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Non-Containment of l^1 in Projective Tensor Products of Banach Spaces

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ABSTRACT. Two properties on projective tensor products are introduced and briefly studied. We apply them to give sufficient conditions to assure the non-containment of l^1 in a projective tensor product of Banach spaces.

Our notation is standard and we refer the reader to [3]. Let E and F be Banach spaces. $L(E, F)$ and $V(E, F)$ denote respectively the Banach spaces of bounded (linear) operators and of fully complete operators (i.e. those operators sending weakly convergent sequences into norm convergent ones) from E to F. Recall that every compact operator from E to F is fully complete and the converse holds whenever E has no copy of l^1 (cf. [3,17.1, and 17.7]). $E\hat{\otimes}F$ denotes the projective tensor product of E and F. As usual, we make the canonical identification of the dual space $(E\hat{\otimes}F)'$ with the Banach spaces $L(E, F')$ and $L(F, E')$.

Let $(x_n)_n$ and $(y_n)_n$ denote sequences in the Banach spaces E and F respectively. We consider the following properties on $E\hat{\otimes}F$:

(a) *$(x_n\otimes y_n)_n$ is weakly null whenever $(x_n)_n$ and $(y_n)_n$ are weakly null.*

(a') *$(x_n\otimes y_n)_n$ is weakly null if $(x_n)_n$ is weakly null and $(y_n)_n$ is weak-Cauchy.*

(b) *$(x_n\otimes y_n)_n$ is weakly null if $(x_n)_n$ is weakly null and $(y_n)_n$ is bounded.*

Note that the property (b) is not symmetric, e.g. $l^1\hat{\otimes}c_0$ enjoys (b) and $c_0\hat{\otimes}l^1$ does not do it as can be easily checked.

The following result summarizes some basic facts on (a), (a′) and (b).

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1. **Proposition:** *Let E and F be Banach spaces and consider the following assertions:*

(i) $L(E, F') = V(E, F')$.

(ii) $E\hat{\otimes}F$ *has the property* (b).

(iii) $E\hat{\otimes}F$ *has the property* (a').

(iv) $E\hat{\otimes}F$ *has the property* (a).

Then one has the chain of implications (i)$\Longleftrightarrow$(ii)$\Longrightarrow$(iii)$\Longleftrightarrow$(iv). *Moreover* (iii)$\Longrightarrow$(ii) *whenever F has no copy of* l^1.

Proof. (ii)$\Longrightarrow$(i). Assume that there is a continuous linear mapping f from E into F' which does not belong to $V(E, F')$. Then, there is a weakly null sequence, say $(x_n)_n$, such that $(f(x_n))_n$ does not converge to 0 in norm. By taking a subsequence if it is necessary, we can choose a bounded sequence $(y_n)_n$ in F such that

$$<y_n, f(x_n)> = <x_n \otimes y_n, f> = 1$$

so contradicting (ii).

(iv)$\Longrightarrow$(iii) follows by a standard argument (e.g. see [1, Theorem 1. (c)$\Longrightarrow$(d)]). If F has no copy of l^1, then (ii) follows from (iii) by using the celebrated Rosenthal's l^1-theorem ([4]). The remaining implications are straightforward. ■

We use (a) and (b) to give a characterization of the classical Dunford-Pettis and Schur properties. We first recall the definitions: (D-P) A Banach space E is said to have the Dunford-Pettis property provided $\lim_{n\to\infty} <x_n, x_n^*> = 0$ whenever $(x_n)_n$ is weakly null in E and $(x_n^*)_n$ is weakly null in E'. (S) We say that a Banach space E has the Schur property if weak Cauchy sequences in E are norm convergent.

2. **Proposition:** *Let E be a Banach space. The following are equivalent:*

(i) $E\hat{\otimes}E'$ *has the property* (a).

(ii) *E has the Dunford-Pettis property,*

(iii) $E\hat{\otimes}F$ *has the property* (a) *for every Banach space F.*

(iv) $E\hat{\otimes}F$ *has the property* (a) *for every reflexive Banach space F.*

Proof. (i)$\Longrightarrow$(ii). Take $(x_n)_n$ and $(x_n^*)_n$ weakly null sequences in E and E' respectively. We denote by $I_{E'}$ the identity map of E' and set $(<x_n, x_n^*>)_n = (<x_n, I_{E'}(x_n^*)>)_n = (<x_n \otimes x_n^*, I_{E'}>)_n$ which is a null sequence by (a).

(ii) $\Rightarrow$ (iii). Let $(x_n)_n$ and $(y_n)_n$ weakly null sequences in E and F respectively and take any $f \in L(F, E')$, then $(f(y_n))_n$ is weakly null in E' hence $(<x_n \otimes y_n, f>)_n = (<x_n, f(y_n)>)_n$ converges to 0. Thus $(x_n \otimes y_n)_n$ is weakly null.

It is clear that (iii) implies (i) and (iv). we finish the proof by showing that (iv) implies (ii). Indeed, by (iv) and by Proposition 1 it follows that $L(E, F) = V(E, F)$ for every reflexive Banach space F. Thus every weakly compact operator from E into any Banach space X is fully complete (use [3, 17.2.9]) and this already implies that E has the Dunford-Pettis property ([1, Theorem 1. (a)]). ∎

We omit the proof of our next result since it is quite similar to the above one.

3. **Proposition.** *Let E be a Banach space. The following are equivalent:*

(i) $E \hat{\otimes} E'$ *has the property* (b),

(ii) *E has the Schur property,*

(iii) $E \hat{\otimes} F$ *has the property* (b) *for every Banach space F,*

We are now ready to provide sufficient conditions for the non-containment of l^1 in a projective tensor product of Banach spaces. Recall that a subset A of a Banach space E is said to be *weakly conditionally compact* (wcc) if every sequence in A has a weak-Cauchy subsequence. From Rosenthal's Theorem, E does not contain a copy of l^1 if and only if all bounded sets of E are wcc. So the next lemma is the key to our main result.

4. **Lemma.** *Let E and F be Banach spaces and let A and B be wcc sets in E and F respectively. Then* $\overline{\Gamma(A \otimes B)}$ *is wcc whenever* $E \hat{\otimes} F$ *has the property* (a).

Proof. According to the results of [5] it is enough to show that $A \hat{\otimes} B$ is wcc. Indeed, let $(x_n \otimes y_n)_n$ be any sequence in $A \otimes B$. By passing to subsequences we assume that $(x_n)_n$ and $(y_n)_n$ are weak-Cauchy. We are done if we show that $(x_n \otimes y_n)_n$ is weak-Cauchy. Indeed, in other case there would be $\epsilon > 0$, $f \in L(E, F')$ and a sequence $n_1 < n_2 < \ldots$, such that

$$|<x_{n_k} \otimes y_{n_k} - x_{n_{k+1}} \otimes y_{n_{k+1}}, f>| > \epsilon$$

However, we can set

$$<x_{n_k} \otimes y_{n_k} - x_{n_{k+1}} \otimes y_{n_{k+1}}, f> = <(x_{n_k} - x_{n_{k+1}}) \otimes y_{n_k}, f> +$$
$$+ <x_{n_{k+1}} \otimes (y_{n_k} - y_{n_{k+1}}), f>$$

and these sequences are null by (a′) (recall that (a′)⇔(a)). This contradiction establishes our assertion. ■

The theorem below is our main result. It has been independently obtained by G. Emmanuele [2, Theorem 15], and it was already proved in [6, 4.4] under the additional hypotheses that E' has the Radon-Nikodym property and the approximation property.

5. **Theorem.** *Let E and F be Banach spaces which do not contain a copy of l^1 and such that $E\hat{\otimes}F$ has the property* (a). *Then $E\hat{\otimes}F$ does not contain a copy of l^1.*

Proof. It readily follows by Lemma 4. ■

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