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Primariness of some spaces of continuous functions

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ABSTRACT. J. Roberts and the author have recently shown that, under the Continuum Hypothesis, the Banach space l_∞/c_0 is primary. Since this space is isometrically isomorphic to the space $C(\omega^*)$ of continuous *scalar* functions on $\omega^*=\beta\omega-\omega$, it is quite natural to consider the question of primariness also for the spaces of continuous *vector* functions on ω^*. The present paper contains some partial results in that direction. In particular, from our results it follows that $C(\omega^*,C(K))$ is primary for any infinite metrizable compact space K (without assuming the CH).

A Banach space X is said to be *primary* if, whenever we have a (topological) direct sum decomposition $X=E\oplus F$, then either E or F is isomorphic to X. Many Banach spaces are known to be primary; among them are the spaces $C(K)$ of continuous scalar functions on infinite metrizable compact spaces K ([3],[1]). In a recent paper [2], answering a question posed by Leonard and Whitfield. James Roberts and the author have shown that, under the Continuum Hypothesis (CH), also the Banach space l_∞/c_0, which is isometrically isomorphic to $C(\omega^*)$, is primary. (Throughout this paper, ω^* denotes the remainder $\beta\omega-\omega$ of the Stone-Čech compactification of $\omega=\{1,2,...\}$). The present paper originated from an attempt, not very successful so far, to generalize this result to the spaces $C(\omega^*,X)$, where X is a Banach space.

For the purpose of this paper let us agree to say that a Banach space X is *nice* if for every (continuous linear) operator $T:X\rightarrow X$ there exists a subspace Y of X which is isomorphic to X and which is mapped isomorphically by one of the operators T or id_X-T onto a complemented subspace of X. Clearly, if X is nice and $X=E\oplus F$, then either E or F contains a complemented isomorph of X.

The approach in [2] is essentially standard and consists in showing that

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(i) the space $C=C(\omega^*)$ is nice;

(ii) (under the CH) the l_∞ – sum of (infinitely many isometric copies of) C, $l_\infty(C):=(C\oplus C\oplus\ldots)_{l_\infty}$, is isomorphic to C;

and then proving that C is primary by an application of Pelczynski's decomposition method.

In the present paper we first give an alternative proof of (i), and then obtain a vector analogue of (i): *if X is separable and nice, then also $C(\omega^*,X)$ is nice.* We also have a vector analogue of (ii), but with a suitable modification of the l_∞-sums used in (ii). Unfortunately, one of the crucial properties of the l_∞-sums that makes the Pelczynski method work in [2], viz., $l_\infty(E\oplus F)\approx l_\infty(E)\oplus l_\infty(F)$, does not seem to hold for our modification. In consequence, we were unable to show that if X is nice (or primary?), then $C(\omega^*,X)$ is primary, a result which is (more or less) what one tends to expect. Nevertheless, there is something positive we can prove: *If X is a separable nice Banach space which is isomorphic to its c_o-sum, $c_o(X)$, then $C(\omega^*,X)$ is primary* (without assuming the CH!). In particular, for every infinite metrizable compact K, the space $C(\omega^*,C(K))\cong C(\omega^*\times K)$ is primary.

Let us introduce some notation and recall some facts about ω^*. (References can be found in [2].) We denote by $\mathcal{A}$ the algebra of clopen subsets of ω^*; $\mathcal{A}_o=\mathcal{A}-\{\emptyset\}$. If $A\in\mathcal{A}$, then 1_A denotes the characteristic function of A relative to ω^*, $\mathcal{A}(A)=\{B\in\mathcal{A}:B\subset A\}$, and $\mathcal{A}_o(A)=\mathcal{A}(A)-\{\emptyset\}$. We recall that $\mathcal{A}$ is a base for the topology of ω^*, and that if $A\in\mathcal{A}_o$, then A is homeomorphic to ω^*; hence, for every Banach space X, $C(A,X)\cong C(\omega^*,X)$. In what follows we often identify $C(A,X)$ with the subspace $\{f\colon 1_Af=f\}$ of $C(\omega^*,X)$. We also recall that the algebra $\mathcal{A}$ has the following property (sometimes called *Cantor separability):* For every decreasing sequence (A_n) in $\mathcal{A}_o$ there exists $A\in\mathcal{A}_o$ which is contained in all A_n. Finally, there is a result of Negrepontis that, under the CH, if A is an open F_σ-subset of ω^*, then its closure $\bar{A}$ is a retract of ω^*.

1. Lemma ([2]). *Let $\lambda:\mathcal{A}_o\to\mathbb{R}$ be a nondecreasing set function. Then for every $A\in\mathcal{A}_o$ there exist $B\in\mathcal{A}_o(A)$ and $\beta\in\mathbb{R}$ such that*

$$\lambda(E)=\beta \text{ for all } E\in\mathcal{A}_o(B).$$

2. Theorem ([2]). If *$T:C(\omega^*)\to C(\omega^*)$ is an operator, then for every $A\in\mathcal{A}_o$ there exists a $B\in\mathcal{A}_o(A)$ and a scalar γ such that*

$$(Tf)1_B=\gamma f \text{ for all } f\in C(B)$$

As in [2], it will be convenient to prove this theorem in its equivalent form stated below. The proof presented here is somewhat different from that in [2], and we first give some explanations.

We recall that there is a one-to-one correspondence between the operators $T:C(\omega^*)\to C(\omega^*)$ and the bounded finitely additive vector measures $\mu:\mathcal{A}\to C(\omega^*)$; If T is given, then the corresponding (representing) measure μ is defined by $\mu(E)=T(1_E)$; if μ is given, then the corresponding operator T is defined by $Tf=\int f\,d\mu$.

Now suppose that T and μ are related to each other in the above manner, and consider the conjugate operator $T^*:M(\omega^*)\to M(\omega^*)$, where $M(\omega^*)$ is the space of regular Borel measures on ω^* (identified with the dual of $C(\omega^*)$). For each $p\in\omega^*$ let $\mu_p=T^*\delta_p$, where δ_p is the Dirac measure at p. Then it is readily seen that

$$\mu(E)(p)=\mu_p(E) \text{ for all } E\in\mathcal{A} \text{ and } p\in w^* .$$

Let a measure $\nu\in M(\omega^*)$ be real-valued, and let ν^+ be its positive part. Then ν^+ is given for every Borel set $E\subset\omega^*$ by

$$\nu^+(E)=\sup_B \nu(B),$$

where the supremum is taken over all Borel subsets B of E. Now, using regularity, it is easy to verify that

$$\nu^+(E)=\sup_{F\in\mathcal{A}(E)} \nu(F) \text{ for all } E\in\mathcal{A}$$

In particular, for the *real* space $C(\omega^*)$, if $\mu:\mathcal{A}\to C(\omega^*)$ is a bounded measure, then

$$\mu_p^+(E)=\sup_{F\in\mathcal{A}(E)} \mu_p(F) \text{ for all } E\in\mathcal{A} \text{ and } p\in\omega^*$$

Hence, for every $E\in A$, the function $p\to\mu_p^+(E)$ is lower semi-continuous on ω^*, and the same is of course true of the negative-part function $p\to\mu_p^-(E)=(-\mu)_p^+(E)$. (The lower semicontinuity of the function $p\to(T^*\delta_p)^+(E)$ holds in fact for every operator $T:C(K)\to C(K)$ and every open set $E\subset K$)

Now we restate the above theorem in an equivalent form.

3. Theorem ([2]). *Let $\mu:\mathcal{A}\to C(\omega^*)$ be a bounded finitely additive vector measure. Then for every $A\in\mathcal{A}_o$ there exist a B in $\mathcal{A}_o(A)$ and a scalar γ such that*

$$\mu(E)1_B=\gamma 1_E \textit{ for all } E\in\mathcal{A}(B).$$

Proof. We may (and will) assume that $C(\omega^*)$ is real. We start by defining two nondecreasing set functions $\lambda_\mu, \lambda_{-\mu}:\mathcal{A}\to\mathbb{R}_+$ by

$$\lambda_\mu(E)=\sup_{p\in E} \mu_p^+(E) \text{ and } \lambda_{-\mu}(E)=\sup_{p\in E} \mu_p^-(E)$$

(It is easy to verify, using the formula for $\nu^+(E)$, $E\in\mathcal{A}$, given above, that these two functions coincide with those used in [2].) Let $A\in\mathcal{A}_o$. Applying Lemma 1 twice, we find $B\in\mathcal{A}_o(A)$ and $\alpha,\beta\in\mathbb{R}$ such that

$$\lambda_\mu(E)=\beta \text{ and } \lambda_{-\mu}(E)=\alpha \text{ for all } E\in\mathcal{A}_o(B).$$

Let $E\in\mathcal{A}_o(B)$. If $F\in\mathcal{A}_o(E)$ and $p\in\omega^*$, then $\mu^+_p(F)\leq\mu^+_p(E)$. But $\sup_{p\in F}\mu^+_p(F)=\lambda_\mu(F)=\beta$; so

$$\sup_{p\in F}\mu^+_p(E)=\beta \text{ for all } F\in\mathcal{A}_o(E)$$

From this and the lower semicontinuity of the function $p\to\mu^+_p(E)$ it follows that for every $\beta'<\beta$ the set $\{p\in E:\mu^+_p(E)>\beta'\}$ is open and dense in E. Hence the set

$$E^\beta:=\{p\in E:\ \mu^+_p(E)=\beta\}$$

is a dense G_δ-subset of E.

Next, if $E=B$, then

$$\beta=\lambda_\mu(B)\geq\ \mu^+_p(E)+\mu^+_p(B-E)=\mu^+_p(E)+\beta \text{ for all } p\in(B-E)^\beta$$

so that

$$\mu^+_p(E)=\ 0 \text{ for all } p\in(B-E)^\beta$$

But, by the lower semicontinuity again, the set $\{p\in B-E:\mu^+_p(E)=0\}$ is closed in $B-E$, and it contains the set $(B-E)^\beta$ which is dense in $B-E$; therefore, $\mu^+_p(E)=0$ for all $p\in B-E$. Thus

$$\mu^+_p(E)=\begin{cases}\beta & \text{for } p\in E^\beta,\\ 0 & \text{for } p\in B-E.\end{cases}$$

By a similar argument, the set

$$E_\alpha:=\{p\in E:\ \mu^-_p(E)=\alpha\}$$

is a dense G_δ-subset of E, and

$$\mu^-_p(E)=\begin{cases}\alpha & \text{for } p\in E_\alpha,\\ 0 & \text{for } p\in B-E.\end{cases}$$

Hence

$$\mu_p(E)=\mu_p^+(E)-\mu_p^-(E)=\begin{cases}\beta-\alpha=:\gamma \text{ for } p\in E^\beta\cap E_\alpha,\\ 0 \quad \text{for } p\in B-E.\end{cases}$$

But the function $\mu(E):p\to\mu_p(E)$ is continuous, and the set $E^\beta\cap E_\alpha$ is dense in E, hence $\mu_p(E)=\gamma$ for all $p\in E$.

We have thus shown that for every $E\in\mathcal{A}_o(B)$,

$$\mu(E)(p)=\mu_p(E)=\gamma 1_E(p) \text{ for all } p\in B,$$

which is precisely what was to be proved.□

4. Corollary. *$C(\omega^*)$ is a nice Banach space.*

Proof. See [2], Proof of Corollary 2.4; see also Proof of Corollary 6 below.□

Now we give an extension of Theorem 3 to the case of vector valued functions.

5. Theorem. *Let X be a separable Banach space, Y a Banach space whose dual Y^* is weak* separable, and let*

$$T:C(\omega^*,X)\to C(\omega^*, Y)$$

be an operator. Then for every $A\in\mathcal{A}_o$ there exist $B\in\mathcal{A}_o(A)$ and $u\in L(X,Y)$ such that

$$(Tf)1_B=u\circ f \text{ for all } f\in C(B,X).$$

Proof. Let (x_m) be a sequence dense in X, and (y_n^*) a sequence in Y^* separating the points of Y.

Given $x\in X$ and $y^*\in Y^*$, consider the bounded finitely additive measure

$$\mu_{x,y^*}:\mathcal{A}\to C(\omega^*);\ A\to y^*.T(1_Ax)$$

Then, by Theorem 3, for every $A\in\mathcal{A}_o$ there exists a $B\in\mathcal{A}_o(A)$ and a scalar γ such that

$$\mu_{x,y^*}(E)1_B=\gamma 1_E \text{ for all } E\in\mathcal{A}(B).$$

Applying this inductively when $y^*=y_n^*(n=1,2...)$ and x is held fixed, and then making use of the Cantor separability of $\mathcal{A}$, we see that for every $x\in X$ and $A\in\mathcal{A}_o$, there exists a $B\in\mathcal{A}_0(A)$ and a sequence of scalars (γ_n) such that

$$\mu_{x,y_n^*}(E)\ 1_B=\gamma_n 1_E \text{ for all } E\in\mathcal{A}(B) \text{ and } n\in\mathbb{N}.$$

Since the sequence (y_n^*) is total on Y, it follows that there exists a (unique) $y\in Y$ such that

$$T(1_E x)1_B=\ 1_E y \text{ for all } E\in\mathcal{A}(B).$$

Now, applying this inductively when $x=x_m(m=1,2,...)$ and then using the Cantor separability of $\mathcal{A}$ again, we find that for every $A\in\mathcal{A}_o$ there exists a $B\in\mathcal{A}_0(A)$ and a sequence (y_m) in Y such that

$$T(1_E x_m)1_B=1_E y_m \text{ for all } E\in\mathcal{A}(B) \text{ and } m\in\mathbb{N}.$$

If $x\in X$ and (x_{k_m}) is a subsequence of (x_m) converging to x, then by the continuity of T there is a $y=u(x)\in Y$ such that the sequence (y_{k_m}) converges to y (and this y does not depend on a particular choice of (x_{k_m})). Thus

$$T(1_E x)1_B=1_E u(x) \text{ for all } E\in\mathcal{A}(B) \text{ and } x\in X.$$

Clearly, the mapping $u{:}X\to Y$ is linear, and

$$\|u(x)\|=\|1_B u\ (x)\|_\infty\leq\|T(1_B x)\ \|_\infty\leq\|T\|.\|x\| \text{ for all } x\in X$$

so that $u\in L(X,Y)$ (and $\|u\|\leq\|\ T\|$.)

It follows that

$$(Tf)1_B=u\circ f$$

for every $\mathcal{A}$-simple function f in $C(B,X)$; since such functions are dense in $C(B,X)$, the last equality holds for all f in $C(B,X)$.□

6. Corollary. *If is a separable nice Banach space, then also the space $C(\omega^*, X)$ is nice.*

Proof. Let I denote the identity operator in $C(\omega^*,X)$ and i the identity operator in X. Let $T\in L(C(\omega^*,X))$. By Theorem 5, we can find $B\in\mathcal{A}_o$ and $u\in L(X)$ such that.

$$(Tf)1_B=u\circ f \text{ for all } f\in C(B,X).$$

It is then easily checked that

$$(I-T)f\, 1_B=(i\ -u)\circ f \text{ for all } f\in C(B,X).$$

Since X is nice, there exists a subspace Y of X which is isomorphic to X and which is mapped isomorphically by u or $i-u$ onto a complemented subspace of X. Let's assume this holds for u so that $\upsilon=u|Y$ is an isomorphic embedding and $\upsilon(Y)\ =u(Y)=:\ Z$ is complemented in X. Let p be a projection from X onto Z.

If $f\in C(B,Y)$, then $(Tf)1_B=\upsilon\circ f$ and so

$$\|\upsilon^{-1}\|^{-1}\ \|f\|_\infty\leq\|\upsilon\circ f\|_\infty\leq\|Tf\|_{\infty}\leq\|T\|.\|f\|_\infty$$

which shows that $T|C(B,Y)$ is an isomorphic embedding of $C(B,Y)$ into $C(\omega^*,X)$. Define an operator $P{:}C(\omega^*,X)\to C(\omega\ ^*,X)$ by

$$Pg=T(\upsilon^{-1}\circ p\ \circ g1_B)$$

Clearly, the range of P is contained in $T[C(B,Y)]$. If $g\in T[C(B,Y)]$, i.e., $g=Tf$ for some $f\in C(B,Y)$, then $g1_B=(Tf)1_B=\upsilon\circ f$ and hence $Pg=T(\upsilon^{-1}\circ p\circ\upsilon\circ f)=T(\upsilon^{-1}\ \circ\upsilon\circ f)=Tf=g$. Thus P is a projection from $C(\omega^*,X)$ onto its subspace $T[C(B,Y)]\approx\ C(B,Y)\approx C(B,X)\approx\ C(\omega^*,X)$.□

As easily seen, for every compact space K and every Banach space X, there is a natural isometric isomorphism between the spaces $C(K,c_o(X))$ and $c_o(C(K,X))$ so that

$$C(K,c_o(X))\cong c_o(C(K,X))$$

We use this fact in our next result.

7. Corollary. *If X is a separable nice Banach space which is isomorphic to its c_o-sum $c_o(X)$, then the space $C(\omega^*,X)$ is primary.*

Proof. We first observe that

$$C(\omega^*,X)\approx C\ (\omega^*,c_o(X))\approx c_o\ (C(\omega^*,X));$$

thus, denoting shortly $C(\omega^*,X)=:C$, we have $C\approx c_o(C)$.

Now let $C=E\oplus F$. By Corollary 6, one of the summands, E say, contains a complemented subspace V which is somorphic to C. Thus there is a subspace U in E such that

$$E = U \oplus V, \text{ where } V \approx C \approx c_o(C).$$

Applying Pelczynski's decomposition method, we now get

$E \approx U \oplus c_o(C) \approx U \oplus C \oplus c_o(C) \approx E \oplus c_o(E \oplus F) \approx E \oplus c_o(E) \oplus c_o(F) \approx c_o(E) \oplus c_o(F) \approx c_o(E \oplus F) \approx c_o(C) \approx C.$□

In particular, we have the following

8. Corollary. *For every infinite metrizable compact space* K, *the space* $C(\omega^*, C(K)) \cong C(\omega^* \times K)$ *is primary.*

Proof. This follows directly from the preceding corollary because such spaces $C(K)$ are known to be nice ([3], [1]) and isomorphic with their c_o-sums [4].□

9. Remark. Let X be an arbitrary Banach space. Define $\kappa(X)$ to be the Banach space of all relatively norm compact sequences (x_n) in X, endowed with the supremum norm. Then

$$\kappa(X)/c_0(X) \cong C(\omega^*, X).$$

This can be verified precisely as in the scalar case, using the Stone-Čech isometric isomorphism between $\kappa(X)$ and $C(\beta\omega, X)$, and the fact (surely well known) that Tietze's type extensions from ω^* to $\beta\omega$ exist for continuous X-valued functions. For the sake of completeness, we give a sketch of that fact:

Let $g \in C(\omega^*, X)$. Then there exists a sequence (g_n) of $\mathcal{A}_o$-simple functions in $C(\omega^*, X)$ converging uniformly to g. For each n choose a finite $\mathcal{A}_o$-partition $\mathcal{A}_n = \{A_1^n, \ldots, A_{k_n}^n\}$ so that g_n assumes constant (not necessarily distinct) values on each of the sets A_i^n; this can be done so that $\mathcal{A}_{n+1}$ is a refirement of $\mathcal{A}_n$. Then it is easily seen that we can define a sequence of partitions of ω, $\mathcal{M}_n = \{M_1^n, \ldots, M_{k_n}^n\}$ consisting of infinite sets and such that $\mathcal{M}_{n+1}$ is a refinement of $\mathcal{M}_n$ and that $A_i^n =$ (the closure of M_i^n in $\beta\omega) - M_i^n$. Let $x^n \in \kappa(X)$ be the sequence which takes the constant value x_i^n on the set M_i^n, where $\{x_i^n\} = g_n(A_i^n)$, $i = 1, \ldots, k_n$. Finally, let f_n be the continuous extension of x^n to $\beta\omega$. Then $f_n|\omega^* = g_n$ and $\|f_n - f_m\|_\infty = \|g_n - g_m\|_\infty$ for all m and n so that the sequence (f_n) converges uniformly to a function $f \in C(\beta\omega, X)$, $f| \omega^* = g$, and $\|f\|_\infty = \|g\|_\infty$.

10. Remark. For a compact space K and a Banach space X, let $l_\kappa(C(K,X))$ denote the Banach space consisting of all sequences (f_n) such that $f_n \in C(K,X)$ for each n and the joint range of the functions f_n, that is, $\cup_{n=1}^\infty f_n(K)$, is a relatively norm compact subset of X, with the norm defined by $\|(f_n)\| = \sup_n \|f_n\|_\infty$. Then the same argument as in the proof of Proposition 3.2 in [2] shows that, under the CH (which enters here via the result of Negrepontis mentioned before Lemma 1), $l_\kappa(C(\omega^*, X))$ is isometric to a complemented subspace of $C(\omega^*,X)$ from which, as a consequence, we have that $l_\kappa(C(\omega^*,X)) \approx C(\omega^*,X)$. Unfortunately, we cannot apply this result to the primariness problem of the spaces $C(\omega^*,X)$ because we do not know if any analog of the fact that $l_\infty(E \oplus F) \approx l_\infty(E) \oplus l_\infty(F)$ holds for our l_κ-sums.

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