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Fréchet spaces of Moscatelli type

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ABSTRACT. A certain class of Fréchet spaces, called of Moscatelli type, is introduced and studied. Using some shifting device these Fréchet spaces are defined as projective limits of Banach spaces $L((X_k)_{k\in\mathbb{N}})$, where L is a normal Banach sequence space and the X_k's are Banach spaces. The duality between Fréchet and (LB)-spaces of Moscatelli type is established and the following properties of Fréchet spaces are characterized in the present context: distinguishedness, quasinormability, Heinrich's density condition, existence of a continuous norm in the space or the bidual, and the properties (DN) and (Ω) of Vogt.

The aim of this article is to study a class of Fréchet spaces which has been used recently quite often to find counterexamples that solved several open questions in the theory of Fréchet and (DF)-spaces.

In 1980, Moscatelli [13] introduced a certain type of Fréchet and (LB)-spaces to find a twisted quojection, i.e., a Fréchet space which is a surjective limit of Banach spaces, without a continuous norm but not isomorphic to a product of Fréchet spaces each having a continuous norm. Such a space cannot have an unconditional basis, according to [7].

The natural extension of the classical idea of Moscatelli yields the following construction. We start with a normal Banach sequence space $(L,\|\ \|)$, two sequences of Banach spaces $(X_k)_{k\in\mathbb{N}}$, $(Y_k)_{k\in\mathbb{N}}$ and linear continuous maps $f_k: Y_k \to X_k$ $(k\in\mathbb{N})$. Then the Banach spaces $F_n := L((Y_k)_{k<n},(X_k)_{k\geq n})$, $n\in\mathbb{N}$ constitute a projective sequence. The Fréchet space of Moscatelli type is defined by $F:=\text{proj } F_n$, *i.e.*, the projective limit of the sequence $(F_n)_{n\in\mathbb{N}}$.

This type of construction was used by the second author in [5] to find reflexive Fréchet spaces F such that $l_1\hat{\otimes}_\pi F = l_1(F)$ is not distinguished, hence $L_b(l_1, F'_b)$ is not quasibarrelled. Bierstedt and the first author [2] characterized the class of Fréchet spaces F such that $l_1(F)$ is distinguished in terms of the density condition introduced by Heinrich [9]. Our proposition 2.10 shows that

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a Fréchet space of Moscatelli type has the density condition if and only if it is a quojection.

In 1985 Moscatelli and the second author [6] used spaces of the type described above to provide examples of Fréchet spaces F with a continuous norm, whose strong bidual F_b'' does not have a continuous norm. Recently, Terzioglu and Vogt [17] have characterized the Köthe echelon spaces of order 1, $\lambda_1(A)$, such that $\lambda_1(A)_b''$ have a continuous norm. In doing this they introduce the class of locally normable locally convex spaces.

We characterize the Fréchet spaces of Moscatelli type F such that (a) F has a continous norm, (b) F_b'' has a continuous norm and (c) F is locally normable (see 2.16 and 2.17).

Taskinen's first counterexample to the problem of topologies of Grothendieck [15] is also a very elaborate Fréchet space of Moscatelli type. Our characterization of distinguished Fréchet spaces of this type (see 2.4 and 2.5) yields that Taskinen's Fréchet space is not distinguished. On the other hand Taskinen proves in [16] that the space $C(R) \cap L^1(R)$ is not distinguished. In fact he constructs a complemented subspace of $C(R) \cap L^1(R)$ which is a Fréchet space of Moscatelli type to which our result 2.5 can be directly applied.

The duality between Fréchet and (LB)-spaces of Moscatelli type is presented, as well as the caracterization of the properties (DN) and (Ω) of Vogt [20] for this class (they only occur in the trivial cases).

Finally we mention that in [3] the authors used a variant of the Moscatelli device to construct strict (LF)-spaces whose biduals are not (LF)-spaces, answering in the negative a problem of Grothendieck [8, Question non-resolues no. 8].

Our notations are standard and we refer the reader to [11], [14] and [18]. The present article is closely related to our previous one [4].

1. DEFINITIONS AND PRELIMINARIES

In what follows $(L, \| \ \|)$ denotes a normal Banach sequence space, i.e., a Banach space that satisfies
(α) $\varphi \subset L \subset \omega$ algebraically and the inclusion $(L, \| \ \|) \to \omega$ is continuous.
(β) $\forall a=(a_k)_{k\in\mathbb{N}} \in L \ \forall b=(b_k)_{k\in\mathbb{N}} \in \omega$ such that $|b_k| \leqslant |a_k| \ \forall k \in \mathbb{N}$, we have $b \in L$ and $\|b\| \leqslant \|a\|$.

Clearly every projection $P_n: \omega \to \omega$, $(a_k)_{k\in\mathbb{N}} \to ((a_k)_{k\leqslant n}, (0))$ onto the first n coordinates induces a $\| \ \|$-decreasing endomorphism of L. We will consider the following properties on $(L, \| \ \|)$.

(γ) $\|a\| = \lim_{n\to\infty} \|P_n(a)\| \ \forall a \in L$.

(δ) if $a \in \omega$, $\sup_{n\in\mathbb{N}} \|P_n(a)\| < \infty$, *then* $a \in L$ *and* $\|a\| = \lim_{n\to\infty} \|P_n(a)\|$.

(ε) $\lim_{n\to\infty} \|a - P_n(a)\| = 0 \ \forall a \in L$.

Typical examples of $(L, \| \|)$ are the spaces $(l_p, \| \|_p)$, $1 \leqslant p \leqslant \infty$, $(c_o, \| \|_\infty)$ and their diagonal transforms. The space $(c_o, \| \|_\infty)$ has (ε) but not (δ), whereas $(l_\infty, \| \|_\infty)$ has (δ) but not (ε).

1.1. Definition. *Let* $(X_k, r_k)_{k\in\mathbb{N}}$ *be a sequence of Banach spaces and* $(L, \| \|)$ *a normal Banach sequence space. We put*
$L((X_k, r_k)_{k\in\mathbb{N}}) := \{x = (x_k)_{k\in\mathbb{N}} \in \prod_{k\in\mathbb{N}} X_k | (r_k(x_k))_{k\in\mathbb{N}} \in L\}$
endowed with the norm $x \to \|(r_k(x_k))_{k\in\mathbb{N}}\|$.

The amalgams of [10] are particular cases of this type of spaces.

1.2. Proposition. *Let* $(L, \| \|)$ *be a normal Banach sequence space and let* $(X_k, r_k)_{k\in\mathbb{N}}$ *be a sequence of Banach spaces. Then* $L((X_k, r_k)_{k\in\mathbb{N}})$ *is a Banach space.*

Proof. The completeness of $(L, \| \|)$ and (α) yield the following property (*) if $(a^n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $(L, \| \|)$ and $\lim_{n\to\infty} a_k^n = 0$ for all $k \in \mathbb{N}$, then $\lim_{n\to\infty} a^n = 0$ in $(L, \| \|)$.

Let $(x^n)_{n\in\mathbb{N}}$ be a Cauchy sequence in $L((X_k, r_k)_{k\in\mathbb{N}})$. From (α) we obtain $\forall\ k \in \mathbb{N}\ \exists x_k \in X_k: x^n_k \to x_k (n \to \infty)$ in (X_k, r_k). For $u_k^n := r_k(x_k^n)$, $k, n \in \mathbb{N}$, it follows from (β) that $\forall m, n \in \mathbb{N}$: $\|u^m - u^n\| \leqslant \|(r_k(x_k^m - x_k^n))_{k\in\mathbb{N}}\|$, hence $(u^n)_{n\in\mathbb{N}}$ is a Cauchy sequence in L. Consequently there is $u \in L$ with $u^n \to u (n \to \infty)$ in $(L, \| \|)$. We can apply (α) to obtain $u_k = r_k(x_k)$ for all $k \in \mathbb{N}$ and therefore $x \in L((X_k, r_k)_{k\in\mathbb{N}})$.

Now set $v_k^n := r_k(x_k^n - x_k)$, $k, n \in \mathbb{N}$. Clearly $v_k^n \to 0 (n \to \infty)$ for all $k \in \mathbb{N}$. On the other hand, for $n, m \in \mathbb{N}$ we can apply (β) to obtain $\|v^m - v^n\| \leqslant \|(r_k(x_k^m - x_k^n))_{k\in\mathbb{N}}\|$, therefore $(v^n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $(L, \| \|)$. According to (*) we have $v^n \to 0$ $(n \to \infty)$ in $(L, \| \|)$. Thus $\|v^n\| = \|(r_k(x_k^n - x_k))_{k\in\mathbb{N}}\| \to 0 (n \to \infty)$ and the proof is complete. ■

The proposition above was proved in [4] under the additional assumption that $(L, \| \|)$ has (γ). The proof presented above was provided by P. Dierolf.

1.3. Definition. (*Fréchet spaces of Moscatelli type*)

Let $(L, \| \|)$ *be a normal Banach sequence space and let* $(X_k, r_k)_{k\in\mathbb{N}}$ $(Y_k, s_k)_{k\in\mathbb{N}}$ *be two sequences of Banach spaces with unit balls* A_k *and* B_k, *respectively, and* *for every* $k\in\mathbb{N}$ *let* $f_k:Y_k\to X_k$ *be a linear map such that* $f_k(B_k)\subset A_k$. *For every* $n\in\mathbb{N}$, *the space* $F_n:=L((Y_k,s_k)_{k<n},(X_k,r_k)_{k\geq n})$ *is a Banach space according to 1.2, and the linear map* $g_n:F_{n+1}\to F_n$, $(z_k)_{k\in\mathbb{N}}\to((z_k)_{k<n},f_n(z_n),(z_k)_{k>n})$ *is norm decreasing.*

The Fréchet space of Moscatelli type associated to (or with respect to -- w.r.t.) $(L,\|\ \|)$, $(X_k,r_k)_{k\in\mathbb{N}}$, $(Y_k,s_k)_{k\in\mathbb{N}}$ *and* $f_k:Y_k\to X_k (k\in\mathbb{N})$ *is the Fréchet space defined by* $F:=\operatorname{proj}_{\leftarrow}((F_n)_{n\in\mathbb{N}},(g_n)_{n\in\mathbb{N}})$ *i.e., the projective limit of the projective sequence of Banach spaces* $(F_n)_{n\in\mathbb{N}}$ *with linking maps* $(g_n)_{n\in\mathbb{N}}$.

1.4. Proposition. *The Fréchet space of Moscatelli type defined above coincides algebraically with* $\{y=(y_k)_{k\in\mathbb{N}}\in\prod_{k\in\mathbb{N}}Y_k:(f_k(y_k))_{k\in\mathbb{N}}\in L((X_k,r_k)_{k\in\mathbb{N}})\}$. *Moreover F has the initial topology w.r.t. the inclusion* $j:F\to\prod_{k\in\mathbb{N}}(Y_k,s_k)$ *and the linear map* $f:F\to L((X_k,r_k)_{k\in\mathbb{N}})$, $f(y):=(f_k(y_k))_{k\in\mathbb{N}}$.

Proof. F coincides with the space $\{(z^j)_{k\in\mathbb{N}}\in\prod_{j\in\mathbb{N}}F_j: g_j(z^{j+1})=z^j \text{ for all } j\in\mathbb{N}\}$, endowed with the topology induced by $\prod_{j\in\mathbb{N}}F_j$. We denote by H the space $\{y\in\prod_{k\in\mathbb{N}}Y_k:(f_k(y_k))_{k\in\mathbb{N}}\in L((X_k,r_k)_{k\in\mathbb{N}})\}$ endowed with the initial topology w.r.t. the linear maps j and f. Define $\psi:F\to H$ by $\psi((z^j)_{j\in\mathbb{N}}):=(z_k^{k+1})_{k\in\mathbb{N}}$. It is a direct matter to check that ψ is linear, bijective and continuous (see [4,(1.3)(3)]). Since F and H are both Fréchet spaces, ψ is also open the proof is complete. ■

From now on we shall make the identification indicated in Proposition 1.4.

We close this section recalling the definition of (LB)-spaces of Moscatelli type and some of their properties from [4].

Let $(L,\|\ \|)$ be a normal Banach sequence space. Let $(X_k,r_k)_{k\in\mathbb{N}}$, $(Y_k,s_k)_{k\in\mathbb{N}}$ be two sequences of Banach spaces such that, for every $k\in\mathbb{N}$, Y_k is a subspace of X_k and $s_k\geq r_k|Y_k$. Then the unit ball B_k of Y_k is contained in the unit ball A_k of X_k. According to 1.2, $E_n:=L((X_k,r_k)_{k<n},(Y_k,s_k)_{k\geq n})$ is a Banach space, $n\in\mathbb{N}$. The corresponding (LB)-space of Moscatelli type is defined by $E:=\operatorname{ind}_{\rightarrow}E_n$. The closed unit ball of E_n will be denoted by $\mathfrak{B}_n$. A basis of 0-neighbourhoods (0-nghbs) in E is given by the sets of the form $\bigoplus_{k\in\mathbb{N}}\varepsilon_kA_k+\delta\mathfrak{B}_1$, $\delta>0$, $\varepsilon_k>0 (k\in\mathbb{N})$.

If $(L,\|\ \|)$ satisfies property (γ), then E is regular if and only if $\exists n\in\mathbb{N}\ \exists\rho\geq 1\ \forall k\geq n\ \bar{B}_k\subset\rho B_k$, where $\bar{B}_k$ denotes the closure of B_k in (X_k,r_k).

In [4,Section 3] we associated to E a projective limit in the following way. Given $\delta>0$ and $\varepsilon_k>0 (k\in\mathbb{N})$, the Minkowski functional of $\varepsilon_kA_k+\delta B_k$ is denoted by $p_{\varepsilon_k\delta}$, and it is a norm on X_k equivalent to r_k. Then $\check{E}$ is the projective limit

$$\check{E}:=\bigcap_{\delta,(\varepsilon_k)} L((X_k,p_{\varepsilon_k\delta})_{k\in\mathbb{N}}).$$

The (LB)-space E is continuously injected in Ě and Ě is a complete (DF)-space. A basis of 0-nghbs in Ě is given by the sets

$$(\prod_{k\in\mathbb{N}}\varepsilon_k A_k+\delta\mathfrak{B}_1)\cap\check{E},\ \varepsilon_k>0\ (k\in\mathbb{N}),\ \delta>0.$$

If E is regular, then Ě and E coincide algebraically.

2. STRUCTURE OF FRECHET SPACES OF MOSCATELLI TYPE

We keep the notations of 1.3 and let F denote the Fréchet space of Moscatelli type w.r.t. $(L,\|\ \|)$, $(X_k,r_k)_{k\in\mathbb{N}}$, $(Y_k,s_k)_{k\in\mathbb{N}}$ and $f_k:Y_k\to X_k(k\in\mathbb{N})$. We will assume in this section that $(L,\|\ \|)$ satisfies the property (γ).

Let $\lambda>0,\mu_k>0(k\in\mathbb{N})$ be given. The Minkowski functional q_{λ,μ_k} of the subset $\lambda f_k^{-1}(A_k)\cap\mu_k B_k$ of Y_k is a norm on Y_k equivalent to s_k. Therefore the space $F_{\lambda,(\mu_k)}:=L((Y_k,q_{\lambda,\mu_k})_{k\in\mathbb{N}})$ is a Banach space. According to [4,(1.3)(3)],$F_{\lambda,(\mu_k)}$ is continuously injected in F (see Proposition 1.4).

2.1. Proposition. *For every bounded subset* $\mathfrak{B}$ *of* F *there are* $\lambda>0,\mu_k>0(k\in\mathbb{N})$ *such that* $\mathfrak{B}$ *is a bounded subset of* $F_{\lambda,(\mu_k)}$. *In particular F can be represented as an (uncountable) inductive limit* $F=\operatorname{ind}_{\lambda,(\mu_k)} F_{\lambda,(\mu_k)}$.

Proof. Let $\mathfrak{B}\subset F$ be bounded. According to 1.4, there are $\lambda>0,\lambda_k>0(k\in\mathbb{N})$ such that $\sup\{\|(r_k(f_k(y_k)))_{k\in\mathbb{N}}\|:y\in\mathfrak{B}\}\leqslant\lambda$, $\sup\{s_m(y_m):y\in\mathfrak{B}\}\leqslant\lambda_m(m\in\mathbb{N})$. We choose $(\eta_k)\in L$ such that $\eta_k>0(k\in\mathbb{N})$ and $\|(\eta_k)\|=1$. We put $\mu_k:=\lambda_k\eta_k^{-1}$ and we show that $\mathfrak{B}$ is bounded in $F_{\lambda,(\mu_k)}$.

Take $y\in\mathfrak{B}$. For every $k\in\mathbb{N}$ one has $y_k\in r_k(f_k(y_k))f_k^{-1}(A_k)\cap s_k(y_k)B_k$. We put

$$I_1:=\{k\in\mathbb{N}:\mu_k^{-1}s_k(y_k)\leqslant\lambda^{-1}r_k(f_k(y_k))\},$$

$$I_2:=\{k\in\mathbb{N}:\lambda^{-1}r_k(f_k(y_k))<\mu_k^{-1}s_k(y_k)\}.$$

If $k\in I_1$, $y_k\in\lambda^{-1}r_k(f_k(y_k))(\lambda f_k^{-1}(A_k)\cap\mu_k B_k)$, whence

$$\|((q_{\lambda,\mu_k}(y_k))_{k\in I_1},(0)_{k\in I_2}\|\leqslant\lambda^{-1}\|((r_k(f_k(y_k)))_{k\in I_1},(0)_{k\in I_2}\|\leqslant 1$$

For every $k\in I_2$, $y_k\in\mu_k^{-1}s_k(y_k)(\lambda f_k^{-1}(A_k)\cap\mu_k B_k)$, whence

$$\|((0)_{k\in I_1},(q_{\lambda,\mu_k}(y_k))_{k\in I_2}\|\leqslant\|((0)_{k\in I_1},(\mu_k^{-1}s_k(y_k))_{k\in I_2})\|\leqslant\|((0)_{k\in I_1},(\eta_k)_{k\in I_2})\|\leqslant 1.$$

Thus $\|(q_{\lambda,\mu_k}(y_k))_{k\in\mathbb{N}}\|\leqslant 2$, which proves the assertion. ■

To establish the duality between Fréchet and (LB)-spaces of Moscatelli type we consider a normal Banach sequence space $(L,\|\ \|)$ satisfying property (ε). Then the map $A:L'\to\omega, u\to u|\varphi\in\omega$ maps L' onto the α-dual $L^{\times}$ of L (cf. [11,§30,1]), hence we may identify L' with $L^{\times}$. Moreover the dual norm $\|\ \|'$ of L' satisfies (β). Therefore $(L',\|\ \|')$ is again a normal Banach sequence space which even has property (δ), as can easily be seen.

2.2. Lemma. *Let $(X_k,r_k)_{k\in\mathbb{N}}$ be a sequence of Banach spaces. Then we may naturally identify (algebraically and topologically) the dual of $L((X_k,r_k)_{k\in\mathbb{N}})$ with $L'((X_k',r_k')_{k\in\mathbb{N}})$, where r_k' denotes the dual norm of X_k'.*

Proof. For $f=(f_k)_{k\in\mathbb{N}}\in L'((X_k',r_k')_{k\in\mathbb{N}})$ and $x=(x_k)_{k\in\mathbb{N}}\in L((X_k,r_k)_{k\in\mathbb{N}})$ we define $<\varnothing(f),x>:=\sum_{k\in\mathbb{N}}f_k(x_k)$. Since $\sum_{k\in\mathbb{N}}|f_k(x_k)|\leqslant\|(r_k'(f_k))_{k\in\mathbb{N}}\|'\|(r_k(x_k))_{k\in\mathbb{N}}\|$, we have that $\varnothing$ is a well-defined, linear, injective and continuous map from $L'((X_k',r_k')_{k\in\mathbb{N}})$ into $L((X_k,r_k)_{k\in\mathbb{N}})'$. We show that $\varnothing$ is also surjective, and hence open. Let $f\in L((X_k,r_k)_{k\in\mathbb{N}})'$ of norm 1 be given. For each $n\in\mathbb{N}$ the map $j_n:X_n\to L((X_k,r_k)_{k\in\mathbb{N}})$, $x\to(\delta_{kn}x)_{k\in\mathbb{N}}$ is linear and continuous. We put $f_k:=f\circ j_k\in X_k'$ for all $k\in\mathbb{N}$. We must show that $(r_k'(f_k))_{k\in\mathbb{N}}\in L'$ or equivalently that $\sum_{k\in\mathbb{N}}r_k'(f_k)|\alpha_k|<\infty$ for all $\alpha=(\alpha_k)_{k\in\mathbb{N}}\in L$. Let $\alpha\in L$ be given such that $\alpha_k\geqslant 0$ for all $k\in\mathbb{N}$. It is enough to show that for all $x=(x_k)_{k\in\mathbb{N}}\in\prod_{k\in\mathbb{N}}r_k^{-1}([0,1])$ one has $\sum_{k\in\mathbb{N}}|f_k(x_k)|\alpha_k\leqslant\|(\alpha_k)_{k\in\mathbb{N}}\|$. Given x as above, we obtain for arbitrary $n\in\mathbb{N}$,

$$\sum_{k=1}^{n}|f_k(x_k)|\alpha_k=\sum_{k=1}^{n}|f_k(\alpha_k x_k)|=\sum_{k=1}^{n}f_k(\alpha_k\varepsilon_k x_k)$$

(for suitable $\varepsilon_k\in\mathrm{K}$, $|\varepsilon_k|=1\ (k\in\mathbb{N}))=f(\sum_{k=1}^{n}j_k(\alpha_k\varepsilon_k x_k))\leqslant\|((r_k(\alpha_k\varepsilon_k x_k))_{k\leqslant n},(0)_{k>n})\|$ $\leqslant\|(\alpha_k)_{k\in\mathbb{N}}\|$.

Clearly $\varnothing((f_k)_{k\in\mathbb{N}})$ and f coincide on $\bigoplus_{k\in\mathbb{N}}X_k$, hence $\varnothing(f_k)_{k\in\mathbb{N}}=f$, because $(L,\|\ \|)$ satisfies (ε). ■

Let $(L,\|\ \|)$ satisfy (ε) and let F be the Fréchet space of Moscatelli type associated to $(L,\|\ \|)$, $(X_k,r_k)_{k\in\mathbb{N}}$, $(Y_k,s_k)_{k\in\mathbb{N}}$ and $f_k:Y_k\to X_k$ $(k\in\mathbb{N})$. We assume that $f_k(Y_k)$ is dense in X_k for all $k\in\mathbb{N}$. This is no loss of generality (indeed, take the closure of $f_k(Y_k)$ in (X_k,r_k) instead of $X_k (k\in\mathbb{N})$ to obtain the same Fréchet space F). Moreover in this case the projective limit defining F is reduced. Then for all $k\in\mathbb{N}$, the transpose $f_k^t:(X'_k,r'_k)\to(Y'_k,s'_k)$ is injective and maps the unit ball A_k' of (X_k',r_k') into the unit ball B'_k of (Y'_k,s'_k) (recall that $f_k(B_k)\subset A_k$). Thus we may form the (LB)-space $E=ind\ E_n$ of Moscatelli type w.r.t. $(L',\ \|\ \|')$,

$(Y_k', s_k')_{k\in\mathbb{N}}$, $(X_k', r_k')_{k\in\mathbb{N}}$. According to 2.2, the strong dual $F_{n,b}'$ of F_n coincides algebraically and topologically with the space E_n. Observe that E is regular. Keeping the notations just stablished, we have proved the first part of the following result.

2.3. Proposition. *Let F be a Fréchet space of Moscatelli type with $f_k(Y_k)$ dense in X_k for all $k \in \mathbb{N}$. Let E be the corresponding (LB)-space of Moscatelli type w.r.t. the duals. Then there is a continuous identity map $E \to F_b'$. Moreover E is the bornological space associated to F_b', and F_b' coincides topologically with the projective limit $\check{E}$ associated to E.*

Proof. Only the last assertion needs a proof. Let $\mathfrak{B} \subset F$ be bounded. By 2.1, there are $\lambda > 0, \mu_k > 0$ ($k \in \mathbb{N}$) such that $sup\{\|(q_{\lambda,\mu_k}(y_k))_{k\in\mathbb{N}}\| : y \in \mathfrak{B}\} \leqslant 1$. Put $\delta := \lambda^{-1}$, $\varepsilon_k := \mu_k^{-1}$ ($k \in \mathbb{N}$). We prove that

$$\mathfrak{U} := \{(f_k)_{k\in\mathbb{N}} \in E : \|(p_{\varepsilon_k,\delta}(f_k))_{k\in\mathbb{N}}\|' \leqslant 2^{-1}\} \subset \mathfrak{B}^\circ.$$

Let $f \in \mathfrak{U}$ and $y \in \mathfrak{B}$. Since $p_{\varepsilon_k,\delta}$ and the dual norm on Y_k' of the norm q_{λ,μ_k} on Y_k, q'_{λ,μ_k}, satisfy $2^{-1}q'_{\lambda,\mu_k} \leqslant p_{\varepsilon_k,\delta} \leqslant 2q'_{\lambda,\mu_k}$, we have

$$|\sum_{k\in\mathbb{N}} f_k(y_k)| \leqslant 2\sum_{k\in\mathbb{N}} p_{\varepsilon_k,\delta}(f_k) q_{\lambda,\mu_k}(y_k) \leqslant 2\|(p_{\varepsilon_k,\delta}(f_k))_{k\in\mathbb{N}}\|' \|(q_{\lambda,\mu_k}(y_k))_{k\in\mathbb{N}}\| \leqslant 1.$$

This proves that the injection $\check{E} \to F_b'$ is continuous. Conversely, let $\delta > 0$, $\varepsilon_k > 0$ ($k \in \mathbb{N}$) be given and put $\lambda = \delta^{-1}$, $\mu_k = \varepsilon_k^{-1}$ ($k \in \mathbb{N}$). We prove that the polar of the bounded set $\mathfrak{B} := \{y \in \prod_{k\in\mathbb{N}} Y_k : \|(q_{\lambda,\mu_k}(y_k))_{k\in\mathbb{N}}\| \leqslant 1\}$ in F' is contained in $\mathfrak{U} := \{f \in E : \|(p_{\varepsilon_k,\delta}(f_k))_{k\in\mathbb{N}}\|' \leqslant 2\}$. For $f \in \mathfrak{B}^\circ$ it is enough to show that $\sum_{k\in\mathbb{N}} p_{\varepsilon_k,\delta}(f_k)|\alpha_k| \leqslant 2$ for all $\alpha = (\alpha_k) \in L$ such that $\|\alpha\| \leqslant 1$. Given $\alpha \in L$ with $\|\alpha\| \leqslant 1$ and $\alpha_k \geqslant 0$ for all $k \in \mathbb{N}$ we take $y \in \prod_{k\in\mathbb{N}} Y_k$ with $q_{\lambda,\mu_k}(y_k) \leqslant 1$ ($k \in \mathbb{N}$). Clearly $(\alpha_k y_k)_{k\in\mathbb{N}} \in \mathfrak{B}$, therefore

$$\sum_{k\in\mathbb{N}} \alpha_k |f_k(y_k)| = \sum_{k\in\mathbb{N}} |f_k(\alpha_k y_k)| \leqslant 1.$$

This implies $\sum_{k\in\mathbb{N}} p_{\varepsilon_k,\delta}(\alpha_k f_k) \leqslant 2$. ■

According to the previous Proposition we have that a Fréchet space of Moscatelli type F as in 2.3 is distinguished if and only if the corresponding (LB)-space E formed w.r.t. the duals satisfies $E = \check{E}$ topologically. Now we characterize when E is a topological subspace of $\check{E}$ for arbitrary (LB)-spaces of Moscatelli type which complements [4,(3.3) and (3.4)].

2.4. Lemma. *Let E be an (LB)-space of Moscatelli type w.r.t. $(L, \|\ \|)$, $(X_k, r_k)_{k\in\mathbb{N}}$, $(Y_k, s_k)_{k\in\mathbb{N}}$ and $\check{E}$ the projective limit associated to E. Then E is topological subspace of $\check{E}$ if and only if the following condition is satisfied*

(*) $\exists \varepsilon_k > 0 \ (k \in \mathbb{N}) \ \forall y \in L((Y_k, s_k)_{k \in \mathbb{N}}) \cap \prod_{k \in \mathbb{N}} \varepsilon_k A_k \quad \exists m \in \mathbb{N}: \|((0)_{k<m}, (s_k(y_k))_{k \geq m})\| \leq 1.$

Proof. Assume first that E is a topological subspace of $\check{E}$. Given the 0-nghb $U := \bigoplus_{k \in \mathbb{N}} A_k + \mathfrak{B}_1$ in E there are $\varepsilon_k > 0 (k \in \mathbb{N})$ such that $(\prod_{k \in \mathbb{N}} \varepsilon_k A_k) \cap E \subset U$ (cf.[4,(3.1)].We fix any element $y \in L((Y_k, s_k)_{k \in \mathbb{N}}) \cap \prod_{k \in \mathbb{N}} \varepsilon_k A_k$. Since $y \in E_1$, we have $y \in E$, whence $y = x + z$, with $x \in \bigoplus_{k \in \mathbb{N}} A_k$, $z \in \mathfrak{B}_1$. Then there is $m \in \mathbb{N}$ such that $x_k = 0$ if $k \geq m$. Consequently $y_k = z_k \ (k \geq m)$. This implies $\|((0)_{k<m}, (s_k(y_k))_{k \geq m})\| \leq 1$.

Suppose now that (*) is satisfied and let $V := \bigoplus_{k \in \mathbb{N}} \eta_k A_k + \delta \mathfrak{B}_1$ be a 0-nghb in E. We prove that $(\prod_{k \in \mathbb{N}} \min(\eta_k, 2^{-1}\varepsilon_k \delta) A_k + 2^{-1}\delta \mathfrak{B}_1) \cap E \subset V$.

It is enough to show that $(\prod_{k \in \mathbb{N}} \min(\eta_k, 2^{-1}\varepsilon_k \delta) A_k) \cap E \subset \bigoplus_{k \in \mathbb{N}} \eta_k A_k + 2^{-1}\delta \mathfrak{B}_1$.

Take x in the left hand side. There is $n \in \mathbb{N}$ with $x \in E_n$, hence $((0)_{k<n}, (s_k(x_k))_{k \geq n}) \in L$ and moreover $2\delta^{-1} x_k \in \varepsilon_k A_k$ if $k \geq n$. According to (*) there is $m > n$ such that $\|((0)_{k<m}, (2\delta^{-1} s_k(x_k))_{k \geq m})\| \leq 1$, hence $((0)_{k<m}, (x_k)_{k \geq m}) \in 2^{-1}\delta \mathfrak{B}_1$ and $((x_k)_{k<m}, (0)_{k \geq m}) \in \bigoplus_{k \in \mathbb{N}} \eta_k A_k$.

This completes the proof. ∎

Clearly the condition (*) of 2.4 is satisfied if $(L, \| \ \|)$ has property (ε), taking $\varepsilon_k = 1 (k \in \mathbb{N})$. If there is $m \in \mathbb{N}$ such that Y_k is a topological subspace of X_k for $k \geq m$, then (*) is satisfied. The converse is true if $(L, \| \ \|) = (l_\infty, \| \ \|_\infty)$ by [4,(3.4)].

2.5. Corollary. *Let F be a Fréchet space of Moscatelli type w.r.t. $(L, \| \ \|)$ satisfying (ε), $(X_k, r_k)_{k \in \mathbb{N}}$, $(Y_k, s_k)_{k \in \mathbb{N}}$ and $f_k: Y_k \to X_k$ with dense range $(k \in \mathbb{N})$. Let E be the corresponding (LB)-space w.r.t. the duals. Then*

(a) if $(L', \| \ \|')$ has (ε), then F is distinguished.

(b) if $(L, \| \ \|) = (l_1, \| \ \|_1)$, the following conditons are equivalent (TFAE):

(1) F is distinguished.

(2) $\exists n \in \mathbb{N}: E = \operatorname{ind} E_{n+k}$ is a strict (LB)-space.

(3) $\exists\, n \in N \ \forall k \geq n \ f_k: Y_k \to X_k$ is surjective.

(4) F is a quojection.

Proof. Since (a) follows from our previous remarks, we only prove (b). F is distinguished if and only if $E = \check{E}$ holds topologically. According to the remark above (1) implies (2). (2) implies (3) by the closed range theorem (cf. [11, 33,4,(1)]). (3) implies that F is a strict projective limit of Banach spaces, hence a quojection. Clearly (4) implies (1). ∎

The Fréchet spaces constructed by Taskinen in his thesis [15] to provide counterexamples to the "problème des topologies' of Grothendieck are all non-distinguished. Indeed, Taskinen starts with a Banach space (G,g) and constructs suitable equivalent norms $g_{nk} \geq g$ on G $(n,k \in \mathbb{N})$. He sets $(Y,s):=l_1((G,g_{nk})_{(n,k)\in\mathbb{N}\times\mathbb{N}})$, $(X,r):=l_1((G,g)_{(n,k)\in\mathbb{N}\times\mathbb{N}})$ and then he forms the Fréchet space of Moscatelli type F w.r.t. $(l_1,\|\ \|_1)$, $(X,r)_{k\in\mathbb{N}}$, $(Y,s)_{k\in\mathbb{N}}$ and the non-surjective continuous injections $j:Y\to X$. According to 2.5, F is not distinguished.

In [16] Taskinen proves that $C(\mathbb{R})\cap L^1(\mathbb{R})$ is not distinguished. In fact he shows that this space has a complemented subspace isomorphic to a Moscateli type Fréchet space which is not a quojection, hence we can deduce that it is not distinguished form 2.5.

Lemma 2.2 can be used to completė the duality between (LB) and Fréchet spaces of Moscatelli type, if $(L,\|\ \|)$ has (ε).

2.6. Proposition. *The strong dual of an (LB)-space of Moscatelli type is the corresponding Fréchet space formed w.r.t. the duals.*

As a consequence of 2.3 and 2.6 we observe that there are plenty of reflexive Fréchet spaces of Moscatelli type. Take, for instance, $(L,\|\ \|)=(l_p,\|\ \|_p)$, $1<p<\infty$, and all the Banach spaces (X_k,r_k), (Y_k,s_k) reflexive. The situation is different for the properties of being Montel or quasinormable.

2.7. Proposition. *The Fréchet space of Moscatelli type w.r.t.* $(L,\|\ \|)$, $(X_k,r_k)_{k\in\mathbb{N}}$, $(Y_k,s_k)_{k\in\mathbb{N}}$ *and* $f_k:Y_k\to X_k$ $(k\in\mathbb{N})$ *is Montel if and only if the dimension of every* Y_k *is finite and there is* $k(o)\in\mathbb{N}$ *with* $f_k(Y_k)=\{0\}$ *for* $k\geq k(o)$.

Proof. We assume w.l.o.g. that f_k has dense range for all $k\in\mathbb{N}$. If every Y_k is finite-dimensional, then $\prod_{k\in\mathbb{N}}(Y_k,s_k)$ is isomorphic to ω. If $f_k(Y_k)=\{0\}$ for $k\geq k(o)$, then the map $f:F\to L((X_k,r_k)_{k\in\mathbb{N}})$, $f(y)=(f_k(y_k))_{k\in\mathbb{N}}$ has finite-dimensional range. We apply 1.4 to obtain that F carries a weak topology. Since F is Fréchet, it is Montel. Conversely, the continuity of the inclusions

$$L((Y_k,s_k)_{k\in\mathbb{N}})\to F\to\prod_{k\in\mathbb{N}}(Y_k,s_k)$$

yields that each (Y_k,s_k) is a topological subspace of F, hence finite-dimensional. Consequently $f_k(Y_k)$ is finite-dimensional too, hence it must coincide with X_k, because we assume that f_k has dense range. Therefore F is a quojection. Since F is Montel, it is either finite-dimensional or isomorphic to ω. In both cases the embedding $F\to\prod_{k\in\mathbb{N}}(Y_k,s_k)$ is a topological isomorphism onto its range, which contains $\bigoplus_{n\in\mathbb{N}} Y_k$. Since F is complete, $F=\prod_{k\in\mathbb{N}}(Y_k,s_k)$ holds topologically. Therefore $f:F\to L((X_k,r_k)_{k\in\mathbb{N}})$ is surjective and the Banach space $L((X_k,r_k)_{k\in\mathbb{N}})$ is a quotient of F. Since F carries a weak topology, the Banach space is finite-dimensional. Thus there is $k(0)\in\mathbb{N}$ with $f_k(Y_k)=\{0\}$ for $k\geq k(0)$. ∎

The density condition, (DC), was introduced by Heinrich in [9] in the study of ultraproducts of locally convex spaces. In [2] the density condition was studied for Fréchet spaces. A Fréchet space F with a basis of absolutely convex 0-nghbs $(U_n)_{n\in\mathbb{N}}$ has (DC) if $\forall(\lambda_j)_{j\in\mathbb{N}}, \lambda_j>0, \forall U\in \mathfrak{U}_0(E)\ \exists m\in\mathbb{N}\ \exists B\subset E$ bounded such that $\bigcap_{j=1}^{m}\lambda_j U_j\subset B+U$.

According to Vogt [19] a Fréchet space F with basis of 0-nghbs $(U_n)_{n\in\mathbb{N}}$ satisfies property (Ω_φ) for an increasing continuous function $\varphi:]0,\infty[\rightarrow]0,\infty[$ if $\forall p\ \exists q\ \forall k\ \exists C>0\ \forall r>0: U_q\subset C\varphi(r)U_k+r^{-1}U_p$.

By [12] a Fréchet space is quasinormable if and only if it has (Ω_φ) for some φ. By [21, 0.3] a Fréchet space F has (Ω_1), for $1(r)=1$ for all $r\in]0,\infty[$ if and only if F'' is a quojection, and this is equivalent to the fact that F does not satisfy the condition (*) of Bellenot and Dubinsky (cf. [1]). A Fréchet space F with a basis of 0-nghbs $(U_n)_{n\in\mathbb{N}}$ is said to satisfy property (Ω) (cf. [20]) if $\forall p\ \exists q\ \forall k\ \exists n, C>0: U_q\subset Cr^nU_k+r^{-1}U_p\ \forall r>0$

To study all these properties in the context of Fréchet spaces of Moscatelli type, we need two Lemmata, perhaps well-known.

2.8. Lemma. *Let X,Y be Banach spaces and $f:Y\rightarrow X$ a continuous linear map. Let A and B denote the closed unit balls of X and Y respectively. If the following condition is satisfied $(+)\forall\varepsilon>0\ \exists\mu>0: A\subset\varepsilon A+\mu f(B)$, then f is surjective.*

Proof. Condition $(+)$ readily implies that $f(Y)$ is dense in X. Therefore we have the continuous injection $j:=f^t: X_b'\rightarrow Y_b'$.
Forming polars w.r.t. X' we obtain from $(+)$
$\forall\varepsilon>0\ \exists\mu>0: \varepsilon^{-1}A^\circ\cap\mu^{-1}f(B)^\circ\subset A^\circ$ or
$\forall\varepsilon>0\ \exists\delta>0: \delta j^{-1}(B^{\circ Y})\cap A^\circ\subset\varepsilon A^\circ$.
This implies that the topologies $\beta(X',X)$ and the initial topology w.r.t. $j:X'\rightarrow Y_b'$ coincide on A°. Since A° is a 0-nghb in X_b', we obtain that $j:X_b'\rightarrow Y_b'$ is a topological isomorphism onto its range. By the closed range theorem, f is surjective. ■

2.9. Lemma. *Let X,Y be Banach spaces and $f:Y\rightarrow X$ a continuous linear map with dense range. Let A and B denote the closed unit balls of X and Y, respectively, and let $C:=A\cap f(Y)$. TFAE:*
(i) $\forall\varepsilon>0\ \exists\mu>0: A\subset\varepsilon A+\mu f(B)$,
(ii) $\forall\varepsilon>0\ \exists\mu>0: C\subset\varepsilon C+\mu f(B)$.

Proof. (i) clearly implies (ii). Assume that (ii) holds. Then if $C\subset\varepsilon 2^{-1}C+\mu f(B)$, we have
$A=C^{-X}\subset(\varepsilon 2^{-1}C+\mu f(B))^{-X}\subset\varepsilon 2^{-1}C+\mu f(B)+\varepsilon 2^{-1}A\subset\ \varepsilon A+\mu f(B)$. ■

2.10. Proposition. *Let F be a Fréchet space of Moscatelli type w.r.t.* $(L,\| \|)$, $(X_k,r_k)_{k\in\mathbb{N}}$, $(Y_k,s_k)_{k\in\mathbb{N}}$ *and* $f_k\colon Y_k\to X_k$ *with dense range* $(k\in\mathbb{N})$. *TFAE*
(1) F has the density condition.
(1)' The bounded subsets of F_b' *are metrizable.*
(1)'' $l_1\hat{\otimes}_\pi F$ *is distinguished.*
(2) F is quasinormable.
(2)' F satisfies property (Ω_φ) *for some* φ.
(3) F has property (Ω).
(3)' F is a quotient of $l_1(I)\hat{\otimes}_\pi s$ *for some index set I.*
(4) F'' *is a quojection.*
(4)' F has property (Ω_1).
(5) F is a quojection.
(6) $\exists m\in\mathbb{N}\ \forall k\geq m\ f_k$ *is surjective.*

Proof. Clearly (6) implies (5) and (5) implies (4). (4) is equivalent to (4)' according to [21,0.3]. By the very definition, (4)' implies (3). The equivalence of (3) and (3)' for arbitrary Fréchet spaces is proved in [20,3.1]. Clearly (3) implies (2)'. The equivalence of (2) and (2)' follows from the result for Fréchet spaces in [12]. For the equivalences of (1), (1)' and (1)'' we refer to [2,1.4]. Clearly (2) implies (1)'. It remains to show that (1) implies (6).

Put $\mathfrak{A}:=\{y\in F:\|(r_k(f_k(y_k)))_{k\in\mathbb{N}}\|\leqslant 1\}$. A basis of 0-nghbs in F is given by the sets

$$\mathfrak{U}_n:=(1/n)(\mathfrak{A}\cap(\Pi_{k<n}B_k\times\Pi_{k\geqslant n}Y_k))(n\in\mathbb{N}).$$

Put $\lambda_n:=n(n\in\mathbb{N})$. We apply (DC) to $(\lambda_n)_{n\in\mathbb{N}}$ and $2^{-1}\mathfrak{A}$ to obtain $m\in\mathbb{N}$ and a bounded subset $\mathfrak{B}$ of F such that

$$\bigcap_{n=1}^{m}\lambda_n\mathfrak{U}_n\subset 2^{-1}\mathfrak{A}+\mathfrak{B}.$$

Consequently, there are $m\in\mathbb{N}$ and $\mathfrak{B}$ bounded in F such that

$$\mathfrak{A}\cap(\Pi_{k<m}B_k\times\Pi_{k\geqslant m}Y_k)\subset 2^{-1}\mathfrak{A}+\mathfrak{B}.$$

Since the k-th projection of $\mathfrak{A}$ in Y_k is $\rho_k^{-1}f_k^{-1}(A_k)$, where $\rho_k:=\|(\delta_{kj})_{j\in\mathbb{N}}\|$, and the k-th projection of $\mathfrak{B}$ in Y_k is bounded in $(Y_k,s_k)(k\in\mathbb{N})$, we obtain

$\forall k\geqslant m\ \ \exists\alpha_k>0: f_k^{-1}(A_k)\subset 2^{-1}f_k^{-1}(A_k)+\alpha_kB_k$. This implies
$\forall k\geqslant m\ \ \forall p\in\mathbb{N}\ \exists\alpha_{k,p}>0: f_k^{-1}(A_k)\subset 2^{-p}f_k^{-1}(A_k)+\alpha_{k,p}B_k$ hence
$\forall k\geqslant m\ \ \forall\varepsilon>0\ \exists\mu_k>0: f_k^{-1}(A_k)\subset\varepsilon f_k^{-1}(A_k)+\mu_kB_k$ whence
$\forall k\geqslant m\ \forall\varepsilon>0\ \exists\mu_k>0: A_k\cap f_k(Y_k)\subset\varepsilon(A_k\cap f_k(Y_k))+\mu_kf_k(B_k)$. According to 2.8 and 2.9, f_k is surjective for $k\geqslant m$ and (6) is satisfied. ■

Let F be a Fréchet space of Moscatelli type w.r.t. $(L,\| \|)$, such that $(L',\| \|')$ has property (ε) (e.g. $(L,\| \|)=(l_p,\| \|_p)$, $1<p<\infty$), and two sequences of Banach spaces $(X_n,r_k)_{k\in\mathbb{N}}$ and $(Y_k,s_k)_{k\in\mathbb{N}}$ such that F is not a quojection. Then F is distinguished by 2.5 (a), but $l_1\hat{\otimes}_\pi F$ is not distinguished, or equivalently $L_b(l_1,F_b')$ is not quasibarrelled, according to Proposition 2.10. The first example of this type was given in [5].

In [17] Terzioglu and Vogt introduced the following definition. A locally convex (l.c.) space E is called locally normable if there is a continuous norm on E such that on every bounded set in E the norm topology and the space topology coincide. They proved that if E is locally normable, then its bidual provided with its natural topology E_n'' has a continuous norm. We recall that if E is quasibarrelled, then $E_n''=E_b''$ (the strong bidual). We now characterize the l.c. spaces such that E_n'' admits a continuous norm.

2.11. Proposition. *Let E be a l.c. space. TFAE*
(i) E_n'' has a continuous norm.
(ii) there is a continuous norm p on E such that the norm topology induces on every bounded subset of E a stronger topology than the weak topology $\sigma(E,E')$.

Proof. (ii) implies (i). Let U be the unit ball of the norm p in E. We have to show that for every bounded subset B of E, E' is included in $\bigcup_{n\in\mathbb{N}}(nU^\circ)+B^\circ$.

Let B be a closed, absolutely convex bounded subset of E and $u\in E'$. By assumption, there is $n\in\mathbb{N}$ such that $n^{-1}U\cap B\subset\{u\}^\circ$. Thus

$$u\in\{u\}^{\circ\circ}\subset(n^{-1}U\cap B)^\circ\subset nU^\circ+B^\circ.$$

(i) implies (ii). By assumption there is an absolutely convex 0-nghb U in E such that the linear span H of U° is dense in E_b'. Since H is also dense in E_σ', U defines a continuous norm on E. Now let B an absolutely convex bounded subset of E and $u_i\in E'$, $1\leqslant i\leqslant k$. There is $m\in\mathbb{N}$ such that $2u_i\in mU^\circ+B^\circ$, $1\leqslant i\leqslant k$.
Therefore $\{u_1,\dots,u_k\}^\circ\supset(2^{-1}(mU^\circ+B^\circ))^\circ\supset m^{-1}U^{\circ\circ}\cap B^{\circ\circ}=m^{-1}U\cap B$. ∎

A locally convex space E is called a Schur space if every sequence in E which converges to the origin w.r.t. the weak topology $\sigma(E,E')$ also converges to the origin w.r.t. the original topology of E. Every Köthe echelon space of order 1 is a Schur space (cf. [18,p. 181,(5)]).

2.12. Corollary. *Let E be a Schur l.c. space. Then E_n'' has a continuous norm if and only if E is locally normable. In particular, a Köthe echelon space of order 1 is locally normable if and only if its strong bidual has a continuous norm (cf. [17]).*

Proof. According to 2.11, E_n'' has a continuous norm if and only if there is a continuous norm p on E such that if $(x_n)_{n\in\mathbb{N}}$ is a bounded sequence in E with $\lim_{n\to\infty}p(x_n)=0$, then $(x_n)_{n\in\mathbb{N}}$ tends to 0 for the weak topology $\sigma(E,E')$, hence for the original topology of E, since E is a Schur space. This implies that E is locally normable. ■

Let $\varphi:]0,\infty[\to]0,\infty[$ denote a strictly increasing continuous function with $\lim \varphi(r)=+\infty$. Let F be a Fréchet space with a sequence of seminorms $(\|\ \|_n)_{n\in\mathbb{N}}$ such that $U_n:=\{x\in F:\|x\|_n\leqslant 1\}(n\in\mathbb{N})$ form a basis of 0-nghbs in F. F is said to satisfy property (DN_φ) (cf. [19,p.373]) if

$$\exists n(o)\ \forall m\ \exists n, C>0\ \forall x\in F\ \forall r>0: \|x\|_m\leqslant C\varphi(r)\|x\|_{n(o)}+r^{-1}\|x\|_n.$$

Clearly $\|\ \|_{n(o)}$ is a continous norm on F. If $\varphi(r)=r$ for all $r>0$, (DN_φ) is called (DN) (cf. [20]).

2.13. Proposition. *Let F be a Fréchet space. if F satifies property* (DN_φ) *for some* φ, *then F is locally normable.*

Proof. According to [19,5.6] F has (DN_φ) for some φ if and only if $\exists\ n(o)$ $\forall m\geqslant n(o)\ \exists n\geqslant m$: every sequence in F which is bounded w.r.t. $\|\ \|_n$ and converges to 0 w.r.t. $\|\ \|_{n(o)}$ even converges to 0 w.r.t. $\|\ \|_m$. This is equivalent to the following condition $\exists n(o)\ \forall m\geqslant n(o)\ \exists n\geqslant m$: the norm topologies associated to $\|\ \|_{n(o)}$ and $\|\ \|_m$ coincide on the unit ball U_n.

Assume now that this condition is satisfied and fix an absolutely convex bounded subset B of F. To prove that the norm topology associated to $\|\ \|_{n(o)}$ coincides with the original topology on B, we fix $m\geqslant n(o)$. According to the assumption we select $n\geqslant m$ and then $\lambda>0$ such that $B\subset\lambda U_n$. Since the norm topologies associated to $\|\ \|_{n(o)}$ and $\|\ \|_m$ coincide on U_n, they also coincide on B. Thus F is locally normable. ■

The converse of Proposition 2.13 does not hold in general. Indeed, clearly every Fréchet Montel space with a continuous norm is locally normable, and, by [19,5.7], a Fréchet Schwartz space has (DN_φ) for some φ if and only if it is countably normed. There are Fréchet nuclear spaces with a continuous norm, hence locally normable, which are not countably normed (see e.g. [22]).

We now characterize the properties mentioned above in the context of Fréchet spaces of Moscatelli type. We keep the notations at the beginning of this section.

2.14. Remark. *A Fréchet space of Moscatelli type F admits a continuous norm if and only if there is* $n\in\mathbb{N}$ *such that* $f_k:Y_k\to X_k$ *is injective for* $k\geqslant n$.

Proof. First suppose f_k injective for $k\geqslant n$. Then

$$q(y) := \max_{1 \leq k < n} s_k(y_k) + \|(r_k(f_k(y_k)))_{k\in\mathbb{N}}\|.$$

is a continuous norm on F. Conversely, if q is a continuous norm on F, we find $n \in \mathbb{N}$ and $C > 0$ with

$$q(y) \leq C\,[\max_{1 \leq k < n} s_k(y_k) + \|(r_k(f_k(y_k)))_{k\in\mathbb{N}}\|].$$

One easily checks that f_k is injective for $k \geqslant n$. ■

2.15. Lemma. *Let E be a regular (LB)-space of Moscatelli type w.r.t. $(L, \| \|)$, $(X_k, r_k)_{k\in\mathbb{N}}$ $(Y_k, s_k)_{k\in\mathbb{N}}$. Let $\check{E}$ be the projective limit associated with E. There is $n \in N$ such that E_n is dense in $\check{E}$ if and only if there is $n \in \mathbb{N}$ such that Y_k is dense in (X_k, r_k) for $k \geqslant n$.*

Proof. Suppose first $E_n = L((X_k, r_k)_{k<n}, (Y_k, s_k)_{k\geqslant n})$ dense in $\check{E}$. Fix $m \geqslant n$ and take $x_m \in X_m, \varepsilon > 0$. Then $x = (x_m \delta_{km})_{k\in\mathbb{N}} \in E = \check{E}$. There is $y = (y_k)_{k\in\mathbb{N}} \in E_n$ with $x\text{-}y \in (\prod_{k\in\mathbb{N}} \varepsilon A_k + \mathfrak{B}_1) \cap \check{E}$. This implies $x_m - y_m \in \varepsilon A_m + Y_m$. Since $y_m \in Y_m$, we have $x_m \in \varepsilon A_m + Y_m$, and Y_m is dense in (X_m, r_m). Conversely, suppose Y_k dense in (X_k, r_k) for $k \geqslant n$. We prove that E_n is dense in $\check{E}$. Fix $x = (x_k)_{k\in\mathbb{N}} \in \check{E}$, $\varepsilon_k > 0 (k \in \mathbb{N})$, $\delta > 0$. We select $m > n$ such that $x \in E_m$. For $n \leqslant k < m$, we take $\bar{y}_k \in Y_k$ with $x_k - \bar{y}_k \in \varepsilon_k A_k$. Then define $y_k := x_k (1 \leqslant k < n$ or $k \geqslant m)$, $y_k := \bar{y}_k$ $(n \leqslant k < m)$ and $y := (y_k)_{k\in\mathbb{N}}$. Clearly $y \in E_n$ and $x - y \in (\prod_{k\in\mathbb{N}} \varepsilon_k A_k + \delta \mathfrak{B}_1) \cap \check{E}$. The proof is complete. ■

2.16. Corollary. *Let F be a Fréchet space of Moscatelli type w.r.t. $(L, \| \|)$ with (ε), $(X_k, r_k)_{k\in\mathbb{N}}$, $(Y_k, s_k)_{k\in\mathbb{N}}$ and $f_k: Y_k \to X_k$ with dense range. TFAE*
(i) F_b'' has a continuous norm.
(ii) $\exists\, n \in \mathbb{N}$: $f_k^t(X_k') $ is dense in (Y_k', s_k') for $k \geqslant n$.

Proof. This follows from 2.15 applied to the corresponding (LB)-space E formed w.r.t. the duals, since F_b'' has a continuous norm if and only if there is an equicontinuous subset of F' such that its linear span is dense in $F_b' = \check{E}$ (see 2.3). ■

2.17. Proposition. *Let F be the Fréchet space of Moscatelli type w.r.t. $(L, \| \|)$, $(X_k, r_k)_{k\in\mathbb{N}}$, $(Y_k, s_k)_{k\in\mathbb{N}}$ and $f_k: Y_k \to X_k (k \in \mathbb{N})$. TFAE*
(1) F is locally normable.
(2) F has (DN_φ) for some φ.
(3) F has property (DN).
(4) $\exists m \in \mathbb{N}$ $\forall k \geqslant m$ f_k is a topological isomorphism onto its range.
(5) F is a Banach space.

Proof. By 2.13, (1) is a consequence of (2). We prove that (1) implies (4).

There is $m \in \mathbb{N}$ such that the seminorm

$$y \to \|((s_k(y_k))_{k<m}, (r_k(f_k(y_k)))_{k \geq m})\|$$

is a norm on F which induces on every bounded subset of F the original topology. Let B_k the unit ball of (Y_k, s_k) $(k \in \mathbb{N})$. Then for all $k \geq m$, the set $\mathfrak{B}_k := \{(\delta_{jk}x)_{j\in\mathbb{N}} : x \in B_k\}$ is bounded in F. Thus the topologies generated by s_k and $r_k \circ f_k$ on Y_k coincide on B_k and hence on Y_k, since B_k is a 0-nghb in (Y_k, s_k). Therefore (4) is satisfied. Now (5) clearly follows from (4), because (4) implies that F is a topological subspace of F_m. Finally the implications (5)⟹(3) and (3)⟹(2) are trivial. ■

2.18. Remark. (1) Take Banach spaces (X,r) and (Y,s) such that the injection $j : (Y,s) \to (X,r)$ is continuous and has dense range but $j^t(X')$ is not dense in (Y',s') (e.g. $j:(l_1, \|\ \|_1) \to (l_2, \|\ \|_2)$) and any normal Banach sequence space $(L, \|\ \|)$ with property (ε). Then the Fréchet space of Moscatelli type F w.r.t. $(L, \|\ \|), (X,r)_{k\in\mathbb{N}}, (Y,s)_{k\in\mathbb{N}}$ and $f_k = j$ $(k \in \mathbb{N})$ has a continuous norm but F_b'' does not have a continuous norm. The first example of this type can be found in [6].

(2) There are Köthe echelon spaces of order 1, $\lambda_1(A)$, which are, via some rearrangement, Fréchet spaces of Moscatelli type. One of the most relevant examples is given by the Köthe matrix $A = (a_n)_{n\in\mathbb{N}}$, $a_n(i,j) := 1$ $(i \geq n)$, $a_n(i,j) := j (i < n)$, for $(i,j) \in \mathbb{N} \times \mathbb{N}$. This example, due to Köthe and Grothendieck, is a non-distinguished Fréchet space. Terzioglu and Vogt [17] gave the following example. $a_n(i,j) := j^i$ $(i > n)$, $a_n(i,j) = j^n$ $(i \leq n)$. The bidual of $\lambda_1(A)$ has a continuous norm, hence it is locally normable, but it is not distinguished (cf. [17]). According to 2.17, $\lambda_1(A)$ is a non-distinguished Köthe echelon space which cannot be isomorphic to a Fréchet space of Moscatelli type.

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