REVISTA MATEMATICA de la
Universidad Complutense de Madrid
Volumen 3, número 1; 1990.

# *Symmetric stochastic matrices with given row sums*

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**ABSTRACT.** Characterizations of extreme infinite symmetric stochastic matrices with respect to arbitrary non-negative vector $\bar{r}$ are given.

## 1. INTRODUCTION

Let $\bar{r}= (r_1, r_2, r_3, ...)$ be a sequences of non-negative reals. We say that a matrix $P=(p_{ij})$ is symmetric stochastic (substochastic) with respect to $\bar{r}$ if

$$p_{ij}\geq 0,$$

$$p_{ij}=p_{ji},$$

$$\sum_j p_{ij}=r_i \quad (\sum_j p_{ij}\leq r_i)$$

Note that every symmetric stochastic matrix with respect to $\bar{r}$ is doubly stochastic with respect to $(\bar{r}, \bar{r})$. In this paper we consider the affine structure of the set of symmetric stochastic matrices, in particular we identify extreme points of the set of symmetric stochastic and substochastic matrices with respect to any arbitrary non-negative vector $\bar{r}$. Note that to each result in symmetric stochastic matrices there corresponds a result for doubly stochastic matrices (d.s.m.). Proofs of this facts have some similarity, thought symmetric stochastic matrices have their own specificity.

We denote by $\mathcal{S}(\bar{r})$ and $\mathcal{S}(\leq \bar{r})$ the sets of all symmetric stochastic and substochastic matrices with respect to $\bar{r}$, respectively. The sets of their extreme points ext $\mathcal{S}(\bar{r})$, ext $\mathcal{S}(\leq \bar{r})$ were considered in certain particular cases. For $\bar{r}=(1, 1, ..., 1, 0, 0, ...)$ the extreme points of $\mathcal{S}(\bar{r})$ and $\mathcal{S}(\leq \bar{r})$ were shown by M. Katz [13] and [14]; for a corresponding to d.s.m. result see Birkhoff [2] and Mirsky [18]. This was generalized to the infinite matrix case ($r_i=1$, $i=1,2,...$) in [8] (for d.s.m. see Kendal [15] and Isbel [11]). The second direction of generalizations is to consider any arbitrary finite sequence

1980 Mathematics Subject Classification (1985 revision): 47D20, 52A05.

Editorial de la Universidad Complutense. Madrid, 1990.

(i.e. $r_i = 0$ for $i > n$). In this case a characterization of ext $\mathscr{S}(\bar{r})$ and $\mathscr{S}(\leq \bar{r})$ was presented by Brualdi [4] (for d.s.m. see [17], [12], [3]). Note that the characterization of extreme matrices is the same in the case of infinite matrices (see Converse and Katz [6]) under the natural assumption that $\sum_i r_i$ is finite (for d.s.m. see [16], [1], [7], [19]). The characterization of ext $\mathscr{S}(\bar{r})$ and ext $\mathscr{S}(\leq \bar{r})$ in the general case is more complicated and it is presented in Section 2 and Section 3 (for d.s.m. see [10]). In Section 4 we investigate the dimension of the faces of the convex polytopes $\mathscr{S}(\bar{r})$ and $\mathscr{S}(\leq \bar{r})$ (for d.s.m. see [5], see also [9]).

Let $P = (p_{ij}) \in \mathscr{S}(\leq \bar{r})$. We define a graph $G(P)$ associated with $P$ as follows. Corresponding to row $i$ (and column $i$) we have a node $i$. There is an edge joining node $m$ and node $n$ if and only if $p_{mn} = p_{nm} > 0$. If a diagonal entry $p_{nn} > 0$, then the node $n$ of the graph $G(P)$ has a loop (an edge joing a node to itself), cf. [4].

For example the matrix

$$P_0 = \begin{bmatrix} 2 & 1 & 3 & 0 & 0 & 0 & 0 \\ 1 & 0 & 4 & 0 & 0 & 0 & 0 \\ 3 & 4 & 0 & 5 & 0 & 0 & 0 \\ 0 & 0 & 5 & 0 & 0 & 7 & 8 \\ 0 & 0 & 0 & 0 & 6 & 9 & 0 \\ 0 & 0 & 0 & 7 & 9 & 0 & 0 \\ 0 & 0 & 0 & 8 & 0 & 0 & 0 \end{bmatrix}$$

has the graph:

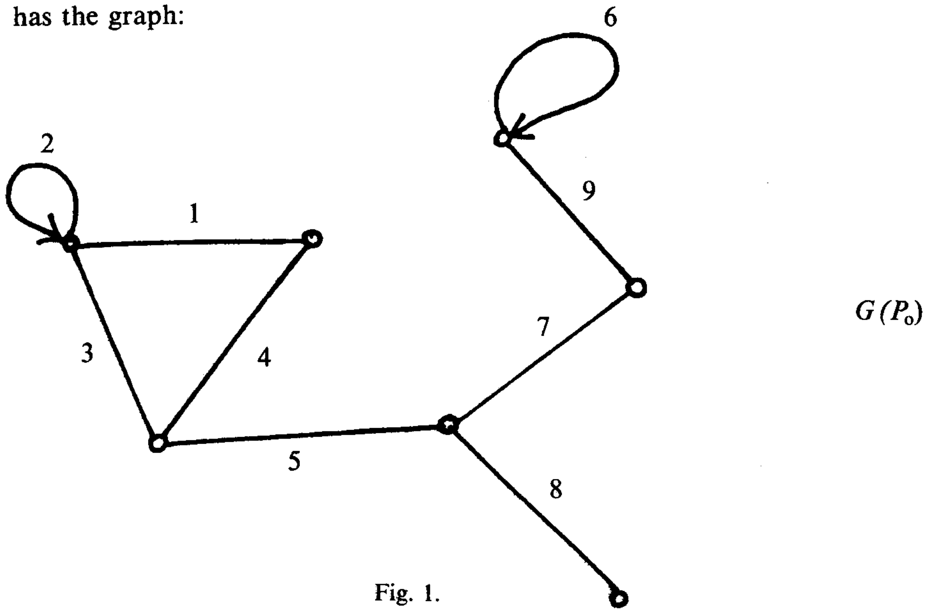


Fig. 1.

We say that $P=(p_{ij})\in\mathscr{S}(\leq\bar{r})$ is an elementary matrix if there are no nonzero matrices $P_1$ and $P_2$ such that $P=P_1+P_2$ and supp $P_1\cap$ supp $P_2=\phi$ (supp $P=\{j$: there exists $i$ such that $p_{ij}\neq 0\}$). Each matrix can be represented as a sum of some elementary matrices $P_k$, $P=\sum_k P_k$, with supp $P_k$ disjoint. In that case we say that $P$ is the direct sum of $P_k$'s. Note that $P$ is an elementary matrix if and only if the graph $G(P)$ is connected. The graph $G(P)$, $P\in\mathscr{S}(\leq\bar{r})$, has $k_0$ ($k_o$ is finite or infinite) connected components if and only if the matrix $P$ can be decomposed into $k_o$ elementary matrices. For instance the matrix $P_o$ is elementary.

If a connected graph $G$ has two distinct cycles of odd length, then $G$ has also a cycle of even length (not necessarily an elementary cycle). If $p_{nn}>0$ for some $n$, then the graph $G((p_{ij}))$ has an odd cycle (of length one). In particular, if $P=(p_{ij})$ is a simple matrix and $p_{nn}>0$, $p_{mm}>0$, $m\neq n$, then the graph $G(P)$ has an even cycle.

In the case $\sum_i r_i=\infty$ the extremality of $P\in\mathscr{S}(\bar{r})$ depends not only on its graph $G(P)$. For example consider matrices $P=(p_{ij})$ and $Q=(q_{ij})$, where

$$P_{ij}=\begin{cases}\dfrac{1}{2} & \text{if } |i\text{-}j|=1\\[2ex] \dfrac{1}{2} & \text{if } i=j=1\\[1ex] 0 & \text{otherwise}\end{cases}$$

$$q_{ij}=\begin{cases}1 & \text{if } i=j=1\\[1ex] \dfrac{1}{i+j} & \text{if } |i\text{-}j|=1\\[1ex] 0 & \text{otherwise}\end{cases}$$

Obviously $G(P)=G(Q)$. It is easy to check that $Q\in\text{ext }\mathscr{S}(\bar{r})$ (where $r_i=\dfrac{4i}{4i^2-1}$) and $P\notin\text{ ext }\mathscr{S}(\bar{r})$ (where $r_i=1$).

The methods we use in this paper are similar to those from [10]. That is why we simplify some parts of the proofs.

## 2. STOCHASTIC MATRICES

For $P \in \mathscr{S}(\leq r)$ we say that the graph $G(P)$ has an $\epsilon$-bitree if there exists a family $\{A_k\}_{k \in K}$ of disjoint non-empty subsets of $\mathbb{N}$ and a family $\{\epsilon_k\}_{k \in K}$ of positive numbers and $m, n \in K \subset \mathbb{N}$ such that

$$m \neq n, k \notin A_k \subset K$$

$$0 < \epsilon = \epsilon_m = \epsilon_n \leq p_{mn},$$

$$0 < \epsilon_i \leq p_{ki}, \qquad i \in A_k,$$

$$\sum_{i \in A_k} \epsilon_i = \epsilon_k, \quad k \in K$$

(cf. [10]). Note that the sets $A_k$ are connected with indexes of rows (and columns) of the matrix $p = (p_{ij})$. In fact $p_{ik} = p_{ki} > 0$ for all $i \in A_k$ (see Fig. 2 for example of $\epsilon$-bitree). If in a graph $G(P)$ there exists an $\epsilon$-bitree then there exists also in $G(P)$ an $\epsilon$-bitree $H$ such that every node of $H$ is joined with a finite number of edges of $H$.

We say that the graph $G(P)$ has an infinite $\epsilon$-path if there exists a sequence of distinct natural numbers $\{i_k\}_{k=1}^{\infty}$ ($i_n \neq i_m$, if $m \neq n$) such that

$$\inf\{p_{i_k i_{k+1}} : k = 1, 2, \dots\} \geq \epsilon > 0$$

**Example.** Let $r = (2, 2, 1, 1, 1, 1, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \dots)$.

The matrix

$$P = \left[\begin{array}{cccccccccccccccccccc}
0 & 1 & 1/2 & 1/2 \\
1 & 0 & & & 1/2 & 1/2 \\
1/2 & & & & & & 1/4 & 1/4 \\
1/2 & & & & & & & & 1/4 & 1/4 \\
 & 1/2 & & & & & & & & & 1/4 & 1/4 \\
 & 1/2 & & & & & & & & & & & 1/4 & 1/4 \\
 & & 1/4 & & & & & & & & & & & & 1/8 & 1/8 \\
 & & 1/4 & & & & & & & & & & & & & & 1/8 & 1/8 \\
 & & & 1/4 & & & & & & & & & & & & & & & 1/8 & \dots \\
 & & & 1/4 \\
 & & & & 1/4 & & & & & & & 0 \\
 & & & & 1/4 \\
 & & & & & 1/4 \\
 & & & & & 1/4 \\
 & & & & & & 1/8 \\
 & & & & & & 1/8 \\
 & & & & & & & 1/8 \\
 & & & & & & & & \ddots
\end{array}\right]$$

has the graph $G(P)$ given by Fig. 2. The graph $G(P)$ has $\epsilon$-bitree ($\epsilon=1$). In view of Theorem 1 $P\notin$ ext $\mathscr{S}(\bar{r})$.

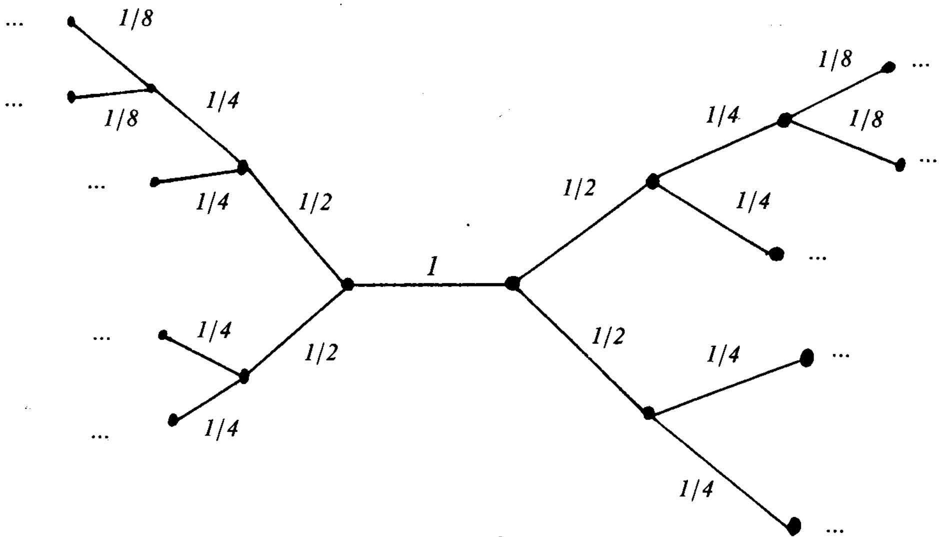


*Fig. 2.*

**Theorem 1.** *Let $P\in\mathscr{S}(\bar{r})$. Then $P\in$ ext $\mathscr{S}(\bar{r})$ if and only if the following conditions are satysfied:*

(i) *the graph $G(P)$ has no even cycle,*

(ii) *the graph $G(P)$ has no $\epsilon$-bitree,*

(iii) *each connected component of $G(P)$ which has an odd cycle has no $\epsilon$-path.*

**Proof.** It is not difficult to check that, if $G(P)$ has a cycle of even length then $P\notin$ ext $\mathscr{S}(\bar{r})$.

Suppose that $G(P)$ has $\epsilon$-bitree. We define a matrix

$T=(t_{ij})$ inductively:

<a> $t_{nm}=t_{mn}=\epsilon$,

<b> $\begin{cases} t_{ni}=t_{in}=-\epsilon_i, i\in A_n \\ t_{mi}=t_{im}=-\epsilon_i, i\in A_m \end{cases}$,

<c> $t_{i_1}=t_{ii_1}=\epsilon_i, i\in A_{i_1}, i_1\in A_m\cup A_n$,

<d> $t_{i_2 i} = t_{ii_2} = -\epsilon_i, i \in A_{i_2}, i_2 \in A_{i_1}, i_1 \in A_m \cup A_n,$

...

...

The other entries $t_{ij}$ are equal to 0. It is easy to verify that $P \pm T \in \mathscr{S}(\bar{r})$. Thus if $G(P)$ has an $\epsilon$-bitree then $T \notin$ ext $\mathscr{S}(\bar{r})$.

Suppose that there exists a connected component of $G(P)$ which has an odd cycle $(i_1, i_2, ..., i_{2n-1}, i_1)$ and has an infinite $\epsilon$-path $(j_1, j_2, j_3, ...)$. We may and do assume that $i_1 = j_1$ and $\min (p_{i_1 i_2}, p_{i_2 i_3}, ..., p_{i_{2n-1} i_1}) \geq \frac{\epsilon}{2} > 0$. We define $R = (r_{ij})$ by

$$r_{ij} = r_{ji} = \begin{cases} \frac{\epsilon}{2} & (i,j) = (i_{2k-1}, i_{2k}), \quad k = 1, 2, ..., n-1), \\ \frac{\epsilon}{2} & (i,j) = (i_{2n-1}, i_1) \quad , \\ -\frac{\epsilon}{2} & (i,j) = (i_{2k}, i_{2k+1}), \quad k = 1, 2, ..., n-1), \\ \epsilon & (i,j) = (j_{2k}, j_{2k+1}), \quad k = 1, 2, ... \quad , \\ -\epsilon & (i,j) = (j_{2k-1}, j_{2k}), \quad k = 1, 2, ... \quad , \\ 0 & \text{otherwise.} \end{cases}$$

One can easily verify, that $P \pm R \in \mathscr{S}(\bar{r})$ i.e. $P \notin$ ext $\mathscr{S}(\bar{r})$. Therefore we get that $P \in$ ext $\mathscr{S}(\bar{r})$ implies (i), (ii) and (iii).

Now assume that (i), (ii) and (iii) are satysfied and $P \notin$ ext $\mathscr{S}(\bar{r})$. Then there exists $T = (t_{ij}) \neq 0$ such that $P \pm T \in \mathscr{S}(\bar{r})$. The graph $G(|T|)$ is a subgraph of $G(P)$ and $|t_{ij}| \leq p_{ij}$. $T$ is symmetric and $\sum_j t_{ij} = 0$ for all $i$. Let $H$ be a (non-empty) connected component of $G(|T|)$. We have two posibilities:

(1) the graph $H$ as no cycle,

(2) the graph $H$ has exactly one odd cycle.

Indeed, if there are two distinct odd cycles in $H$ then $H$ has an even cycle. Hence $G(P)$ has also an even cycle, what is impossible because of (i).

Now suppose that $H$ has no cycle. Take an arbitrary edge of $H$ and the corresponding to it entry, say $t_{mn} \neq 0$. We construct inductively the $\epsilon$-bitreee in $H$. Put $A_m = \{i \neq n\colon t_{mi} \neq 0\}$ and $A_n = \{i \neq m\colon t_{in} \neq 0\}$.

Since $\sum_i t_{mi} = \sum_i t_{ni} = 0$, the sets $A_m$ and $A_n$ are non-empty. Put $\epsilon = |t_{mn}| = \epsilon_m = \epsilon_n$. We choose $\epsilon_i$, $i \in A_m \cup A_n$ such that $\sum_{i \in A_m} \epsilon_i = \sum_{i \in A_n} \epsilon_i = \epsilon$ and $0 < \epsilon_i \leq |t_{im}|$, $i \in A_m$, and $0 < \epsilon_i \leq |t_{in}|$, $i \in A_n$.

Now we put $A_k = \{i \neq m\colon t_{ik} \neq 0\}$, $k \in A_m$, and $A_k = \{i \neq n\colon t_{ik} \neq 0\}$, $k \in A_n$. And so forth. The sets $A_k$ are disjoint, because $H$ has no cycle. Thus we get an $\epsilon$-bitree in $H$, so also in $G(P)$. This contradiction implies that (1) does not hold.

Suppose that $H$ has exactly one odd cycle, say $(i_1, i_2, \ldots, i_{2n-1}, i_{2n} = i_1)$. Then there exists $k$ such that $t_{i_{k-1}i_k} + t_{i_k i_{k+1}} \neq 0$, say $k = 1$. Put

$$B_{i_1} = \{i\colon t_{ii_1} \neq 0,\ i \neq i_2,\ i \neq i_{2n-1}\}$$

$$B_k = \{i \neq i_1\colon t_{ik} \neq 0\},\ k \in B_{i_1},$$

$$B_k = \{i \neq j_1\colon t_{ii_1} \neq 0\},\ k \in B_{j_1},\ j_1 \in B_{i_1},$$

and so forth. The sets $B_k$ are non-empty. If all $B_k$ have exactly one element then $0 < |t_{i_1 j_1}| = |t_{j_1 j_2}| = |t_{j_2 j_3}| = \ldots$, where $\{j_1\} = B_{i_1}$, $\{j_2\} = B_{j_1}$, $\{j_3\} = B_{j_2}$: ..., i.e. $H$ has an infinite $\epsilon$-path, so $G(P)$ has an infinite $\epsilon$-path, what is impossible in view of (iii). It is also not difficult to see that if some of $B_k$ has more than one element, $H$ (so also $G(P)$) has an $\epsilon$-bitree, what contradicts with (ii). Therefore also (2) does not hold.

The above presented contradictions prove that the conditions (i), (ii) and (iii) imply $P \in \operatorname{ext} \mathscr{S}(\mathfrak{r})$.

We say that a connected graph $G$ is a simple odd cactus if $G$ consists of exactly one cycle of odd (and finite) length (cf. [4]). Using the same arguments as in [10] we obtain the following facts.

**Corollary 1.** *Let $\sum_i r_i$ be finite, and let $P \in \mathscr{S}(\mathfrak{r})$. Then $P \in \operatorname{ext} \mathscr{S}(\mathfrak{r})$ if and only if the connected components of the graph $G(P)$ are tree or simple odd cacti.*

We say that a matrix $P \in \mathscr{S}(\mathfrak{r})$ is uniquely determined in $\mathscr{S}(\mathfrak{r})$ by its graph if there is no matrix $Q \in \mathscr{S}(\mathfrak{r})$, $P \neq Q$, such that the graph $G(Q)$ is a subgraph of $G(P)$.

**Proposition 1.** *The extreme points of $\mathscr{S}(\bar{r})$ are those matrices in $\mathscr{S}(\bar{r})$ which are uniquely determined in $\mathscr{S}(\bar{r})$ by their graphs.*

**Proposition 2.** *The set of all extreme points of $\mathscr{S}(\bar{r})$ coincides with the set of all exposed points of $\mathscr{S}(\bar{r})$.*

Note that Corollary 1 can be also obtained as a consequence of two results: of Theorem in [6, p. 174] and of Theorem 3.1 in [4].

## 3. SUBSTOCHASTIC MATRICES

We say that the $i$-th row sum of a matrix $P=(p_{ij})\in\mathscr{S}(\leq\bar{r})$ is unattained if $\sum_j p_{ij}<r_i$.

**Theorem 2.** *Let $P\in\mathscr{S}(\leq\bar{r})$. Then $P\in ext\ \mathscr{S}(\leq\bar{r})$ if and only if the following conditions are satysfied:*

(i) *the graph $G(P)$ has no even cycle,*

(ii) *the graph has no $\epsilon$-bitree,*

(iii) *each connected component $G_o$ of the graph $G(P)$ has at most one node corresponding to row of $P$ whose sum in $P$ is unattained, and $G_o$ satisfies at most one of the following conditions:*

(a) *$G_o$ has an infinite $\epsilon$-path,*

(b) *$G_o$ has an odd cycle,*

(c) *$G_o$ has a node corresponding to the row of $P$ whose sum in $P$ is unattained.*

**Proof.** Obviously, if the graph $G(P)$ has a cycle of even length then $P\notin$ ext $\mathscr{S}(\leq\bar{r})$. By the proof of Theorem 1, if $G(P)$ has an $\epsilon$-bitree then also $P\notin$ ext $\mathscr{S}(\leq\bar{r})$. Suppose that in some connected component $G_o$ of $G(P)$ there are two nodes corresponding to rows whose sum in $P$ is unattained, say $m_1$ and $m_2$. Then there exists a path $(i_1, i_2, ..., i_n)$ in $G(P)$ such that $i_1=m_1$ and $i_n=m_2$. We define $R=(r_{ij})$ by

$$r_{ij}=r_{ji}=\begin{cases}\epsilon & \text{if } (i,j)=(i_{2k-1}, i_{2k})\\ -\epsilon & \text{if } (i,j)=(i_{2k}, i_{2k+1})\\ 0 & \text{otherwise,}\end{cases}$$

where $0<\epsilon=\min\,(p_{i_1i_2}, p_{i_2i_3}, ..., p_{i_{n-1}i_n}, r_{i_1}-\sum_j p_{i_1j}, r_{i_n}-\sum_j p_{i_nj})$. We have $P\pm R\in\mathcal{S}(\leq\bar{r})$, so $P\notin$ ext $\mathcal{S}(\leq\bar{r})$.

Now suppose that two from three conditions (a), (b), and (c) are satisfied. If (a) and (b) are satisfied then, by the proof of Theorem 1, $P$ is not extreme. Now assume that some connected component $G_o$ of $G(P)$ satisfies (c), i.e. there exists a node $m$ in $G_o$ such that $\sum_j p_{mj}<r_j$. Supposed that also (a) is satisfied by $G_o$. Let $\{j_k\}_{k=1}^{\infty}$ be an infinite $\epsilon$-path in $G_o$. We may and do assume that $m=j_1$. We define $R=(r_{ij})$ by

$$r_{ij}=r_{ji}=\begin{cases}\epsilon_1 & \text{if } (i,j)=(j_{2k-1}, j_{2k}) \\ -\epsilon_1 & \text{if } (i,j)=(j_{2k}, j_{2k+1}) \\ 0 & \text{otherwise}\end{cases}\qquad k=1,2,...$$

where $\epsilon_1=\min\,(\epsilon,\ r_m-\sum_j p_{mj})>0$. We get $P\pm R\in\mathcal{S}(\leq\bar{r})$, i.e. $P$ is not extreme.

Suppose now that (b) (and (c)) holds. Let $C=(i_1, i_2, ..., i_{2n-1}, i_{2n}=i_1)$ be a cycle of odd length in $G_o$. We find a path $D=(j_1, j_2, ..., j_l)$ joining the node $m$ with the odd cycle $C$. We may assume that $D$ and $C$ have exactly one common node, say $i_1=j_l$. Then we have $j_1=m$. We have $P\pm R\in\mathcal{S}(\leq\bar{r})$, where $R=(r_{ij})$,

$$r_{ij}=r_{ji}=\begin{cases}\epsilon & (i,j)=(i_{2k}, i_{2k+1}),\quad k=1,2,...,n \\ -\epsilon & (i,j)=(i_{2k-1}, i_{2k}),\quad k=1,2,...,n \\ 2\epsilon & (i,j)=(j_{2k-1}, j_{2k}),\quad 1\leq k\leq\left[\frac{l}{2}\right] \\ -2\epsilon & (i,j)=(j_{2k}, j_{2k+1}),\quad 1\leq k\leq\left[\frac{l-1}{2}\right] \\ 0 & \text{otherwise}\end{cases}$$

and $\epsilon>0$ is sufficently small. Hence $P$ is not extreme. From the above part of the proof it follows that $P\in$ext $\mathcal{S}(\leq\bar{r})$ implies the conditions (i), (ii) and (iii).

Now let $T=(t_{ij})$ be such that $P\pm T\in\mathcal{S}(\leq\bar{r})$ and suppose the conditions (i), (ii) and (iii) are satisfied. We have $|t_{ij}|\leq p_{ij}$ and the graph $G(|T|)$ also satisfies the conditions (i) and (ii). We have $\sum_j t_{ij}=$ o for all $i$ such that $\sum_j p_{ij}=r_j$. Let $H$ be a (non-empty) connected component of $G(|T|)$. For all nodes $i$ of

$H$, except at most one, we have $\sum_j t_{ij}=0$. Moreover at most one of the following conditions holds:

(a$_1$) $H$ has an infinite $\epsilon$-path,

(b$_1$) $H$ has an odd cycle,

(c$_1$) $H$ has a node (at most one) corresponding to a row $i$ such that $\sum_j t_{ij}=0$.

If the condition (c$_1$) does not hold then using the arguments presented in the proof of Theorem 1 we get a conntradiction. Thus $H$ is a tree (has no cycle) which has no infinite $\epsilon$-path, and for all nodes $i$ of $H$, except $i_o$ ($i_o$ is a node of $H$) we have $\sum_j t_{ij}=0$. Put $A=\{i: t_{ii_o}=0\}$.

Note that $i_o \in A$, because $H$ has no cycle (even those of length one). Take $i_1 \in A$. Put

$$B_{i_1}=\{i \neq i_0: t_{ii_1} \neq 0\},$$

$$B_k=\{i \neq i_1: t_{ik} \neq 0\}, \quad k \in B_{i_1}$$

$$B_k=\{i \neq i_2: t_{ik} \neq 0\}, \quad k \in B_{i_2}, \quad i_2 \in B_{i_1}$$

and so forth. The sets $B_k$ are non-empty (because of $\sum_j t_{ij}=0$) and disjoint ($H$ has no cycle).

If all $B_k$ have exactly one element then $H$ has an infinite $\epsilon$-path (what is impossible). If some of $B_k$ has more than one element then $H$ has an $\epsilon$-bitree (what is also impossible). This contradiction proves that the conditions (i), (ii) and (iii) imply $P \in \text{ext}\ \mathcal{S}(\leq \mathfrak{r})$.

The following corollary is a generalization of Theorem 3.3 in [4].

**Corollary 2.** *Let $\sum_j r_j$ be finite, and let $P \in \mathcal{S}(\leq \mathfrak{r})$. Then $P \in \text{ext}\,\mathcal{S}(\leq \mathfrak{r})$ if and only if the connected components of the graph $G(P)$ are trees with at most one node corresponding to the row whose sum in $P$ is unattained or simple odd cacti all of whose nodes correspond to rows whose sum in $P$ is attained.*

## 4. THE FACIAL STRUCTURE IN THE FINITE DIMENSIONAL CASE

As already pointed out, the vertices (extreme points) of $\mathcal{S}(\mathfrak{r})$ and $\mathcal{S}(\leq \mathfrak{r})$ are described. Now we consider the dimension of the faces of the convex polytopes $\mathcal{S}(\mathfrak{r})$, $\mathcal{S}(\leq \mathfrak{r})$ in the finite dimensional case.

For $q \in Q$ ($Q$ being a convex set) we define $\dim_Q q$, the dimension of $q$ in $Q$, as the affine dimension of the face

$$\{q_1 \in Q\text{: there exists } q_2 \in Q \text{ and } 0 < \alpha \leq 1 \text{ such that } q = \alpha q_1 + (1-\alpha) q_2\}$$

generated by $q$ in $Q$.

**Theorem 3.** *Let $\bar{r} = (r_1, r_2, \ldots, r_n)$ ($r_i > 0$, $i \leq n$, $n$ is finite) and let $P = (p_{ij}) \in \mathcal{S}(\bar{r})$ be $n \times n$ elementary matrix. Then.*

$$\dim_{\mathcal{S}(\bar{r})} P = \begin{cases} \delta(P) - n + 1 & \text{if the graph } G(P) \text{ has no odd cycle} \\ \delta(P) - n & \text{if the graph } G(P) \text{ has an odd cycle (including the length one)} \end{cases}$$

*where* $\delta(P) = \sum_{1 \leq i \leq j \leq n} \text{sign } p_{ij}$

**Proof.** We define the functionals $\varphi_i(T) = \sum_{j=1}^{n} t_{ij}$, $T \in \mathcal{S}(\bar{r})$. We have

$\dim_{\mathcal{S}(\bar{r})} P = \dim\{T \in X\colon \varphi_i(T) = 0, 1, 2, \ldots, n\}$, where $X = \{T\colon T = (t_{ij})$ is $n \times n$ matrix, $t_{ij} = t_{ji}$, $t_{ij} = 0$ for all $(i, j)$ such that $p_{ij} = 0\}$. Thus $\dim_{\mathcal{S}(\bar{r})} P$ is equal to $\dim X = \delta(P)$ minus the number of linearly independent functionals in the set $\{\varphi_1, \varphi_2, \ldots, \varphi_n\}$ considered as the set of linear functionals on $X$.

Let $\alpha_i$ be real scalars. Put $\xi = \sum_{j=1}^{n} \alpha_i \varphi_i$. Suppose that $\xi = 0$. Take a subgraph $H$ of $G(P)$ such that $H$ is a connected tree and all nodes of $G(P)$ belong to $H$. The graph $H$ exists, because $P$ is an elementary matrix. The tree $H$ has $n-1$ edges $(a_1, b_1)$, $(a_2, b_2)$, ... $(a_{n-1}, b_{n-1})$ ($a_i$ and $b_i$ are nodes of $G(P)$) and $n$ nodes $i_1, i_2, \ldots, i_n$. We have $\xi(Q^{mn}) = \alpha_m + \alpha_n = 0$, where $Q^{mn} = (q_{ij})$, $q_{ij} = \delta_{mi}\delta_{nj} + \delta_{ni}\delta_{mj}$.

Therefore we have $n-1$ equalities:

$$\begin{cases} \alpha_{a_1} + \alpha_{b_1} = 0 \\ \alpha_{a_2} + \alpha_{b_2} = 0 \\ \ldots \\ \alpha_{a_{n-1}} + \alpha_{b_{n-1}} = 0 \end{cases}$$

Because $H$ is a tree, the above equalities are independent. Hence, if we put $\alpha_1=\alpha$ then all other $\alpha_i$ are equal to $\alpha$ or $-\alpha$. More precisely $\alpha_k=(-1)^L\alpha$, where $L$ is the length of the path joining $i_k$ and $i_1$. Thus we get

$$\dim \operatorname{lin} \{\varphi_1, \varphi_2, \ldots, \varphi_n\} \geq n-1.$$

Put

$$A=\{i\colon \alpha_i=\alpha\} \text{ and } B=\{i\colon \alpha_i=-\alpha\}.$$

We have

$$\xi=\alpha(\sum_{i\in A} \varphi_i-\sum_{i\in B} \varphi_i)$$

Suppose that there exists in $G(P)$ an edge which joints some element of $A$ and some element of $B$. Let $m\in A$, $k\in B$, and let edge $(m, k)$ belong to $G(P)$ ($p_{mk}=0$, $Q^{mk}\in X$). Then $\xi(Q^{mk})=2\alpha=0$. Hence $\alpha=0$ i.e. $\dim \operatorname{lin} \{\varphi_i\colon i\leq n\}=n$. Note that the existence of such $m$ and $k$ means that: $G(P)$ has an odd cycle (including of length one), whereas if there is no such edge in $G(P)$ we have

$$\sum_{i\in A} \varphi_i = \sum_{i\in B} \varphi_i$$

for all $T\in X$. Hence $\{\varphi_i\}$ are not linearly independent as a functional on $X$. Thus

$$\dim \operatorname{lin}\{\varphi_i : i\leq n\}= \begin{cases} n \text{ if } G(P) \text{ has an odd cycle,} \\ n-1 \text{ if } G(P) \text{ has no odd cycle.} \end{cases}$$

Let $P=\sum_{k=1}^{k_0} P_k$ be the decomposition into elementary matrices $P_k$. Then $\dim_{\mathscr{S}(\bar{r})} P=\sum_{k=1}^{k_0} \dim P_k$. Hence we have the following fact.

**Corollary 3.** *Let $r=(r_1, r_2, \ldots, r_n)$ $(r_i>0,\ i\leq n)$ and let $P=(p_{ij})\in \mathscr{S}(\bar{r})$ be $n\times n$ matrix: Then*

$$\dim_{\mathscr{S}(\bar{r})} P=\delta(P)-n+k_1,$$

*where $k_1\leq k_0$ denotes the number of connected components of $G(P)$ which have no odd cycle.*

**Theorem 4.** *Let $\bar{r}=(r_1, r_2, \ldots, r_n)$ $(r_i>0,\ i\leq n)$ and let $P=(p_{ij})\in\mathcal{S}(\leq\bar{r})$ be $n\times n$ elementary matrix. Then*

$$\dim_{\mathcal{S}(\leq\bar{r})} P=\begin{cases}\delta(P)-n+1 & \text{if } G(P) \text{ has no odd cycle and all row sums in } P \text{ are attained } (P\in\mathcal{S}(\bar{r})),\\ \delta(P)-n_0 & \text{otherwise}\end{cases}$$

$$\text{where } \delta(P)=\sum_{1\leq i\leq j\leq n} \operatorname{sign} p_{ij},\quad n_0=\operatorname{card}\{i: \sum_{j=1}^{n} p_{ij}=r_i\}.$$

**Proof.** We use facts from the proof of Theorem 3. We have $\dim_{\mathcal{S}(\leq\bar{r})} P=\dim\{T\in X: \varphi_i(T)=0,\ i\in A\}$, where $A=\{i: \sum_{j=1}^{n} p_{ij}=r_i\}$ (card $A=n_0$).

We should consider only the case $n_0<n$.

Because $G(P)$ is connected and $\dim \operatorname{lin}\{\varphi_i\}_{i=}^{n}\geq n-1$ any arbitrary proper subset of $\{\varphi_i\}_{i=1}^{n}$ is linearly independent. Hence

$$\dim \operatorname{lin}\{\varphi_i: i\in A\}=\begin{cases} n \text{ or } n-1 & \text{if } n_0=n\\ n_0 & \text{if } n_0<n\end{cases}$$

This with Theorem 3 ends the proof.

**Corollary 4.** *Let $\bar{r}=(r_1, r_2, \ldots, r_n)$ and let $P\in\mathcal{S}(\leq\bar{r})$. Then*

$$\dim_{\mathcal{S}(\leq\bar{r})} P=\delta(P)-n_0+k_2$$

*where $k_2$ denotes the number of connected components $H$ of the graph $G(P)$ such that* 1° *all nodes of $H$ correspond to rows whose sum in $P$ is attained, and* 2° *$H$ has no odd cycle.*

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Recibido: 3 de febrero de 1988
Revisado: 18 de junio de 1989