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Multiplicative functionals on function algebras

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ABSTRACT. Let X be a completely regular Hausdorff space and $C(X)$ the algebra of all continuous $\mathbb{K}$-valued functions on X ($\mathbb{K}=\mathbb{R}$ o $\mathbb{C}$). If $A\subseteq C(X)$ is a subalgebra, in [4] can be found conditions on A under which each character of A, i.e., each non-zero $\mathbb{K}$-linear multiplicative functional $\phi:A\to\mathbb{K}$, is given by a point evaluation at some point of X.

In this paper we present a «Michael» type theorem for the particular case in which X is a real Banach space. As consequence it is showed that if E is a separable Banach space or E is the topological dual space of a separable Banach space and A is the algebra of all real analytic or the algebra of all real C^m-functions, $m=0, 1, ..., \infty$, on E, then every character ϕ of A is a point evaluation at some point of E.

Let E be a real Banach space with topological dual E' and let $C(E)$ be the algebra of all continuous $\mathbb{R}$-valued functions on E. Let $l^1(\mathbb{N})=\{\alpha=(\alpha_n)\in\mathbb{R}^{\mathbb{N}}: \sum_{n=1}^{\infty}|\alpha_n|<\infty\}$.

Theorem 1. *Assume that there exists* $(\phi_n)_{n=1}^{\infty}\subset E'$, $\|\phi_n\|\leqslant 1$ *for every* $n\in\mathbb{N}$, *such that* (ϕ_n) *separates points of E. Let* $A\subseteq C(E)$ *be a subalgebra with* $1\in A$. *Assume:*

(i) *If* $f\in A$, $f(x)\neq 0$ *for all* $x\in E$, *then* $1/f\in A$.

(ii) $E'\subset A$ *and for every* $\alpha=(\alpha_n)\in l^1(\mathbb{N})$, *the function* $\sum_{n=1}^{\infty}\alpha_n\cdot\phi_n^2$ *belongs to* A.

Then every character $\phi:A\to\mathbb{R}$, *such that* $\phi(\phi_n)=\phi_n(a)$ *for every* $n\in\mathbb{N}$ *and some* $a\in E$, *is the point evaluation at* a.

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Proof. Let $\alpha=(\alpha_n)\in l^1(\mathbb{N})$ with $\alpha_n>0$ for all $n\in\mathbb{N}$. Condition (*ii*) implies that the functions:

$$f(x)=\sum_{n=1}^{\infty}\alpha_n\phi_n^2(x-a) \text{ and } g(x)=\sum_{n=1}^{\infty}\frac{\alpha_n}{n}\phi_n^2(x-a) \text{ belong to } A$$

For each $N\in\mathbb{N}$, let $x\in E$ such that $\phi(f)=f(x)$; $\phi(g)=g(x)$ and $\phi(\phi_i)=\phi_i(x)$, $i=1, \ldots, N$ (a such x exists after condition (*i*)). For this $x\in E$, we have

$$\phi(f)=\sum_{N+1}^{\infty}\alpha_n\phi_n^2(x-a) \ ; \quad \phi(g)=\sum_{N+1}^{\infty}\frac{\alpha_n}{n}\phi_n^2(x-a)$$

Therefore $0\leqslant N\phi(g)\leqslant\phi(f)$ and it follows that $\phi(g)=0$.

If $h\in A$ is given, let $y\in E$ such that $\phi(h)=h(y)$ and $\phi(g)=g(y)$. Since $\phi(g)=g(y)=0$, it follows that $\phi_n(y)=\phi_n(a)$ for all $n\in\mathbb{N}$, i.e., $y=a$ and $\phi(h)=h(a)$.

Remark 1. The hypothesis on the real Banach space E in Theorem 1 is equivalent to say that E' is $\sigma(E'; E)$-separable. Therefore it holds when E is a separable Banach space and when E is the topological dual space of a separable Banach space.

Consequences

Let $A(E)$ be, respectively $C^m(E)$ ($m=0, 1, \ldots, \infty$), the subalgebra of $C(E)$ of all real analytic functions (see [2]), respectively of all C^m-functions in the Fréchet sense, on E.

Corollary 1. *If E is finite dimensional and $A=A(E)$ or $A=C^m(E)$, then every character $\phi:A\to\mathbb{R}$ is a point evaluation at some point of E.*

Proof. This follows from Theorem 1 if we consider (ϕ_n) as the canonical projections.

Proposition 1. *For every character $\phi:A(E)\to\mathbb{R}$, the restriction $\phi_{|E'}$ is $\sigma(E'; E)$-sequentially continuous.*

Proof. Assume that $(x'_n)\subset E'$ converges to zero for the $\sigma(E'; E)$-topology. If $\phi(x'_n)\not\to 0$, there are $\alpha>0$ and (x'_{n_p}), subsequence of (x'_n), such that

$$\left[\phi\left[\frac{x'_{n_p}}{\sqrt{\alpha}}\right]\right]^2>1$$

for every $p\in\mathbb{N}$. Since $(x'_{n_p})\to 0$ ($p\to\infty$) for the $\sigma(E'; E)$-topology, the function

$$f(x)=\sum_{p=1}^{\infty}\left[\frac{x'_{n_p}(x)}{\sqrt{\alpha}}\right]^{2p}$$

is well defined and $f\in A(E)$. (See ([2], Th. 6)). For each $N\in\mathbb{N}$,

$$\phi(f)\geqslant\phi\left[\sum_{p=1}^{N}\left[\frac{x'_{n_p}}{\sqrt{\alpha}}\right]^{2p}\right]=\sum_{p=1}^{N}\left[\phi\left[\frac{x'_{n_p}}{\sqrt{\alpha}}\right]\right]^{2p}.$$

Therefore $\sum_{p=1}^{\infty}\left[\phi\left[\frac{x'_{n_p}}{\sqrt{\alpha}}\right]\right]^{2p}<\infty$ and then $\left[\phi\left[\frac{x'_{n_p}}{\sqrt{\alpha}}\right]\right]^{2p}\to 0$ $(p\to\infty)$, which is a contradiction because $\left[\phi\left[\frac{x'_{n_p}}{\sqrt{\alpha}}\right]\right]^{2}>1$ for all $p\in\mathbb{N}$.

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Corollary 2. *Let E be a separable Banach space and $\phi:A(E)\to\mathbb{R}$ a character. Then $\phi_{|E'}$ is a point evaluation at some point of E.*

Proof. This is immediate from Prop. 1, since by ([5], Ch. IV; Th. 6.2 and Corollary 3) for $\phi_{|E'}$ to be $\sigma(E'; E)$-continuous it suffices to show that $\phi_{|E'}$ is $\sigma(E'; E)$-sequentially continuous.

Corollary 3. *Let E be a separable Banach space and $\phi:A(E)\to\mathbb{R}$ a character. Then ϕ is a point evaluation at some point of E.*

Proof. This is immediate from Theorem 1, Remark 1 and Corollary 2.

Let F be a separable Banach space and $(y_n)_{n=1}^{\infty}$ a dense subset in $\{y\in F:\|y\|\leqslant 1\}$. Let $E=F'$. Let ϕ_n:E$\to\mathbb{R}$ be defined as $\phi_n(x)=x(y_n)$. Then $\phi_n\in E'$, $\|\phi_n\|\leqslant 1$ and $(\phi_n)_{n=1}^{\infty}$ separates points of E. The mapping $y\to\phi_y$, defined as $\phi_y(x)=x(y)$, allow us identify F with a subspace of $E'=F''$. Thus, if $\phi:A(E)\to\mathbb{R}$ is a character, Prop. 1 implies that $\phi_{|F}$ is $\|\cdot\|$-continuous, therefore $\phi_{|F}\in F'=E$. Then, it follows that there exists $a\in E$ such that $\phi(\phi_n)=\phi_n(a)$ for all $n\in\mathbb{N}$. Now the following Corollary is clear after Theorem 1.

Corollary 4. *Let E be a topological dual space of a separable Banach space and $\phi:A(E)\to\mathbb{R}$ a character. Then ϕ is a point evaluation at some point of E.*

Corollary 5. *Assume that E is a separable Banach space or E is the topological dual space of a separable Banach space. Then every character $\phi:C^m(E)\to\mathbb{R}$, $m=0, 1, \ldots, \infty$, is a point evaluation at some point of E.*

Proof. $\phi_{|A(E)}$ is a point evaluation by Corollary 3 and Corollary 4. Thus, ϕ satisfies conditions of Theorem 1 with $A = C^m(E)$.

Remark 2. The Corollary 5, for the particular case E a separable Banach space and $m = \infty$, can be found in [1]. Also, for E with C^m-partitions of unity and $m < \infty$, see [3].

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