . Let  $T \in B(X, Y)$  and

$$m(T) = \inf \{ \|Tx\| : \|x\| = 1 \}$$

be the minimum modulus of  $T$ , and let

$$q(T) = \sup \{ \varepsilon \geq 0 : TU \supset \varepsilon U \}$$

be the surjection modulus of  $T$ . Recall that both  $m(T)$  and  $q(T)$  are positive if and only if  $T$  is invertible, and in this case  $m(T) = q(T) = \|T^{-1}\|^{-1}$ .

For each  $r = 1, 2, \dots, \infty$  we define the following lower analogues of the approximation numbers [8]:

$$m_r(T) = \sup \{ m(T+F) : \text{rank } F < r \},$$

$$q_r(T) = \sup \{ q(T+F) : \text{rank } F < r \},$$

$$g_r(T) = \max \{ m_r(T), q_r(T) \}.$$

If  $M$  is a subspace of  $X$ , then  $T|_M$  will denote the restriction of  $T$  to  $M$ .  $T$  is a semi-Fredholm operator if either the null space  $N(T)$  is finite-dimensional and the range  $R(T)$  is closed, or the codimension of  $R(T)$  is finite. For such operators the index defined by

$$\text{ind}(T) = \dim N(T) - \text{codim } R(T),$$

and the minimum index by

$$\text{min. ind}(T) = \min \{ \dim N(T), \text{codim } R(T) \},$$

which is always finite. It was shown in [12, Theorem 8.3] that

$$s(T) = \lim_k g_\infty(T^k)^{1/k}$$

is the semi-Fredholm radius of  $T$ , i.e. the supremum of all  $\varepsilon \geq 0$  such that  $T - \lambda I$  is semi-Fredholm for  $|\lambda| < \varepsilon$ . It is well known that the function  $\text{min. ind}(T - \lambda I)$  is constant everywhere in the disk  $|\lambda| < s(T)$  except possibly for a discrete subset  $G$ . We denote by  $n(T)$  this constant, and call it the *stability index* of the semi-Fredholm operator  $T$  [8]. A point  $\omega$  in  $G$  is called a jumping point of the minimum index in the semi-Fredholm domain. For  $\omega$  in  $G$  we have  $\text{min. ind}(T - \omega I) > n(T)$ , and  $X$  decomposes into the direct sum of two closed  $T$ -invariant subspaces  $Y_\omega$  and  $Z_\omega$ , where  $Z_\omega$  is finite-dimensional and  $T - \omega I$  is nilpotent on it, while the restriction on  $T - \lambda I$  to  $Y_\omega$  has constant minimum index on a neighbourhood of  $\omega$  [3, Theorem 4]. Consistently with the matrix case we define the (algebraic) *multiplicity of the jumping point*  $\omega$  to be  $\dim Z_\omega$  [8, pp. 232]. Thus the point in  $G$  can be ordered in such a way that

$$|\omega_1(T)| \leq |\omega_2(T)| \leq \dots < s(T),$$

where each jump appears consecutively according to its multiplicity. If there are only  $p$  ( $= 0, 1, 2, \dots$ ) such jumps, we put  $|\omega_{p+1}(T)| = |\omega_{p+2}(T)| = \dots = s(T)$ . Recall that [8, Theorem 1.1] if  $T$  is a semi-Fredholm operator, then for each  $r = 1, 2, \dots$  we have

$$(1) \quad |\omega_r(T)| = \lim_k g_{kn+r}(T^k)^{1/k},$$

where  $n = n(T)$  is the stability index of  $T$ .

In this note we prove (1) when the stability index of  $T$  is zero, and we believe that in this case the proof is simpler than the mentioned one in the general case. Further, we use a restriction techniques and show how this particular case is related to general case.

## 2. RESULTS

In the following lemma we list some properties of the lower s-numbers.

**Lemma 2.1.** *Let  $T \in B(X, Y)$ . Then*

- (i)  $0 \leq m_1(T) \leq m_2(T) \dots \leq m_\infty(T) \leq \sup_{K \in K(X, Y)} m(T+K) \leq \inf_{K \in K(X, Y)} \|T+K\|,$
- (ii)  $m_n(S+T) \leq m_n(S) + \|T\|$  for  $S, T \in B(X, Y),$
- (iii)  $m_n(RST) \geq m(R)m_n(S)m(T)$  for  $T \in B(X, Y), S \in B(Y, Z)$  and  $R \in B(Z, W),$
- (iv) If  $\dim X \geq n,$  then  $m_n(I) = 1,$
- (v)  $m_{n+m-1}(ST) \geq m_n(S)m_m(T)$  for  $T \in B(X, Y)$  and  $S \in B(Y, Z),$
- (vi)  $m_n(T) > 0 \iff \dim N(T) < n, R(T)$  is closed and  $\text{ind}(T) \leq 0.$

**Proof.** (i) By the definition and [6. pp. 389].

(ii) Let  $F \in B(X, Y)$  and  $\text{rank } F < n.$  By [1, Lemma 2.2] we have

$$m(S+T+F) \leq m(T+F) + \|S\| \leq m_n(T) + \|S\|,$$

and hence  $m_n(S+T) \leq m_n(T) + \|S\|$

(iii) Let  $F \in B(Y, Z)$  and  $\text{rank } F < n.$  Now,  $RFT \in B(X, W), \text{rank } RFT < n$  and by [1, pp. 21] we have

$$m_n(RST) \geq m(R(S+F)T) \geq m(R)m(S+F)m(T).$$

Further, it follows that  $m_n(RST) \geq m(R)m_n(S)m(T).$

(iv) It is clear that  $m_n(I) \geq 1.$  If  $m_n(I) > 1,$  then there is an  $F \in B(X)$  and  $\text{rank } F < n,$  such that  $m(I+F) > 1.$  Since  $m(F) = 0,$  it follows that  $m(I+F) \leq m(F) + \|I\| = 1,$  which is a contradiction. Hence  $m_n(I) = 1.$

(v) Let  $F_1 \in B(X, Y), \text{rank } F_1 < n, F_2 \in B(Y, Z)$  and  $\text{rank } F_2 < m.$  Then  $(S+F_2)(T+F_1) \in B(X, Z), (S+F_2)(T+F_1) = ST + SF_1 + F_2(T+F_1) \in B(X, Z)$  and  $\text{rank } [SF_1 + F_2(T+F_1)] < n+m-1.$  Thus  $m_{n+m-1}(ST) \geq m[(S+F_2)(T+F_1)] > m(S+F_2)m(T+F_1),$  which proves (v).

(vi) Suppose that  $m_n(T) > 0$ ,  $\text{rank } F < n$  and  $\dim N(T) \geq n$ . Now  $\text{codim } N(F) < n$ , and it follows that  $N(T) \cap N(F) \neq \{0\}$ . Thus  $m(T+F) = 0$ , i.e.,  $m_n(T) = 0$ , whence a contradiction. Thus  $m_n(T) > 0$  implies  $\dim N(T) < n$ . That  $R(T)$  is closed and  $\text{ind } (T) \leq 0$  follows by elementary properties of semi-Fredholm operators [9]. Conversely, if  $R(T)$  is closed,  $\dim N(T) < n$  and  $\text{ind } (T) \leq 0$ , then by [11, Theorem 3.9 (2)] there is an operator  $F \in \mathcal{B}(X)$  such that  $\text{rank } (F) < n$  and  $m(T+F) > 0$ . This implies that  $m_n(T) > 0$ .

This completes the proof of the lemma.

**Theorem 2.2.** *Let  $T \in \mathcal{B}(X)$  be a semi-Fredholm operator with the stability index of  $T$  equal to zero and  $\min. \text{ind } (T - \lambda I) = \dim N(T - \lambda I)$  in the disk  $|\lambda| < s(T)$  except possibly for the jumps  $\omega_r(T)$ ,  $r = 1, 2, \dots$ . Then for each  $r = 1, 2, \dots$  we have*

$$|\omega_r(T)| = \lim_k m_r(T^k)^{1/k}.$$

**Proof.** We have to prove two things. First

$$(2) \quad |\omega_r(T)| \leq \lim_k \inf m_r(T^k)^{1/k}.$$

and second

$$(3) \quad \lim_k \sup m_r(T^k)^{1/k} \leq |\omega_r(T)|,$$

Note that  $\omega_1(T) = \lim_k m_1(T^k)^{1/k}$  [4, Theorem 3], and it is clear that (2) and (3) are true for  $r = 1$ . To show the induction step for (2), take the least  $q$  such that  $\omega_{n-q}(T) \neq \omega_n(T)$ . (If such a  $q$  does not exist, then (2) is obvious since  $|\omega_n(T)| = |\omega_1(T)|$  in that case). Let  $Z$  be the direct sum of the finite-dimensional parts in the Kato decompositions corresponding to the points  $\omega_1(T), \dots, \omega_{n-q}(T)$  [3, Theorem 4]. Now  $\dim Z = n - q$ . Let  $Y$  be the intersection of the corresponding Kato complements to the finite-dimensional parts in the Kato decompositions corresponding to the points  $\omega_1(T), \dots, \omega_{n-q}(T)$ . Thus the space  $X$  decomposes into a direct sum of two closed subspace  $Y$  and  $Z$ . These subspaces are  $T$ -invariant. Let  $F$  be the removing operator from the proof of [12, Theorem 7.1], i.e.,  $F$  is zero on  $Y$  and  $\mu_1 I$  on  $Z$ ;  $\mu_1$  is any complex number with  $|\mu_1| > \|T\| + s(T)$ . By the proof of [12, Theorem 7.1] and [4, Theorem 3] we have that

$$\lim_k m((T+F)^k)^{1/k} = |\omega_{n-q+1}(T)|.$$

Further for each  $k = 1, 2, \dots$  we have

$$m_n(T^k) \geq m_{n-q+1}(T^k) \geq m((T+F)^k),$$

and so the proof of (2) is complete.

Now we turn to prove the inequality (3). Let  $W$  be the direct sum of the finite-dimensional parts in the Kato decompositions corresponding to the points  $\omega_1(T), \dots, \omega_n(T)$  [3, Theorem 4]. Now  $\dim W \geq n$ . Let  $V$  be the intersection of the corresponding Kato complements to the finite-dimensional parts in the Kato decompositions corresponding to the points  $\omega_1(T), \dots, \omega_n(T)$ . Thus the space  $X$  decomposes into a direct sum of two closed subspaces  $W$  and  $V$ . These subspaces are  $T$ -invariant. Let  $F \in B(X)$  and  $\text{rank } F < n$ . Hence, there is a vector  $h \in W \cap N(F)$  such that  $h \neq 0$ . Let  $P$  be the projection of  $X$  onto  $W$  along  $V$ . Then

$$\|(T+F)h\| = \|Th\| = \|TPh\| \leq \|T|_W\| \|P\| \|h\|.$$

Thus,  $m(T+F) \leq \|P\| \|T|_W\|$ . It is easy to see that for each  $k = 1, 2, \dots$  we have  $m(T^k+F) \leq \|P\| \|T^k|_W\|$ . Consequently  $m_n(T^k) \leq \|P\| \|T^k|_W\|$ , and since the spectral radius of  $T|_W$  is equal to  $|\omega_n(T)|$ , it follows that

$$\lim_k \sup m_n(T^k)^{1/k} \leq |\omega_n(T)|.$$

This proves (3), and the proof of the theorem is complete.

**Remark 2.3.** *Let us mention that if in Theorem 2.2 we have that  $\omega_j(T) \neq 0$ , then we can prove (3) in the following way (we use the same notations as in the proof of Theorem 2.2):* Now  $T|_W: W \rightarrow W$  is invertible and since  $\dim W \geq n$  we have by Lemma 2.1 (iv) that  $m_n(T^k(T|_W^{-k})) = 1, k = 1, 2, \dots$ . Thus by Lemma 2.1 (v) we have  $1 \geq m_n(T^k)m(T|_W^{-1})^k$ , and so

$$m_n(T^k) \leq 1/m(T|_W^{-1})^k = \|T|_W^k\|.$$

Since the spectral radius of  $T|_W$  is equal to  $|\omega_n(T)|$  we conclude that

$$\lim_k \sup m_n(T^k)^{1/k} \leq |\omega_n(T)|,$$

whence the result.

Next we state properties of  $q_n(T)$  and the dual result of Theorem 2.2. They can be proved similarly, so we leave out details.

**Lemma 2.4.** *Let  $T \in B(X, Y)$ . Then*

- (i)  $0 \leq q_1(T) \leq q_2(T) \leq \dots \leq q_\infty(T) \leq \sup_{K \in K(X, Y)} q(T+K) \leq \inf_{K \in K(X, Y)} \|T+K\|$ .
- (ii)  $q_n(S+T) \leq q_n(S) + \|T\|$  for  $S, T \in B(X, Y)$ ,
- (iii)  $q_n(RST) \geq q(R)q(S)q(T)$  for  $T \in B(X, Y)$ ,  $S \in B(Y, Z)$  and  $R \in B(Z, W)$ ,
- (iv) If  $\dim X \geq n$ , then  $q_n(I) = 1$ ,
- (v)  $q_{n+m-1}(ST) \geq q_n(S)q_m(T)$  for  $T \in B(X, Y)$  and  $S \in B(Y, Z)$ ,
- (vi)  $q_n(T) > 0 \iff \text{codim } R(T) < n$ , and  $\text{ind } (T) \geq 0$ ,
- (vii) If  $m_n(T) > 0$  and  $q_n(T) > 0$ , then  $m_n(T) = q_n(T)$  and  $\text{ind } (T) = 0$ .

**Proof.** We shall prove only (vii). From (vi) and Lemma 2.1 (vi), it follows that  $\dim N(T) < n$ ,  $R(T)$  is closed,  $\text{codim } R(T) < n$  and  $\text{ind } (T) = 0$ . Let  $F \in B(X, Y)$  and  $\text{rank } F < n$ . If  $m(T+F) > 0$ , then  $\dim N(T+F) = 0$ , and it follows that  $\text{codim } R(T+F) = 0$ . Thus,  $m(T+F) = q(T+F) \leq q_n(T)$ , and we have that  $m_n(T) \leq q_n(T)$ . In a similar way, we can prove that  $q_n(T) \leq m_n(T)$ , and the proof is complete.

**Theorem 2.5.** *Let  $T \in B(X)$  be a semi-Fredholm operator with the stability index of  $T$  equal to zero and  $\min. \text{ind } (T - \lambda I) = \text{codim } R(T - \lambda I)$  in the disk  $|\lambda| < s(T)$  except possibly for the jumps  $\omega_r(T)$ ,  $r = 1, 2, \dots$ . Then for each  $r = 1, 2, \dots$  we have*

$$|\omega_r(T)| = \lim_k q_r(T^k)^{1/k}.$$

**Proof.** By Lemma 2.4 and Theorem 2.2.

For  $T$  in  $B(X)$  set  $N(T^\infty) = \bigcup N(T^n)$  and  $R(T^\infty) = \bigcap R(T^n)$ . If  $T$  is a semi-Fredholm, then it is well known ([5, Theorem 4.1] see also [7, Theorem 5.2] for general case) that the function  $\lambda \rightarrow N((T - \lambda)^\infty) + R((T - \lambda)^\infty)$  is constant, say  $W$  everywhere in the disk  $|\lambda| < s(T)$ . Let us remark that  $W$  is closed, hence Banach subspace in  $X$  (see ([5, pp. 517, Corollary 3.2] and [10, Proposition 1.10]) or ([7, Remark 5.3] and [2, Lemma 3.6 (a), Theorem 3.8])) The restriction of  $T$  to the subspace  $W$  has been studied in [2], [5], [7] and [10]. Now we have

**Theorem 2.6.** *Let  $T \in B(X)$  be a semi-Fredholm operator, and  $\omega_r(T)$ ,  $r = 1, 2, \dots$  are as above. Then for each  $\omega_r(T)$ ,  $r = 1, 2, \dots$  we have*

$$|\omega_r(T)| = \lim_k q_r((T|_W)^k)^{1/k}.$$

**Proof.** By [5, Theorem 4.1] and [3, Theorem 4] we know that everywhere in the disk  $|\lambda| < s(T)$  we have that  $W = R((T - \lambda)^\infty) \oplus N_\lambda$ , where  $N_\lambda$  is finite dimensional subspace  $T$ -invariant and  $(T - \lambda)|_{N_\lambda}$  is nilpotent on it (see also [7, Remark 5.3]). Thus by [2, Theorem 3.4] we have that  $(T - \lambda)(W) = (T - \lambda)(R((T - \lambda)^\infty) \oplus N_\lambda) = R((T - \lambda)^\infty) \oplus (T - \lambda)(N_\lambda)$ . Thus,  $(T - \lambda)|_W$  is semi-Fredholm,  $\dim W/R((T - \lambda)|_W) < \infty$  and the stability index of  $T|_W$  is zero ([5, Proposition 2.6]). Let us remark that  $\omega_r(T)$ ,  $r = 1, 2, \dots$  are jumps (with the same multiplicity) in the semi-Fredholm region of  $T|_W$ . Now the proof of the theorem follows by Theorem 2.5.

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