REVISTA MATEMATICA de la
Universidad Complutense de Madrid
Volumen 4, números 2 y 3; 1991.

Monotonicity in Time and Stationary Solutions for a Quasilinear Heat Equation with Source

VICTOR A. GALAKTIONOV and SERGEY A. POSASHKOV

ABSTRACT. We consider the Cauchy problem for the quasilinear parabolic heat equation with source $u_t = \Delta u^{\sigma+1} + u^\beta$ in $R^N \times (0, T)$, $\sigma > 0, \beta > 1$ are fixed constants, with nonnegative bounded symmetric initial function. Two properties of monotone behaviours of the solution $u(|x|, t)$ for $x = 0$ are investigated. 1. Monotonicity of large solutions: there exists a constant $M_k > 0$ such that if $u(0, t_0) \geq M_k$ for some $t_0 \in [0, T)$, then $u_t(0, t) \geq 0$ for all $t \in [t_0, T)$. 2. $u(0, t)$ does not decrease in $(0, T)$. It is shown that sufficient conditions for these properties are quite different for five cases: i) $1 < \beta < \sigma + 1$, ii) $\beta = \sigma + 1$, iii) $\sigma + 1 < \beta < \beta_*$, iv) $\beta = \beta_*$, v) $\beta > \beta_*$, where $\beta_* = (\sigma+1)(N+2)/(N-2)$ for $N > 2$ ($\beta_* = \infty$ for $N = 1,2$) is the critical Sobolev exponent.

1. INTRODUCTION. MAIN RESULTS

In this paper we consider the Cauchy problem for the quasilinear parabolic heat equation with source

$$u_t = \Delta u^{\sigma+1} + u^\beta \quad \text{in} \quad R^N \times (0, T) \tag{1}$$

$$u(x, 0) = u_0(x) \geq 0 \quad \text{in} \quad R^N, \tag{2}$$

where $\sigma > 0$ and $\beta > 1$ are fixed constants. Equation (1) is well-known equation arising in the theory and in different applications of the processes of the heat conduction and combustion in a medium, where the heat conductivity coefficient $k(u) \equiv (\sigma+1)u^\sigma$ and the power of the energy emission $Q(u) \equiv u^\beta$ depend upon the temperature of the medium $u = u(x, t) \geq 0$.

1991 Mathematics Subject Classification: 35K55, 35K60.
Editorial de la Universidad Complutense. Madrid, 1991.

We shall assume the initial function (2) satisfies the following hypotheses:

$$u_0 = u_0(r) \geq 0 \quad \text{in} \quad R^N, r = |x|;\ M_1 = \sup u_0 < \infty;$$

$$u_0(r) \text{ is the continuous function in } R^1_+ = (0, \infty), \tag{3}$$

$$(u_0^\sigma u_0')(0) = 0.$$

Under these hypotheses, there exists the unique weak local (in time) solution $u = u(r, t)$ of the problem (1), (2), which is nonnegative continuous function in $R^N \times (0, T)$, where $T \in (0, \infty]$ is a finite or an infinite *existence time*. See a full list of references given in [11]. Notice that $u(r, t)$ is the classical solution at any point (r, t) where $u(r, t)$ is strictly positive.

The main results of the paper are devoted to the analysis of the behaviour with time of the temperature at the single point $x = 0$. This yields the conditions on the initial temperature for the *ignition* of the combustion process at the single point $x = 0$.

We shall consider two types of the monotone behaviours of $u(0, t)$.

Property (ML) *(Monotonicity of the Large solution):* there exists a constant $M_k > 0$, depending on the initial function, such that if $u(0, t_0) \geq M_k$ for some $t_0 \in [0, T)$, then

$$u_t(0, t) \geq 0 \text{ for all } t \in [t_0, T). \tag{4}$$

Property (M) *(Monotonicity for arbitrary t):*

$$u(0, t) \text{ doesn't decrease in } (0, T). \tag{5}$$

We show that above properties (ML) and (M) depend on the initial function u_0, the dimension of the space $N \geq 1$ and exponents σ, β of equation (1). These properties are quite different for five cases: i) $1 < \beta < \sigma + 1$, ii) $\beta = \sigma + 1$, iii) $\sigma + 1 < \beta < \beta_*$, iv) $\beta = \beta_*$, v) $\beta > \beta_*$, where

$$\beta_* = (\sigma + 1)(N + 2)/(N - 2) \text{ for } N > 2 \ (\beta_* = \infty \text{ for } N = 1,2) \tag{6}$$

is the critical Sobolev exponent for the nonlinear elliptic operator in the right-hand side of (1).

Let $B_\varepsilon = \{r < \varepsilon\}$ be the ball in R^N of a radius $\varepsilon > 0$ with the boundary $S_\varepsilon = \{r = \varepsilon\}$, $D_\varepsilon = R^N \setminus B_\varepsilon$ and denote

$$h^*(r) = c_* r^{-(N-2)/2(\sigma+1)} \text{ for } r > 0,$$

$$c_* = \left[\frac{N(N-2)}{4}\right]^{(N-2)/4(\sigma+1)} \quad (N>2);$$

$$h(r) = c\, r^{-2/[\beta-(\sigma+1)]} \text{ for } r>0,$$

$$c = \left[\frac{2N(\sigma+1)}{\beta-(\sigma+1)}\right]^{1/[\beta-(\sigma+1)]} \quad (\beta>\sigma+1); \tag{7}$$

$$h_\infty(r) = c_\infty r^{-2/[\beta-(\sigma+1)]} \text{ for } r>0,$$

$$c_\infty = \left\{\frac{2(\sigma+1)[\beta(N-2)-N(\sigma+1)]}{[\beta-(\sigma+1)]^2}\right\}^{-1/[\beta-(\sigma+1)]},$$

$$(\beta>(\sigma+1)\, N/(N-2) \quad \text{for } N>2).$$

We now state the main results of the paper.

Theorem 1 *($1<\beta<\sigma+1$). Let $1<\beta<\sigma+1$ and let (3) holds. Then*

(i) If $u_0(r)$ satisfies

$$\textit{there exists a large constant } R>0 \textit{ such that } u_0^{\sigma+1} \textit{ is uniformly Lipschitz continuous in } D_R, \tag{8}$$

then (ML) holds.

(ii) If $u_0^{\sigma+1}(r) \in C^1$ and

$$|[u_0^{\sigma+1}(r)]'| = o\,(r^{(\beta+\sigma+1)/(\sigma+1-\beta)}) \quad \text{as} \quad r\to+\infty, \tag{9}$$

then (ML) holds.

(iii) If $u_0^{\sigma+1}(r) \in C^1$ and

$$(u_0^{\sigma+1}(r))' + u_0^\beta(r)\; r/N > 0 \quad \text{in} \quad \{r>0\}\cap\{u_0>0\}, \tag{10}$$

then (M) holds.

Theorem 2 *($\beta=\sigma+1$). Let $\beta=\sigma+1$ and let (3) holds. Then*

(i) If $u_0(r)$ satisfies

there exists small enough $\varepsilon>0$ *such that*

$$u_0^{\sigma+1} \text{ is Lipschitz continuous in } \{z_N-\varepsilon<r<z_N\}, \tag{11}$$

where $z_N>0$ *is the first zero of Bessel function* $J_{(N-2)/2}$, *then (ML) holds.*

(ii) If $u_0^{\sigma+1}(r)\in C^1$ *and*

$$(u_0^{\sigma+1}(r))'+u_0^{\sigma+1}(r)\,r/N>0 \text{ in } \{0<r<z_N\}\cap\{u_0>0\}, \tag{12}$$

then (M) holds.

(iii) If $u_0(r)$ *is the nondecreasing function in* $\{0<r<z_N\}$, *then (M) holds.*

Theorem 3 *($\sigma+1<\beta<\beta_*$). Let* $\sigma+1<\beta<\beta_*$ *and let (3) holds. Then*

(i) If for some small $\varepsilon>0$

$$u_0^{\sigma+1}(r) \text{ is Lipschitz continuous in } B_\varepsilon, \tag{13}$$

then (ML) holds.

(ii) If $u_0^{\sigma+1}(r)\in C^1$ *and*

$$(u_0^{\sigma+1}(r))'+u_0^{\beta}(r)\,r/N>0 \text{ in } \{r>0\}\cap\{u_0>0\}, \tag{14}$$

then (M) holds.

(iii) If $u_0(r)$ *is the nondecreasing function in* $\{r<l_0\}$,
where

$$l_0=c_1(u_0(0))^{-[\beta-(\sigma+1)]/2} \tag{15}$$

and $c_1=c_1(\sigma,\beta,N)$ *is some positive constant, then (M) holds.*

(iv) If $u_0(r)$ *is the nondecreasing function in* $\{r>0 \mid u_0(r)<h(r)\}$, *then (M) holds.*

(v) Let $l_*=\sup\{a>0 \mid u_0(r)$ *is the nondecreasing function for* $r\in(0,a)\}>0$. *Then (ML) holds for*

$$M_k=(c_1/l_*)^{2/[\beta-(\sigma+1)]}.$$

Theorem 4 *($\beta=\beta_*$). Let $\beta=\beta_*$ for $N\geq 3$ and let (3) holds. Then*

(i) If $u_0(r)$ satisfies (13) for some $\varepsilon>0$ and

$$u_0(r)>0 \text{ for all } r>0, \tag{16}$$

there exists constant $m_0>0$ such that

$$u_0(r)>m_0\cdot r^{-(N-2)/(\sigma+1)} \text{ for any large } r>0, \tag{16'}$$

then (ML) holds.

(ii) Assertion (ii) of Theorem 3 is valid.

(iii) If $u_0(r)$ is the nondecreasing function in $\{r>0\mid u_0(r)<h^*(r)\}$, then (M) holds.

Theorem 5 *($\beta>\beta_*$). Let $\beta>\beta_*$ for $N\geq 3$ and let (3) holds. Then*

(i) Let $u_0(r)=h_\infty(r)$ for a unique point $r=r_*>0$, $u_0\in C^1$ in a small neighbourhood of $r=r_*$ and $u_0'(r_*)>h_\infty'(r_*)$. Let for any sufficiently small $\varepsilon>0$ the set $\{r>0\mid |u_0(r)/h_\infty(r)-1|<\varepsilon\}$ be a finite connected interval containing the point $r=r_*$. Then (ML) holds.

(ii) Assertion (ii) of Theorem 3 is valid.

(iii) If $u_0(r)$ is the nondecreasing function in $\{r>0\mid u_0(r)<h(r)\}$, then (M) holds.

Proofs of Theorems 1-5 are based on intersection comparison of the solution $u(r,t)$ with the set of the stationary solutions $\{U\}$. Therefore, we begin with the analysis of the set $\{U\}$.

2. SOME PROPERTIES OF THE SET OF THE STATIONARY SOLUTIONS

We now describe some well-known properties of the stationary solutions $U(r;\lambda)$ satisfying

$$\begin{gathered}\Delta U^{\sigma+1}+U^\beta\equiv r^{1-N}(r^{N-1}(U^{\sigma+1})_r)_r+U^\beta=0 \text{ for } r>0,\\ U(0;\lambda)=\lambda>0,\\ U_r(0;\lambda)=0,\end{gathered} \tag{17}$$

where $\lambda>0$ is an arbitrary fixed constant.

One can see that equation (17) is invariant under the self-similar transformation and hence

$$U(r;\lambda)=\lambda\, U(r\lambda^{m};1),\ m=[\beta-(\sigma+1)]/2, \text{ for any fixed } \lambda>0. \qquad (18)$$

It is well-known (see e.g. [10]) that for $\beta\geq\beta_*$ any solution of (17) is strictly positive for all $r>0$ and $U(\infty;\lambda)=0$. For $1<\beta<\beta_*$ the function $U(r;\lambda)$ vanishes at some point $r=l(\lambda)$, where

$$l(\lambda)=c_1\cdot\lambda^{-m}, \qquad (19)$$

$c_1(\sigma,\beta,N)\equiv l(1)>0$ is given in (15). In this case we let $U(r;\lambda)\equiv 0$ for $r>l(\lambda)$. For $\beta\geq\beta_*$ we formally let $l(\lambda)=+\infty$.

Thus, for $\beta>1$ we have introduced one parametric set of the stationary solutions $\{U(r;\lambda);\ \lambda\geq 0\}$ $(U(r;0)\equiv 0)$. In all cases for arbitrary fixed $\lambda>0$ $U(r;\lambda)$ is the monotone decreasing function in the domain of positivity. The function $U(r;\lambda)$ is the continuous function with respect to r and λ. From well-known properties of solution of the Cauchy problem for the ordinary differential equation (17) we get that there exists the continuous dependence on λ of the solution $U(r;\lambda)$ in any compact from R^1_+ and the derivative $(U^{\sigma+1})_r$ in any compact from the domain $\omega(\lambda)=[0,l(\lambda))$ of positivity of the stationary solution.

The case $\beta\in(1,\sigma+1)$. For $\beta\in(1,\sigma+1)$ from (19) it follows that

$$l(\lambda)\to+\infty \text{ monotone as } \lambda\to+\infty. \qquad (20)$$

Identity (18) yields the following conditions

$$U(r;\lambda)\to+\infty \text{ uniformly in } [0,\ l/2] \text{ as } \lambda\to+\infty, \qquad (21)$$

$$(U^{\sigma+1})_r(r;\lambda)\to-\infty \text{ uniformly in } [l/2,l] \text{ as } \lambda\to+\infty. \qquad (22)$$

Introduce for a fixed constant $M>0$ the set $g(\lambda,M)=\{r>0\mid 0<U(r;\lambda)<M\}$. From (18), (19) one can obtain the following estimates: if $r\in g(\lambda,M)$ then

$$\begin{aligned} &r=l(\lambda)\,(1+o(1))\to\infty \text{ as } \lambda\to\infty,\\ &|(U^{\sigma+1})_r(r;\lambda)|=k_1 r^{-(\beta+\sigma+1)/2m}(1+o(1))\to\infty \text{ as } \lambda\to\infty, \end{aligned} \qquad (22')$$

where $k_1=k_1(\sigma,\beta,N)>0$ is some constant.

The case $\beta=\sigma+1$. In this case equation (17) has the explicit solution

$$U(r;\lambda)\equiv U_S(r;\lambda)=\lambda\{\Gamma(N/2)2^{\nu}r^{-\nu}J_{\nu}(r)\},\ r\in[0,z_N), \qquad (23)$$

where $v=(N-2)/2$, $\Gamma(N/2)$ is Euler's Gamma function, z_N is the first zero of Bessel function $J_v(r)$ $(U(z_N;\lambda)=0)$.

For any fixed arbitrary small $\varepsilon>0$

$$U_S\to+\infty \text{ uniformly in } r\in[0,z_N-\varepsilon] \text{ as } \lambda\to+\infty, \tag{24}$$

$$(U_S^{\sigma+1})_r\to-\infty \text{ uniformly in } r\in[z_N-\varepsilon,z_N) \text{ as } \lambda\to+\infty. \tag{25}$$

The case $\beta\in(\sigma+1,\ \beta_*)$. Then it follows from (19) that

$$l(\lambda)\to 0 \text{ monotone as } \lambda\to+\infty, \tag{26}$$

and properties (21), (22) are also valid.

The case $\beta=\beta_*(N\geq 3)$. It is well-known [12] that for $\beta=\beta_*$, $N\geq 3$, equation (17) admits the explicit strictly positive solution

$$U(r;\lambda)=\lambda\cdot\left[\frac{N(N-2)}{N(N-2)+\lambda^{4(\sigma+1)/(N-2)}r^2}\right]^{(N-2)/2(\sigma+1)} \tag{27}$$

for $r>0$. It has the following properties:

$$U(r;\lambda)=c_2\lambda^{-1}\,r^{-(N-2)/(\sigma+1)}(1+o(1)) \text{ as } r\to\infty, \tag{28}$$

where $c_2=[N(N-2)]^{(N-2)/2(\sigma+1)}$;

$U(r;\lambda)\to 0$ uniformly in any set $[\delta,+\infty)$ as $\lambda\to\infty$,

$$(U^{\sigma+1})_r(r;\lambda)\to-\infty \text{ uniformly in any set} \tag{29}$$

$$\{r>0 \mid m_1\leq U(r;\lambda)\leq m_2\} \text{ as } \lambda\to\infty,$$

where δ, $m_1<m_2$ are arbitrary fixed positive constants.

The case $\beta>\beta_*$, $N\geq 3$. Then $U(r;\lambda)>0$ for any $r>0$, $U(+\infty;\lambda)=0$ and there hold [10]

$$\begin{gathered} U(r;\lambda)=h_\infty(r)(1+o(1)),\\ U_r(r;\lambda)=h'_\infty(r)(1+o(1)) \text{ as } r\to+\infty, \end{gathered} \tag{30}$$

where the function $h_\infty(r)$ is given by (7). From (18), (30) one can easily verify that for any fixed $\delta>0$

$$U(r;\lambda)/h_\infty(r)\to 1, \tag{31}$$

$$U_r(r;\lambda)/h'_\infty(r)\to 1 \text{ uniformly in any set } [\delta,+\infty) \text{ as } \lambda\to\infty.$$

The estimates of $U(r;\lambda)$. The first estimate holds for arbitrary $\beta>1$. By using monotonicity of the function $U(r;\lambda)$ for $r>0$ we obtain the following inequality

$$r^{1-N}(r^{N-1}(U^{\sigma+1})_r)_r=-U^\beta\geq-\lambda^\beta \text{ for } r>0.$$

Integrating twice this inequality yields the lower estimate

$$U(r;\lambda)\geq U_-(r;\lambda)\equiv\lambda(1-r^2/r_0^2)_+^{1/(\sigma+1)} \text{ for } r>0, \tag{32}$$

where $r_0=r_0(\lambda)\equiv(2N)^{1/2}\cdot\lambda^{-[\beta-(\sigma+1)]/2}$.

By integrating equation (17) and by using the monotonicity of the function $U(r;\lambda)$ for $r>0$ we get the following inequality:

$$(U^{\sigma+1})_r+U^\beta\frac{r}{N}\leq 0 \text{ for } r>0. \tag{33}$$

The second estimate for $\beta\in(\sigma+1,+\infty)$ we shall derive by integrating (33). Then we have for $r>0$

$$U(r;\lambda)\leq\left[\lambda^{-[\beta-(\sigma+1)]}+\frac{r^2[\beta-(\sigma+1)]}{2N(\sigma+1)}\right]^{-1/[\beta-(\sigma+1)]},$$

and hence

$$U(r;\lambda)\leq h(r),\ r>0, \text{ for any fixed } \lambda>0, \tag{34}$$

where $h(r)$ is given in (7).

The tangent curve. Upper estimate (34) implies that for any $\beta>\sigma+1$ there exists the tangent curve $L=L(r)$ of the set $\{U(r;\lambda)\}$ and the following upper estimate

$$L(r)\equiv\sup_{\lambda>0}U(r;\lambda)\leq h(r) \text{ for } r>0 \tag{35}$$

holds. Moreover, from (32) for $\beta>\sigma+1$ we obtain the lower estimate of $L(r)$:

$$L(r)\geq L_-(r)\equiv\sup_{\lambda>0}U_-(r;\lambda)=c_3r^{-2/[\beta-(\sigma+1)]} \text{ for } r>0, \tag{36}$$

where

$$C_3=\left[\frac{\beta-(\sigma+1)}{\beta}\right]^{1/(\sigma+1)}\left[\frac{2N(\sigma+1)}{\beta}\right]^{1/[\beta-(\sigma+1)]}.$$

For $\beta = \beta_*$ the tangent curve can be calculated explicitly:

$$L(r) = h^*(r) \text{ for } r > 0,$$

where the function $h^*(r)$ is determined by (7).

3. INTERSECTION COMPARISON

1. Let $\omega(\lambda) = [0, l(\lambda))$ be the domain of positivity of the function $U(r; \lambda)$ for fixed $\lambda > 0$ ($\omega(\lambda) \equiv [0, +\infty)$ for $\beta \geq \beta_*$). Let $N(t; \lambda)$ for fixed $t \in [0, T)$ be the number of intersections of the functions $u(r, t)$ and $U(r; \lambda)$ in $\omega(\lambda)$ (see e.g. [6]–[8], [15]) or, which is the same, the number of sign changes of the difference $u(r, t) - U(r; \lambda)$ [1], [14], [16]. This implies that the functions $u(r, t)$ and $U(r; \lambda)$ are positive in a small neighbourhood of any intersection. Since the solutions $u(r, t)$ and $U(r; \lambda)$ are classical there, each intersection is an isolated point for any fixed $t > 0$ [1], [13], [16]. Without loss of generality we shall assume that for $t = 0$ and for all $\lambda > 0$ there exist only points of intersection, and $N(0; \lambda) < +\infty$.

The following Lemma 1 is well-known for classical solutions, see [1], [6], [7], [13]-[16]; for weak solutions of degenerate equations see results in [8], [15].

Lemma 1. *Fix arbitrary $\lambda > 0$. Let (3) holds and $N(0; \lambda) < \infty$. Then $N(t; \lambda) \leq N(0; \lambda) + 1$. If $N(0; \lambda) \leq 1$ for some fixed $\lambda > u_0(0)$, then*

$$N(t; \lambda) \leq 1 \text{ for any } t \in (0, T). \tag{37}$$

Proof. Below we use the standard technique of construction of the weak solution of the Cauchy problem (1), (2) [11]. Fix an arbitrary small $\varepsilon > 0$ and denote

$$u_{0\varepsilon}(x) \equiv \max\{u_0(r), \varepsilon\} > 0 \text{ for } r \geq 0. \tag{38}$$

Clearly, $u_{0\varepsilon}(x) \to u_0(r)$ as $\varepsilon \to 0$ uniformly in R^N. The Cauchy problem for (1) with initial function (38) has the unique classical strictly positive solution $u_\varepsilon(r, t) \geq \varepsilon$ in $R^N \times (0, T)$ [3]. Moreover (see [11] and references therein)

$$u_\varepsilon(r, t) \to u(r, t) \text{ as } \varepsilon \to 0 \tag{39}$$

uniformly in any compact set from $R^N \times (0, T)$.

Let $N_\varepsilon(t;\lambda)$ for fixed $t\in[0, T)$ be the number of the points of intersection of the solutions $u_\varepsilon(r,t)$ and $U(r;\lambda)$ in $\omega(\lambda)$. One can see that $N_\varepsilon(0;\lambda)\leq N(0;\lambda)+1$ (one additional point of intersection «given» in the right-hand side of the last inequality can arise on the boundary $r=l(\lambda)$ of $\omega(\lambda)$). Consider for arbitrary fixed $t\in(0,T)$ the domain $\Omega(t)=\omega(\lambda)\times(0,t)$. It is well-know that $N_\varepsilon(t;\lambda)$ is not greater than the number of the sign changes of the difference $w_\varepsilon(r,t;\lambda)\equiv u_\varepsilon(r,t)-U(r;\lambda)$ on the parabolic boundary of $\Omega(t)$ [1], [13], [14], [16]. Since $u_\varepsilon(l(\lambda),t)>0$ for all $t\in(0,T)$ we get the inequality

$$N_\varepsilon(t;\lambda)\leq N_\varepsilon(0;\lambda) \quad \text{for } t\in(0,T).$$

From (39) by using the continuity of $u(r,t)$ we can conclude that

$$N(t;\lambda)\leq N_\varepsilon(t;\lambda) \text{ for any small } \varepsilon>0,$$

and hence $N(t;\lambda)\leq N(0;\lambda)+1$.

If $\lambda>u_0(0)$ and $N(0;\lambda)=0$ or 1, then it is easily seen that $N_\varepsilon(0;\lambda)\leq 1$ for any small $\varepsilon>0$. Then in both cases we have $N(t;\lambda)\leq 1$ for all $t\in(0,T)$, which completes the proof.

2. **Comparison with the set of the stationary solutions.** The following two Lemmas 2, 3 are based on the analysis of the number of intersection of the initial function $u_0(r)$ with the set $\{U(r;\lambda)\}$ of the stationary solutions.

These Lemmas yield the sufficient conditions for the properties (ML) and (M).

Lemma 2 *(property (ML))). Let (3) holds and there exists* $\lambda_*>M_1$ *such that*

$$N(0;\lambda)\leq 1 \text{ for all } \lambda>\lambda_*. \tag{40}$$

Then (ML) holds with $M_k=\lambda_*$.

Lemma 3 *(property (M)). Let (3) holds and*

$$\begin{gathered} N(0;\lambda)\leq 1 \text{ for all } \lambda>\lambda_0=u_0(0), \\ u_0(r)\geq U(r;\lambda_0) \text{ in } R^l_+. \end{gathered} \tag{41}$$

Then (M) holds.

Proof of Lemma 2. Let $u_0 \notin \{U(r;\lambda)\}$ and (40) is valid for some λ_*. Fix arbitrary $\lambda > \lambda_*$. Let $u(0, t_\lambda) = \lambda$ for some $t_\lambda \in (0, T)$ such that $u(0, t) < \lambda$ for all $t \in (0, t_\lambda)$. We shall show that (see also [4], [15, Chapter IV])

$$u(r, t_\lambda) \geq U(r;\lambda) \text{ for all } r \geq 0. \tag{42}$$

Suppose (42) is not valid. Then from (40) and Lemma 1 it follows that $N(t_\lambda;\lambda) \leq 1$.

Consider the first case when $N(t_\lambda;\lambda) = 0$. Then $u(r, t_\lambda) \leq U(r;\lambda)$ in $\omega(\lambda)$. Hence, without loss of generality we may assume supp $u(r, t_\lambda) \subseteq \omega(\lambda)$. Since by well-known properties of the interface [11] supp $u_0 \subseteq$ supp $u(r, t_\lambda)$ and hence by Lemma 1 we have $N(0;\lambda) = 0$. Therefore, $u_0(r) \leq U(r;\lambda)$ in $\omega(\lambda)$ and $u(l(\lambda), t) = 0$ for $t \in [0, t_\lambda]$. Then by the strong maximum principle [3, Chapter II], applied for $u(r, t)$ in the domain of positivity near the origin, we get $u(0, t_\lambda) < U(0;\lambda)$, whence the contradiction.

Consider the second case when $N(t_\lambda;\ \lambda) = 1$, i.e. there exists one sign change of the difference $w(r, t_\lambda;\lambda) \equiv u(r, t_\lambda) - U(r;\lambda)$ in $\omega(\lambda)$. Then by using the continuity of $U(r;\lambda)$ with respect to λ we get that there exists some sufficiently small $|\varepsilon| > 0$, $\lambda + \varepsilon > \lambda_*$, such that the difference $w(r, t_\lambda;\lambda)$ has at least two sign changes in $\omega(\lambda + \varepsilon)$ and $N(t_\lambda;\ \lambda + \varepsilon) \geq 2$. See the similar analysis in [6-8], [15, p. 384]. This contradicts (37) and (40).

Thus, (42) is valid. Then by the comparison theorem $u(r, t) \geq U(r;\lambda)$ in $\omega(\lambda)$ for all $t \in (t_\lambda, T)$. Since $u(0, t_\lambda) = \lambda = U(0;\lambda) > 0$ and $u(0, t) \geq \lambda$ for any $t \in (t_\lambda, T)$ we obtain the inequality $u_t(0, t_\lambda) \geq 0$ and hence (ML) is valid because $\lambda > \lambda_*$ is arbitrary.

Proof of Lemma 3. Since $u_0(r) \geq U(r;\lambda_0)$ in R_+^1 by the comparison theorem $u(r, t) \geq U(r;\lambda_0)$ in $R_+^1 \times (0, T)$. This implies that $u(r, t)$ doesn't decrease with time for $t = 0$ at any point $r = r_*$, where $u_0(r_*) = U(r_*;\lambda_0) > 0$, in particular, at the point $r = 0$. The end of the proof is quite similar to the proof of Lemma 2 with $\lambda_* = \lambda_0$.

4. PROOFS OF THEOREMS 1-5

Proofs of Theorems 1-5 are based on the properties of the set of the stationary solution and on Lemmas 2 and 3.

Proof of Theorem 1. (i) From (20)$-$(22) and (8) it follows that there exists some sufficiently large $\lambda_* = \lambda_*(R) > 0$ such that $N(0;\lambda) \leq 1$ for all $\lambda > \lambda_*$. Then by Lemma 2 (i) holds.

Notice that, roughly speaking, for $\beta \in (1, \sigma+1)$ the property (ML) doesn't depend on the behaviour of the initial function $u_0(r)$ in a neighbourhood of origin.

(ii) Estimates (22') with $M = M_1$ (see (3)) and the condition (9) imply that for any sufficiently large $\lambda > 0$ the inequality $N(0; \lambda) \leq 1$ holds. Hence by Lemma 2 (ii) is valid.

(iii) From estimate (33) and condition (10) we get the ordinary differential inequality of the first order for the difference $z(r) = u_0(r) - U(r; \lambda_0)$, $\lambda_0 = u_0(0) > 0$:

$$(a(r)\, z)_r + b(r) z > 0 \text{ in } \omega(\lambda) \cap \{u_0 > 0\},\ z(0) = 0,$$

where $a(r)$ and $b(r)$ are some smooth nonnegative functions. Then by the comparison theorem $z \geq 0$ in $\omega(\lambda) \cap \{u_0 > 0\}$ and hence $u_0(r) \geq U(r; \lambda_0)$ in R^1_+. Moreover, if for $\lambda > \lambda_0$ there exists some point of intersection $r = r_*$ of the functions $u_0(r)$ and $U(r; \lambda)$ in $\omega(\lambda)$, then $(u_0)_r(r_*) > U_r(r_*; \lambda)$. This implies that $u_0(r) \geq U(r; \lambda)$ for all $r > r_*$ and hence $N(0; \lambda) \leq 1$. Then by Lemma 3 (iii) is valid.

From (i) we have

Corollary 1. *Let $\beta \in (1, \sigma+1)$, (3) holds and let $u_0(r)$ be a compactly supported function. Then (ML) holds.*

Proof of Theorem 2. (i) This is similar to the proof of (i) of Theorem 1. Explicit solution (23) and properties given in (24), (25) are used.

Note that for $\beta = \sigma+1$ the property (ML) at the origin depends on the behaviour of u_0 in a small left neighbourhood of the point $r = z_N$.

(ii) See the proof of (iii) of Theorem 1.

(iii) Denote $\lambda_0 = u_0(0)$. Then $u_0(r) > U(r; \lambda_0) \equiv U_S(r; \lambda_0)$ for all $r \in (0, z_N)$ since $u_0(r)$ is nondecreasing function in $(0, z_N)$. Since $U_r(r; \lambda) < 0$, $r \in (0, z_N)$, for any $\lambda > \lambda_0$ and $U(r; \lambda) \equiv 0$ for $r \geq z_N$, we get $N(0; \lambda) = 1$ and Lemma 3 can be used.

The assertion (i) yields

Corollary 2. *Let $\beta = \sigma+1$, (3) holds and let $u_0(r)$ be a compactly suppoted function such that $\sup \{r > 0 \mid u_0(r) > 0\} < z_N$. Then (ML) is valid.*

Proof of Theorem 3. (i) Properties (21), (22), (26) and the condition (13) guarantee the inequality $N(0;\lambda)\leq 1$ for all sufficiently large $\lambda>0$. Hence Lemma 2 can be applied. Therefore, the property (ML) depends on the behaviour of the initial function near $r=0$.

(ii) See the proof of (iii) of Theorem 1.

(iii) Since $U_r(r;\lambda)<0$ in $\omega(\lambda)$, which is the domain of positivity of the function $U(r;\lambda)$, from (15), (18), (19) we get that $u_0(0)=U(0;u_0(0))$ and $u_0(r)>U(r;u_0(0))$ for $r\in(0,l_0)$. From (26) we obtain the inequality $N(0;\lambda)=1$ for all $\lambda>u_0(0)$. Now one can use Lemma 3.

(iv) Since $h(r)\geq\sup\limits_{\lambda>0} U(r;\lambda)$ (see (35)) and $u_0(r)$ is nondecreasing function in $\{r>0\,|\,u_0(r)<h(r)\}$ we get that $u_0(r)\geq U(r;u_0(0))$ for all $r>0$ and $N(0;\lambda)=1$ for all $\lambda>u_0(0)$. Then by Lemma 3 (iv) holds.

(v) See (19), (26) and the proof of (iii) of this Theorem.

Proof of Theorem 4. (i) Choose $\delta=\varepsilon$ (see (13)). Then from (13) and (29) for all sufficiently large $\lambda>0$ there exists unique intersection of the functions $u_0(r)$ and $U(r;\lambda)$ in $(0,\delta)$. By using (16), (16′), (28), (29) we get that there are no points of intersection in $[\delta,\infty)$, i.e. $N(0;\lambda)=1$ for large $\lambda>0$. Hence, (ML) holds.

(ii) See the proof of assertion (iii) of Theorem 1. Notice that for $\beta\geq\beta_*$ under hypothesis (14) the initial function should be strictly positive.

(iii) See the proof of (iv) of Theorem 3.

Proof of Theorem 5. (i) Fix small $\delta>0$ such that $h_\infty(\delta)>2M_1$. By using (31) for any large $\lambda>0$ we can choose sufficiently small $\varepsilon>0$ such that

$$|U(r;\lambda)/h_\infty(r)-1|<\varepsilon,\quad |U_r(r;\lambda)/h'_\infty(r)-1|<\varepsilon \tag{43}$$

in $[\delta,\infty)$. Since $u_0\in C^1$ in the neighbourhood of unique point $r=r_*$, where $u_0(r_*)=h_\infty(r_*)$ and $u_0'(r_*)>h'_\infty(r_*)$, from (43) for small $\varepsilon>0$ and for all large $\lambda>0$ it follows that there exists a unique intersection of the functions $u_0(r)$ and $U(r;\lambda)$ in the connected interval $\{r>0\,|\,|u_0(r)/h_\infty(r)-1<\varepsilon\}$. Then by Lemma 2 (ML) is valid.

(ii) See the proof of (iii) of Theorem 1.

(iii) See the proof of (iv) of Theorem 3.

References

[1] S. ANGENENT: *The zero set of a solution of a parabolic equation,* J. reine angew. Math., **390,** 78-96 (1988).
[2] A. FRIEDMAN: *On the regularity of the solutions of nonlinear elliptic and parabolic systems of partial differential equations,* J. Math. Mech., 7, 43-59 (1958).
[3] A. FRIEDMAN: *Partial Differential Equations of Parabolic Type,* Prentice-Hall, Englewood Cliffs, New Jersey (1964).
[4] V. A. GALAKTIONOV: *On conditions of localization of unbounded solutions of quasilinear parabolic equations,* Doklady AN SSSR, *264,* 1035-1040 (1982) (in Russian).
[5] V. A. GALAKTIONOV, S. P. KURDYUMOV and A. A. SAMARSKII: *On structure of limiting distributions for blow-up solutions of quasilinear parabolic equations,* Doklady AN SSSR, *301,* 781-785 (1988) (in Russian).
[6] V. A. GALAKTIONOV and S. A. POSASHKOV: *Applications of new comparison theorems for unbounded solutions of nonlinear parabolic equations,* Differentzial'nye Uravnenia, *22,* 1165-1173 (1986) (in Russian).
[7] V. A. GALAKTIONOV and S. A. POSASHKOV: *Any large solution of nonlinear heat conduction equation with source becomes monotome in time,* Preprint Inst. Appl. Math. Acad. Sci. USSR, n.º 109 (1989) (in Russian).
[8] V. A. GALAKTIONOV and S. A. POSASHKOV: *On the property of monotonicity for quasilinear degenerate parabolic equation,* Differentzial'nye Uravnenia, *26,* 1127-1136 (1990) (in Russian).
[9] Y. GIGA and R. V. KOHN: *Characterizing blow-up using similarity variables,* Indiana Univ. Math. J., *36,* 1-40 (1987).
[10] D. D. JOSEPH and T. S. LUNDGREN: *Quasilinear Dirichlet problems driven by positive sources,* Arch. Rational Mech. Anal., *60,* 171-183 (1976).
[11] A. S. KALASHNIKOV: *Some questions of qualitative theory of nonlinear degenerate parabolic equations of the second order,* Uspekhi Matem. Nauk, *42,* 135-176 (1987) (in Russian).
[12] C. LOEWNER and L. NIRENBERG: *Partial differential equations invariant under conformal and projective transformations.* In: Contributions to Analysis, Academic Press, New York, 245-272 (1974).
[13] H. MATANO: *Nonincrease of the lap number of a solution for a one-dimensional semi-linear parabolic equation,* J. Fac. Sci. Univ. Tokyo, Sec. IA, *29,* 401-441 (1982).
[14] K. NICKEL: *Gestaltaussagen uber Losungen Parabolischer Differenzialgleichungen,* J. reine angew. Math., *211,* 78-94 (1962).
[15] A. A. SAMARSKII, V. A. GALAKTIONOV, S. P. KURDYUMOV and A. P. MIKHAILOV: *Blow-up in Problems for Quasilinear Parabolic Equations,* Nauka, Moscow (1987) (in Russian).
[16] D. H. SATTINGER: *On the total variation of solutions of parabolic equations,* Math. Ann., *183,* 78-92 (1969).

Keldysh Institute of Applied Mathematics USSR Acad. Sci.
Miusskaya Sq. 4.
125047 MOSCOW, USSR

Recibido: 3 de septiembre de 1990