

Analytic Functions on c_0

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ABSTRACT. Let F be a space of continuous complex valued functions on a subset of c_0 which contains the standard unit vector basis $\{e_n\}$. Let $R:F \rightarrow C^N$ be the restriction map, given by $R(f) = (f(e_1), \dots, f(e_n), \dots)$. We characterize the ranges $R(F)$ for various "nice" spaces F . For example, if $F = P^n(c_0)$, then $R(F) = l_1$, and if $F = A^\infty(B(c_0))$, then $R(F) = l_\infty$.

Let c_0 be the Banach space of complex null sequences $\vec{x} = (x_n)$, with the normal *sup*-norm and usual basis vectors $\vec{e}_n = (0, \dots, 0, 1, 0, \dots)$, and let F be a space of continuous complex-valued functions on some subset of c_0 which contains the standard basis of c_0 . Let $R:F \rightarrow C^N$ be the mapping which assigns to each function $f \in F$ the sequence $(f(e_1), \dots, f(e_n), \dots)$. Our attention in this article will be focussed on characterizing the range of R for various spaces F of interest. For example, if $F = C(c_0)$, the space of all continuous complex valued functions on c_0 , then a trivial application of the Tietze extension theorem shows that $R(F) = C^N$. On the other hand, c_0 is weakly normal (Corson [6], see also Ferrera [9]). Since $\{0\} \cup \{e_n; n \in N\}$ is weakly compact, we see that $R(F) = c$, the space of convergent sequences, if we take F to be the subspace of $C(c_0)$ consisting of weakly continuous functions. Recently Jaramillo [11] has examined the relationship between reflexivity of the space F and the range of R , for certain spaces of *real* valued infinitely differentiable functions and polynomials on a Banach space E with unconditional basis $\{e_n; n \in N\}$.

We concentrate here on analogous spaces of *complex* valued functions on c_0 . After a review of relevant notation and definitions, we show in Section 1 that $R(F) = l_1$ when $F = P^n(c_0)$, $n \in N$. As a consequence, we prove that if $F = \{f \in H_b(B_R(c_0)): f(0) = 0\}$, then $R(F) = l_1$. Taking $n = 2$ in the above result, we see that every 2-homogeneous polynomial P on c_0 satisfies $\sum_{j=1}^{\infty} |P(e_j)| < \infty$. This result is reminiscent of classical work of Littlewood [13], who proved that every continuous bilinear form A on $c_0 \times c_0$ satisfies $(A(e_p, e_k))_{j,k=1}^{\infty} \in l_{4/3}$. Littlewood's work was extended by Davie [7], who showed

that every continuous n -linear form $A: c_0 \times \dots \times c_0 \rightarrow C$ satisfies $(A(e_{\alpha_1}, \dots, e_{\alpha_n})) \in l_{2n/n+1}$. In Section 2, we prove that $R(A^\infty(B(c_0))) = l_\infty$, and as a corollary of the proof of this result we show $R(A_U(B(c_0))) = l_1$.

Our notation for analytic functions is standard and follows, for example, Dineen [8] and Mujica [14]. For a Banach space E , $B_R(E)$ denotes the open R -ball centered at 0 in E with $B_1(E)$ abbreviated to $B(E)$. $L^n(E)$ denotes the Banach space of continuous n -linear forms $A: E \times \dots \times E \rightarrow C$, equipped with the norm $\|A\| = \sup\{|A(x_1, \dots, x_n)| : x_j \in E, \|x_j\| \leq 1, j = 1, \dots, n\}$. $P^n(E)$ denotes the Banach space of continuous n -homogeneous polynomials on E . Each such polynomial P is associated with a unique symmetric continuous n -linear form A , by $P(x) = A(x, \dots, x)$, and $\|P\|$ is defined to be $\sup_{\|x\| \leq 1} |P(x)|$. A function f from an open subset U of E to C is said to be *holomorphic* if f has a complex Fréchet derivative at each point of U . Equivalently, f is holomorphic if for all points $a \in U$, the Taylor series $f(x) = \sum_{n=0}^{\infty} P_n(x-a)$, converges uniformly for all x in some neighborhood of a , where each $P_n \in P^n(E)$.

$H_b(B_R(E))$ is the space of all holomorphic functions on $B_R(E)$ which are bounded on $B_r(E)$ for every $r < R$. A useful characterization of $H_b(B_R(E))$ is that it consists of all holomorphic functions f on $B_R(E)$ such that $\limsup_{n \rightarrow \infty} \|P_n\|^{1/n} \leq 1/R$, where $\{P_n: n \in N\}$ represents the Taylor polynomials of f at the origin. The spaces $A^\infty(B(E))$ and $A_U(B(E))$ have been studied by Cole and Gamelin [4,5], Globevnik [10] and others [1]. $A^\infty(B(E)) = \{f: B(E) \rightarrow C: f \text{ is holomorphic on } B(E) \text{ and continuous and bounded on } B(E)\}$. Unless E is finite dimensional, this space is always strictly larger than $A_U(B(E)) = \{f: B(E) \rightarrow C: f \text{ is holomorphic and uniformly continuous on } B(E)\}$. Both of these spaces are natural infinite dimensional analogues of the disc algebra.

SECTION 1

We show here that for all $P \in P^n(c_0)$ and all $n \in N$, $\sum_{j=1}^{\infty} |P(e_j)| \leq \|P\|$. This has already been done by K. John [12], in the case $n=2$. In [13], Littlewood showed that for every $A \in L^n(c_0)$, $(A(e_p, e_k))_{p,k=1}^{\infty} \in l_{4/3}$, and that $4/3$ is best possible; thus, Littlewood's $4/3$ result notwithstanding, John's result is that every $A \in L^n(c_0)$ has a trace. Our proof will make use of a generalization of the classical Rademacher functions, which seems to be well-known to probabilists (see, for example, Chatterji [3]).

Definition 1.1. Fix $n \in N$, $n \geq 2$, and let $\alpha_1 = 1, \alpha_2, \dots, \alpha_n$ denote the n^{th} roots of unity. Let $s_j: [0,1] \rightarrow C$ be the step function taking the value α_j on $(j-1/n, j/n)$, for $j=1, \dots, n$. Assuming that s_{k-1} has been defined, define s_k in the following natural way. Fix any of the n^{k-1} sub-intervals I of $[0,1]$ used in the definition of s_{k-1} . Divide I into n equal intervals I_1, \dots, I_n and set $s_k(t) = \alpha_j$ if $t \in I_j$,

(The endpoints of the intervals are irrelevant for this construction and we may, for example, define s_k to be 1 on each endpoint.)

Of course, when $n=2$, Definition 1.1 gives us the classical Rademacher functions. The following lemma lists the basic properties of the functions s_k . Its proof is similar to the usual, induction proof for the Rademacher functions, and is omitted.

Lemma 1.2. For each $n=2,3,\dots$, the associated functions s_k satisfy the following properties:

- (a). $|s_k(t)| = 1$, for all $k \in N$ and all $t \in [0,1]$.
- (b). For any choice of k_1, \dots, k_n

$$\int_0^1 s_{k_1}(t) \dots s_{k_n}(t) dt = \begin{cases} 1 & \text{if } k_1 = \dots = k_n \\ 0 & \text{otherwise} \end{cases}$$

We are grateful to Andrew Tonge for suggesting an improvement in the proof of the following result.

Theorem 1.3. Let $P \in P(c_0)$. Then $\|(P(e_j))\|_{l_1} \leq \|P\|$.

Proof. Let $A \in L(c_0)$ be the symmetric n -linear form associated to P . Fix any $m \in N$. For each $i=1,\dots,m$, let $\lambda_i = |A(e_i, \dots, e_i)| / A(e_i, \dots, e_i)$, if $A(e_i, \dots, e_i) \neq 0$, and 1 otherwise. Furthermore, let β_i denote any n^{th} root of λ_i . Thus, $\lambda_i A(e_i, \dots, e_i) = |P(e_i)|$ for each $i=1,\dots,m$. Adding and applying Lemma 1.2 for the integer n , we get $\sum_{i=1}^m |P(e_i)| = \sum_{i=1}^m \lambda_i A(e_i, \dots, e_i)$

$$\begin{aligned} &= \sum_{i_1, \dots, i_n=1}^m \int_0^1 \lambda_{i_1} s_{i_1}(t) s_{i_2}(t) \dots s_{i_n}(t) A(e_{i_1}, e_{i_2}, \dots, e_{i_n}) dt \\ &= \int_0^1 A(\sum_{i=1}^m \lambda_i s_i(t) e_i, \dots, \sum_{i=1}^m s_i(t) e_i) dt \\ &= \int_0^1 A(\sum_{i=1}^m \beta_i s_i(t) e_i, \dots, \sum_{i=1}^m \beta_i s_i(t) e_i) dt. \end{aligned}$$

Since $\|\sum_{j=1}^m \beta_j s_j(t) e_j\| \leq 1$ for all t , the last expression is clearly less than or equal to $\|P\|$. Since m was arbitrary, the proof is complete. ■

Rephrasing the above result in terms of the mapping R mentioned in the introduction, Theorem 1.3 implies that for any n , $R(P(c_0)) \subset l_1$. In fact, R is onto l_1 , since any $\vec{\lambda} = (\lambda_1, \dots, \lambda_p, \dots) \in l_1$ equals $R(P)$, where $P \in P(c_0)$ is given by $P(x) = \sum_{j=1}^{\infty} \lambda_j x_j^n$

We conclude this section by proving that, up to a normalizing factor, $R(H_b(B_R(c_0)))=l_1$, for every $R>1$. Since $H_b(B_R(c_0))$ "approaches" $A^\infty(B(c_0))$ as $R \downarrow 1$, it is tempting to guess that Corollary 1.4 below is also true for the latter space. We will see in the next section that this is completely false.

Corollary 1.4. *Let $R>1$ and let $f \in H_b(B_R(c_0))$, with $f(0)=0$. Then $(f(e_n))_{n=1}^\infty \in l_1$.*

Proof. By the characterization given earlier of $H_b(B_R(c_0))$, we see that if S is such that $1 < S < R$, then $\|P_m\|^{1/m} < 1/S$, for all large m . Therefore,

$$\begin{aligned} \sum_{n=1}^\infty |f(e_n)| &= \sum_{n=1}^\infty \left| \sum_{m=1}^\infty P_m(e_n) \right| \\ &\leq \sum_{m=1}^\infty \sum_{n=1}^\infty |P_m(e_n)| \leq \sum_{m=1}^\infty \|P_m\| < \infty. \blacksquare \end{aligned}$$

SECTION 2

The following fundamental lemma shows in effect that any sequence of 0's and 1's can be interpolated by a norm one function in $A^\infty(B(c_0))$.

Lemma 2.1. (i). *Let $S \subset N$ be an arbitrary set. There exists a function $F \in A^\infty(B(c_0))$ with the following properties:*

$$\|F\| = \sup_{x \in B(c_0)} |F(x)| = 1,$$

$$F(e_n) = \begin{cases} 1 & \text{if } n \in S \\ 0 & \text{if } n \notin S \end{cases}$$

(ii). *If S is finite, then a function $F \in A_c(B(c_0))$ can be found which satisfies the above conditions.*

Proof. Let $\alpha_j \uparrow \infty$ so quickly that the following three conditions are satisfied:

(i). The function $\Phi(x) \equiv \prod_{j \in S} (1 - x_j)^{1/\alpha_j}$, converges for all $x \in \overline{B(c_0)}$,

(ii). $\operatorname{Re} \Phi(x) \geq 0$, for all $x \in B(c_0)$,

(iii). $\Phi(x) = 0$ for some $x \in \overline{B(c_0)}$ if and only if $\operatorname{Re} \Phi(x) = 0$.

Note that $\Phi \in A^\infty(B(c_0))$ and, if S is finite then in fact $\Phi \in A_c(B(c_0))$. Also,

$$\Phi(e_n) = \begin{cases} 0 & \text{for } n \in S \\ 1 & \text{for } n \notin S \end{cases}$$

Now, let $G(x) \equiv e^{-\Phi(x)}$. From the above, it is clear that $G \in A^\infty(B(c_0))$ for arbitrary S and that $G \in A_c(B(c_0))$ for finite S . In addition, $|G(x)| \leq 1$ for all x and

$$G(e_n) = \begin{cases} 1 & \text{for } n \in S \\ 1/e & \text{for } n \notin S \end{cases}$$

Finally, let $T: \bar{\Delta} \rightarrow \bar{\Delta}$ be the Mobius transformation $T(z) = \frac{z-1/e}{1-z/e}$ (where Δ is the complex unit disc.) It is clear that $F \equiv T \circ G$ satisfies all the conditions of the lemma. ■

We come now to the analogue of Corollary 1.4, for the polydisc algebras $A^\infty(B(c_0))$ and $A_c(B(c_0))$. Note that here the situation is completely different from the situation in Section 1.

Theorem 2.2. (i). $R(A^\infty(B(c_0))) = l_\infty$. In fact, given $(\alpha_n) \in l_\infty$, there is $F \in A^\infty(B(c_0))$ such that $F(e_n) = \alpha_n$ for all $n \in N$ and such that $\|F\| \leq 4\|(\alpha_n)\|_1$.

(ii). $R(A_c(B(c_0))) = c$. In fact, given $(\alpha_n) \in c$, there is $F \in A_c(B(c_0))$ such that $F(e_n) = \alpha_n$ for all $n \in N$ and such that $\|F\| \leq 8\|(\alpha_n)\|_1$.

Proof. (i). Without loss of generality, $\|(\alpha_n)\| \leq 1$. Let us first suppose that $\alpha_n \geq 0$ for all n . Write $\alpha_n = \sum_{j=1}^\infty 2^{-j} \alpha_{nj}$, where each $\alpha_{nj} = 0$ or 1. Let $S_j = \{n \in N: \alpha_{nj} = 1\}$, and let F_j be the associated function obtained using Lemma 2.1. It is easy to see that $F \equiv \sum_{j=1}^\infty 2^{-j} F_j$ is the required function in this case, and that $\|F\| \leq \|(\alpha_n)\|$. The case of general α_n 's is treated by writing $\alpha_n = p_n - q_n + iu_n - iv_n$.

(ii). Suppose first that $(\alpha_n) \in c$ with $\|(\alpha_n)\| \leq 1$, and write each $\alpha_n = l + \beta_n$, where $l = \lim_{n \rightarrow \infty} \alpha_n$. As above, if each β_n is expressed in binary series form, then each of the associated sets S_j is finite. As a result, each F_j is in $A_c(B(c_0))$ by Lemma 2.1 (ii), so that $F \in A_c(B(c_0))$. The required function is $G \equiv F + l$.

Finally, note that for any $F \in A_c(B(c_0))$, $F(x)$ can be approximated uniformly for $x \in B(c_0)$ by $F(rx) = F(rx)$ for r sufficiently close to 1. Next, $F(rx)$ can be uniformly approximated on the unit ball of c_0 by a finite Taylor series, say $\sum_{k=0}^M P_k(x)$ (where P_0 is a constant). Next, it is well known (see, for example, [15]) that any k -homogeneous polynomial P_k on c_0 can be uniformly approximated on $B(c_0)$ by an k -homogeneous polynomial Q_k which is a finite sum of products of k continuous linear functionals on c_0 . Summarizing, we see that the original function F can be uniformly approximated on $B(c_0)$ by $\sum_{k=0}^M Q_k$. Now, since, $(e_n) \rightarrow 0$ weakly it follows that for each $k = 1, \dots, M$, $Q_k(e_n) \rightarrow 0$ as $n \rightarrow \infty$. Hence $R(F) \in c$, and the proof is complete. ■

It would be interesting to determine the best possible estimates in Theorem 2.2. In [2], we note that in this situation, the best estimate must be strictly

larger than 1. To see this, suppose that there is $F \in A^\infty(B(c_0))$ such that $\|F\| = 1$ and such that $F(e_1) = 1$, $F(e_2) = -1$, and $F(e_j) = 0$ for all $j \geq 3$. Then the function $f_1(z) \equiv F(1, z, 0, \dots)$ would be in the disc algebra $A(\Delta)$, and f_1 would attain its maximum at 0. Hence, f_1 would be a constant and, in particular, $1 = f_1(1) = F(1, 1, 0, \dots)$. Similarly, the function $f_2(z) \equiv F(z, 1, 0, \dots)$ would be constant, and so $-1 = f_2(1) = F(1, 1, 0, \dots)$, a contradiction. In [2], the authors find necessary and sufficient conditions on the sequence $(x_n) \subset c_0$ in order that the mapping $F \in A^\infty(B(c_0)) \rightarrow (F(x_n)) \in l_\infty$ be surjective and satisfy the following condition: For each $(\alpha_n) \in l_\infty$, there is $F \in A^\infty(B(c_0))$ such that $F(x_n) = \alpha_n$ for each $n \in \mathbb{N}$ and $\|F\| = \sup_n |\alpha_n|$.

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