# On a rigidity condition for Berwald spaces

## Ricardo Gallego Torrome and Fernando Etayo

**Abstract**  We show that which that for a Berwald structure, any Riemannian structure that is preserved by the Berwald connection leaves the indicatrix invariant under horizontal parallel transport. We also obtain the converse result: if $(\mathbf{M}, F)$ is a Finsler structure such that there exists a Riemannian structure that leaves invariant the indicatrix under parallel transport of the associated Levi-Civita connection, then the structure $(\mathbf{M}, F)$ is Berwald. As application, a necessary condition for pure Landsberg spaces is formulated. Using this criterion we provide an strategy to solve the existence or not of pure Landsberg surfaces.

### Sobre una condición de rigidez de los espacios de Berwald

**Resumen.**  Se muestra que la conexión de Levi Civita de cualquier métrica Riemanniana afínmente equivalente a una estructura de Berwald deja invariante por transporte paralelo la indicatriz de dicha estructura de Berwald. También se demuestra el resultado recíproco: Si $(\mathbf{M}, F)$ es una estructura de Finsler y existe una estructura Riemanniana cuya conexión de Levi Civita deja invariante por transporte paralelo la indicatriz de la estructura de Finsler, entonces $(\mathbf{M}, F)$ es de Berwald. Como aplicación se obtiene una condición necesaria para que una variedad sea de Landsberg pura. Y usando este criterio se formula una estrategia para resolver el problema de la existencia de superficies de Landsberg puras.

## 1   Introduction

A Riemannian structure on a manifold is given by a Riemannian metric. As is well known, the Levi Civita connection is an important tool associated to the structure. A more general concept is that of a Finsler structure (see Definition 1 below). In general, one cannot define a Levi Civita type connection associated to a Finsler structure. Given a Finsler structure, one can define several linear connections on the pull-back bundle $\pi^*\mathbf{TM} \to \mathbf{N}$ determined by the Finsler function $F$ and additional conditions, usually restrictions to the "torsion". Cartan, Chern and Berwald's linear connections are notable examples ([1]). Although the relevant merits of these connections, compared with affine connections, are quite complicated objects.

This is one of the reasons that make Finsler geometry specially difficult to investigated, compared with the Riemannian case.

One step on the understanding of the structure of Finsler geometry maybe achieved by the theory presented (or better suggested) in reference [2], where it was introduced the averaged connection. The averaged connection is obtained from a linear connection on $\pi^*\mathbf{TM} \to \mathbf{N}$ by an averaged procedure on a suitable subset $\mathbf{\Sigma} \subset \mathbf{N}_x = \pi^{-1}(x) \subset \mathbf{N}$. Usually this subset is defined to be the indicatrix $\mathbf{I}_x$ over $x \in \mathbf{M}$. The average connection is an affine connection on the tangent bundle $\pi\colon \mathbf{TM} \to \mathbf{M}$. If we perform the averaged operation on convex combinations of connections that have the same averaged connection, the result is the same averaged connection. We called this property convex invariance. The connection coefficients of the averaged connection are equal to the average of the connection coefficients of the original connection on $\mathbf{\Sigma}$.

The main purpose of this note is to prove a necessary condition for Berwald spaces. The condition is obtained using the averaged connection and the convex invariance mentioned above. In particular two-dimensional spaces are considered.

In Section 2 we introduce the basic notions of Finsler structures that we need. We will follow the notation from Bao-Chern-Shen [1]. Usually, the concepts of Finsler geometry are introduced by using local coordinates, but we show some intrinsic expressions (e.g., propositions 1 and 2; see also [10]). In Section 3 we recall the notion of average of a linear connection in the pull back bundle $\pi^*\mathbf{TM} \to \mathbf{N}$ and other results from [2]. In Section 4 we obtain Proposition 12 which states that for a Berwald structure, any Riemannian structure that is preserved by the Berwald connection leaves the indicatrix invariant under horizontal parallel transport. We also obtain the converse result, Proposition 13: if $(\mathbf{M}, F)$ is a Finsler structure such that there exists a Riemannian structure that leaves invariant the indicatrix under parallel transport of the associated Levi-Civita connection, then the structure $(\mathbf{M}, F)$ is Berwald. We finish showing that these results together with the notion of convex invariance, can be useful in the research of pure Landsberg spaces through Theorem 1 and a criterion for pure two dimensional Landsberg space is given.

## 2   Basics notions on Finsler geometry

In this section we introduce the notions of Finsler geometry as well the notation that we will use in this work. The main reference that we follow is [1].

Let $\mathbf{M}$ be an $n$-dimensional manifold and $\mathbf{TM}$ its tangent bundle. If $\{x^i\}$ is a local coordinate system on $\mathbf{M}$, the induced local coordinate system on $\mathbf{TM}$ is denoted by $\{(x^i, y^i)\}$. This type of coordinate systems on $\mathbf{TM}$ are called natural coordinate systems. The slit tangent bundle is $\mathbf{N} = \mathbf{TM} \setminus \{0\}$. Then we have,

**Definition 1** *A Finsler structure $F$ on the manifold $\mathbf{M}$ is a non-negative, real function $F\colon \mathbf{TM} \to [0, \infty[$ such that*

1. *It is smooth in the slit tangent bundle $\mathbf{N}$.*

2. *Positive homogeneity holds: $F(x, \lambda y) = \lambda F(x, y)$ for every $\lambda > 0$.*

3. *Strong convexity holds: the Hessian matrix*

$$g_{ij}(x, y) := \frac{1}{2} \frac{\partial^2 F^2(x, y)}{\partial y^i \partial y^j}$$

   *is positive definite in $\mathbf{N}$. The fundamental and the Cartan tensors are defined by the equations:*

$$g(x, y) := \frac{1}{2} \frac{\partial^2 F^2(x, y)}{\partial y^i \partial y^j} \, \mathrm{d}x^i \otimes \mathrm{d}x^j.$$

Given a Finsler structure $(\mathbf{M}, F)$ it is not possible to define a Levi-Civita connection in the general case. In order to obtain a connection related with the structure, one has to go to higher order bundles over $\mathbf{M}$. This is done in the following standard way. First, we introduce a non-linear connection on the bundle $\pi_N \colon \mathbf{TN} \to \mathbf{N}$:

1. there is a splitting of each tangent space $\mathbf{T}_u\mathbf{N}$ in complementary sub-spaces $\mathcal{V}_u$ and $\mathcal{H}_u$

$$\mathbf{T}_u\mathbf{N} = \mathcal{V}_u \oplus \mathcal{H}_u, \qquad \forall u \in \mathbf{N}$$

2. $\ker_u(\pi_N) = \mathcal{V}_u, \qquad \forall u \in \mathbf{N}$.

This decomposition is invariant by the action of $\mathbf{GL}(n, \mathbf{R})$, which is induced by the action of the linear group $\mathbf{Gl}(n, \mathbf{R})$ acting freely and by the right on the tangent bundle manifold $\mathbf{TM}$.

A Local basis for $\mathbf{T}_u\mathbf{N}$ is given by the distributions

$$\left\{ \frac{\delta}{\delta x^1}\big|_u, \dots, \frac{\delta}{\delta x^n}\big|_u, F\frac{\partial}{\partial y^1}\big|_u, \dots, F\frac{\partial}{\partial y^n}\big|_u \right\}, \qquad \frac{\delta}{\delta x^j}\big|_u = \frac{\partial}{\partial x^j}\big|_u - N_j^i \frac{\partial}{\partial y^i}\big|_u;$$

where the non-linear connection coefficients $N_j^i$ must be specified. The first $n$ elements develop the horizontal subspace $\mathcal{H}_u$ while the second half the vertical subspace $\mathcal{V}_u$. Similarly, for the cotangent space $\mathbf{T}_u^*\mathbf{N}$ a dual basis is defined by

$$\left\{ \mathrm{d}x^1\big|_u, \dots, \mathrm{d}x^n\big|_u, \frac{\delta y^1}{F}\big|_u, \dots, \frac{\delta y^n}{F}\big|_u \right\}, \qquad \frac{\delta y^i}{F}\big|_u = \frac{1}{F}(\mathrm{d}y^i + N_j^i\,\mathrm{d}x^j)\big|_u.$$

The manifold $\pi^*\mathbf{TM}$ is a subset of the cartesian product $\mathbf{TM} \times \mathbf{N}$. One has the pull-back bundle $\pi^*\mathbf{TM} \to \mathbf{N}$ given by the square

$$
\begin{array}{ccc}
\pi^*\mathbf{TM} & \xrightarrow{\ \pi_2\ } & \mathbf{TM} \\
{\scriptstyle \pi_1}\downarrow & & \downarrow{\scriptstyle \pi} \\
\mathbf{N} & \xrightarrow{\ \pi\ } & \mathbf{M}
\end{array}
$$

The projection on the first and second factors are

$$\pi_1 \colon \pi^*\mathbf{TM} \longrightarrow \mathbf{N}, \qquad (u, \xi) \longrightarrow u,$$
$$\pi_2 \colon \pi^*\mathbf{TM} \longrightarrow \mathbf{TM}, \qquad (u, \xi) \longrightarrow \xi, \quad \xi \in \pi_1^{-1}(u).$$

Every vector field $Y$ over $\mathbf{M}$ can be interpreted as a section of the tangent bundle $\mathbf{TM} \to \mathbf{M}$ and has associated a section $\pi^*Y$ of the vector bundle $\pi^*\mathbf{TM} \to \mathbf{N}$. In local coordinates, the associated $\pi^*Y$ to $Y$ is given in the following way:

$$Y = Y^i(x)\frac{\partial}{\partial x^i} \longrightarrow \pi^*Y = Y^i(x)\pi^*\frac{\partial}{\partial x^i}, \qquad \pi_2\left(\pi^*\frac{\partial}{\partial x^i}\right) = \frac{\partial}{\partial x^i}.$$

We also use the following lifted fundamental tensor (or fiber metric):

$$\pi^*g(x, y) := \frac{1}{2}\frac{\partial^2 F^2(x, y)}{\partial y^i \partial y^j}\,\pi^*\,\mathrm{d}x^i \otimes \pi^*\,\mathrm{d}x^j.$$

**Definition 2** *Let $(\mathbf{M}, F)$ be a Finsler structure. The Cartan tensor is defined by*

$$A(x, y) := \frac{F}{2}\frac{\partial g_{ij}}{\partial y^k}\frac{\delta y^i}{F} \otimes \mathrm{d}x^j \otimes \mathrm{d}x^k = A_{ijk}\frac{\delta y^i}{F} \otimes \mathrm{d}x^j \otimes \mathrm{d}x^k.$$

One possible non-linear connection is introduced by defining the non-linear connection coefficients as

$$\frac{N^i_j}{F} = \gamma^i_{jk}\frac{y^k}{F} - A^i_{jk}\gamma^k_{rs}\frac{y^r}{F}\frac{y^s}{F}, \qquad i,j,k,r,s = 1,\ldots,n.$$

The coefficients $\gamma^i_{jk}$ are defined in local coordinates by

$$\gamma^i_{jk} = \frac{1}{2}g^{is}\left(\frac{\partial g_{sj}}{\partial x^k} - \frac{\partial g_{jk}}{\partial x^s} + \frac{\partial g_{sk}}{\partial x^j}\right), \qquad i,j,k,s = 1,\ldots,n;$$

$A^i_{jk} = g^{il}A_{ljk}$ and $g^{il}g_{lj} = \delta^i_j$.

As we have said, there is not a Levi-Civita connection associated to the Finsler structure. However, there are several connections that one can define and that play a similar role to the Levi-Civita connection in Riemannian geometry. One of these connections is Chern's connection, which introduced through the following theorem ([1, p. 38]),

**Theorem 1** *Let $(\mathbf{M}, F)$ be a Finsler structure. The pull-back vector bundle $\pi^*\mathbf{TM} \to \mathbf{N}$ admits a unique linear connection determined by the connection 1-forms $\{\omega^i_j, \ i,j = 1,\ldots,n\}$ such that the following structure equations hold:*

1. *Torsion free condition,*

$$\mathrm{d}(\mathrm{d}x^i) - \mathrm{d}x^j \wedge w^i_j = 0, \qquad i,j = 1,\ldots,n. \tag{1}$$

2. *Almost g-compatibility condition,*

$$\mathrm{d}g_{ij} - g_{kj}w^k_i - g_{ik}w^k_j = 2A_{ijk}\frac{\delta y^k}{F}, \qquad i,j,k = 1,\ldots,n,$$

*where $A_{ijk}$ are the components of the Cartan tensor.*

A coordinate invariant characterization of Chern's connection is given by the following two propositions,

**Proposition 1** *Let $(\mathbf{M}, F)$ be a Finsler structure. Then the almost g-compatibility condition of the Chern's connection is equivalent to the conditions*

$$\nabla^{\mathrm{ch}}_{V(\tilde{X})}\pi^*g = 2A(X,\cdot,\cdot), \tag{2}$$

$$\nabla^{\mathrm{ch}}_{H(\tilde{X})}\pi^*g = 0, \tag{3}$$

*where $V(\tilde{X})$ is the vertical component and $H(\tilde{X})$ the horizontal component of $\tilde{X} \in \mathbf{T}_u\mathbf{N}$.*

PROOF. We write the above equations in local coordinates,

$$\nabla^{\mathrm{ch}}\pi^*g = \mathrm{d}(g_{ij}(x,y))\,\mathrm{d}x^i \otimes \mathrm{d}x^j + g_{ij}\nabla^{\mathrm{ch}}(\mathrm{d}x^i) \otimes \mathrm{d}x^j + g_{ij}\,\mathrm{d}x^i \otimes \nabla^{\mathrm{ch}}\,\mathrm{d}x^j$$

$$= \frac{\partial g_{ij}}{\partial y^k}\delta y^k \otimes \mathrm{d}x^i \otimes \mathrm{d}x^j + (-g_{lj}\omega^l_{ik} - g_{li}\omega^l_{jk} + \frac{\delta g_{ij}}{\delta x^k})\,\mathrm{d}x^k \otimes \mathrm{d}x^i \otimes \mathrm{d}x^j.$$

∎

**Proposition 2** *Let* $(\mathbf{M}, F)$ *be a Finsler structure. The torsion-free condition of the Chern connection is equivalent to the following conditions*

1. *Null vertical covariant derivative of sections of* $\pi^*\mathbf{TM}$: *let* $\tilde{X} \in \mathbf{T}_u\mathbf{N}$ *and* $Y \in \mathbf{\Gamma M}$, *then*

$$\nabla^{ch}_{V(\tilde{X})}\pi^*Y = 0. \tag{4}$$

2. *Let us consider* $X, Y \in \mathbf{TM}$ *and their horizontal lifts* $\tilde{X}$ *and* $\tilde{Y}$. *Then*

$$\nabla^{ch}_{\tilde{X}}\pi^*Y - \nabla^{ch}_{\tilde{Y}}\pi^*X - \pi^*([X,Y]) = 0. \tag{5}$$

PROOF. The expression (5) defines a section of the bundle $\pi^*\mathbf{TM}$ due to the commutator term, as well as (4). Therefore, it is only necessary to write the above equations in local coordinates: the commutator term is zero when the vectors are $X = \frac{\partial}{\partial x^i}, Y = \frac{\partial}{\partial x^j}$. Then

$$\nabla^{ch}_{\frac{\partial}{\partial x^i}|_u}\pi^*\frac{\partial}{\partial x^j} - \nabla^{ch}_{\frac{\partial}{\partial x^j}|_u}\pi^*\frac{\partial}{\partial x^i} = (\Gamma^l_{ij} - \Gamma^l_{ji})\pi^*\frac{\partial}{\partial x^l} = 0,$$

because due to equation (1), one has $\Gamma^i_{jk} = \Gamma^i_{kj}$ ([1, p. 39]). The result follows from the characterization of the Chern connection. ∎

**Definition 3** *A Berwald space is a Finsler structure such that the coefficients of the Chern's connection live on* $\mathbf{M}$.

The non-linear connection of the Cartan type is constructed in the following way ([7]). By Finsler geodesic we mean the parameterized curves in $\mathbf{M}$ that are extremal of the Finsler functional arc-length. They are solutions of the differential equations (in the case of unit parameterized Finslerian geodesics)

$$\frac{\mathrm{d}^2x^i}{\mathrm{d}s^2} + \gamma^i_{jk}(x,y)\frac{\mathrm{d}x^k}{\mathrm{d}s}\frac{\mathrm{d}x^j}{\mathrm{d}s} = 0, \qquad i,j,k = 1,\ldots,n.$$

The associated spray coefficients are

$$G^i := (\gamma^i_{sk}(x,y)y^k y^s), \qquad i,s,k = 1,\ldots,n.$$

The connection coefficients of the non-linear Cartan connection are given by the derivative of the spray coefficients,

$$^C N^i_j = \frac{1}{2}\frac{\partial^2}{\partial y^j}(\gamma^i_{sk}(x,y)y^k y^s).$$

Using this non-linear connection on $\mathbf{TN} \longrightarrow \mathbf{N}$ we can define the linear Berwald connection through the following propositions:

**Proposition 3** *Let* $(\mathbf{M}, F)$ *be a Finsler structure. Then the g-compatibility condition of the Berwald connection is:*

$$\nabla^b_{V(X)}\pi^*g = 2A(X,\cdot,\cdot),$$

$$\nabla^b_{H(X)}\pi^*g = -2\nabla^b_l A(\cdot,\cdot,X), \qquad l = \frac{y^i}{F}\frac{\partial}{\partial x^i}.$$

**Proposition 4** *Let* $(\mathbf{M}, F)$ *be a Finsler structure. Then the Berwald connection is torsion free:*

1. *Null vertical covariant derivative of sections of* $\pi^*(\mathbf{TM})$: *let* $\tilde{X} \in \mathbf{T}_u\mathbf{N}$ *and* $Y \in \pi^*(\mathbf{\Gamma M})$, *then*

$$\nabla^b_{V(\tilde{X})}\pi^*Y = 0.$$

2. *Let us consider* $X, Y \in \mathbf{TM}$ *and the associated vector fields with horizontal components* $X^i$ *and* $Y^i$, $\tilde{X}$ *and* $\tilde{Y}$. *Then*

$$\nabla^b_{\tilde{X}}\pi^*Y - \nabla^b_{\tilde{Y}}\pi^*X - \pi^*([X,Y]) = 0.$$

In the case of Berwald structures, the Chern's connection and the Berwald connection coincide.

## 3  Averaged Connection

We introduced in [2] a method to obtain a linear connection over $\mathbf{M}$ from the Chern connection on $\pi^*\mathbf{TM}$. The structure of this averaged is natural: it is defined using only canonical maps and the given Finsler structure. It is not unique because depend on the measured used and also the sub-manifold $\Sigma_x$ where we perform the integration. Also one can see that the covariant derivative associated with the averaged connection is the limit of a convex sum of covariant derivatives in different directions of the tangent space $\mathbf{T}_x\mathbf{M}$. However, since they are all points on the fiver $\pi^{-1}(x) \subset \mathbf{N}$, the convex sum of covariant derivatives is still covariant under the structure group $\mathbf{GL}(2,\mathbf{R})$. Therefore the averaged operation can be seen as a homomorphism between $\mathrm{Conn}(\pi^*\mathbf{TM})$, the space of linear connections on $\pi^*\mathbf{TM}$ and $\mathrm{Conn}(\mathbf{TM})$, the space of affine connections on $\mathbf{TM}$,

$$\langle\cdot\rangle\colon \mathrm{Conn}(\pi^*\mathbf{TM}) \longrightarrow \mathrm{Conn}(\mathbf{TM})$$
$$\nabla \longrightarrow \langle\nabla\rangle.$$

The averaged connection was introduced in reference [2]. We review briefly this construction. The proof can be found in the original reference [2], although for convenience of the reader we indicate the basics steps here too.

Let $\pi^*$, $\pi_1$, $\pi_2$ be the canonical projections of the pull-back bundle $\pi^*\mathbf{\Gamma M}$, being $\mathbf{\Gamma M}$ a tensor bundle over $\mathbf{M}$:

$$
\begin{array}{ccc}
\pi^*\mathbf{\Gamma M} & \xrightarrow{\ \pi_2\ } & \mathbf{\Gamma M} \\
{\scriptstyle \pi_1}\downarrow & & \downarrow{\scriptstyle \pi} \\
\mathbf{N} & \xrightarrow{\ \pi\ } & \mathbf{M}
\end{array}
$$

$\pi_u^*\mathbf{\Gamma M}$ denotes the fiber over $u \in \mathbf{N}$ of the bundle $\pi^*\mathbf{\Gamma M}$ and $\mathbf{\Gamma}_x\mathbf{M}$ are the tensors over $x \in \mathbf{M}$, being $S_x \in \mathbf{\Gamma}_x\mathbf{M}$ a generic element of $\mathbf{\Gamma}_x\mathbf{M}$. $S_u$ is the evaluation of the section $S$ of the bundle $\pi^*\mathbf{\Gamma M}$ at the point $u \in \mathbf{N}$. The indicatrix at the point $x \in \mathbf{M}$ is the compact submanifold

$$\mathbf{I}_x := \{\, y \in \mathbf{T}_x\mathbf{M} \mid F(x,y) = 1 \,\} \subset \mathbf{T}_x\mathbf{M}.$$

Let us consider the element $S_u \in \pi_u^*\mathbf{\Gamma M}$ and the tangent vector field $\tilde{X}$ of the horizontal path $\tilde{\gamma}\colon [0,1] \to \mathbf{N}$ connecting the points $u \in \mathbf{I}_x$ and $v \in \mathbf{I}_z$. The parallel transport of the Chern connection along $\tilde{\gamma}$ of a section $S \in \pi^*\mathbf{TM}$ is denoted by $\tau_{\tilde{\gamma}}S$; the parallel transport along $\tilde{\gamma}$ of the point $u \in \mathbf{I}_x$ is by definition $\tau_{\tilde{\gamma}}(u) = \tilde{\gamma}(1) \in \pi^{-1}(z)$; the horizontal lift of a path is defined using the non-linear connection in $\mathbf{N}$.

The following is a standard result, although a simpler proof can also be found in [2],

**Proposition 5** *Let $(\mathbf{M}, F)$ be a Finsler structure and let $\tilde{\gamma}\colon [0,1] \to \mathbf{N}$ be the horizontal lift of a path $\gamma\colon [0,1] \to \mathbf{M}$ joining $x$ and $z$ points in $\mathbf{M}$. Then $F(x,y)$ is invariant by the horizontal parallel transport of the Chern's connection. In particular, let us consider the indicatrixes over $x$ and $z$ $\mathbf{I}_x \subset \mathbf{T}_x\mathbf{M}$ and $\mathbf{I}_z \subset \mathbf{T}_z\mathbf{M}$. Then $\tau_{\tilde{\gamma}}\pi^*(\mathbf{I}_x) = \pi^*\mathbf{I}_z$. Therefore the horizontal parallel transport maps $\mathbf{I}_x$ into $\mathbf{I}_z$ as submanifolds of $\mathbf{N}$.*

PROOF.  Let $\tilde{X}$ be the horizontal lift in $\mathbf{TN}$ of the tangent vector field $X$ along the path $\gamma \subset \mathbf{M}$ joining $x$ and $z$, $S_1, S_2 \in \pi^*(\mathbf{T}_x\mathbf{M})$. Then proposition 1 implies $\nabla_{\tilde{X}}g(S_1, S_2) = 2A(\tilde{X}, S_1, S_2) = 0$ because the vector field $\tilde{X}$ is horizontal and the Cartan tensor is evaluated in the first argument. Therefore the value of the Finslerian norm $F(x,y) = \sqrt{g_{ij}(x,y)y^iy^j}$, $y \in \mathbf{T}_x\mathbf{M}$, $Y$ with $Y = \pi^*y$ is conserved by horizontal parallel transport,

$$\nabla_{\tilde{X}}(F^2(x,y)) = \nabla_{\tilde{X}}(g(x,y))(Y,Y) + 2g(x, \nabla_{\tilde{X}}Y) = 0,$$

being $\tilde{X} \in \mathbf{TN}$ an horizontal vector. The first term is zero because the above calculation. The second term is zero because of the definition of parallel transport of sections $\nabla_{\tilde{X}}Y = 0$. In particular the indicatrix $\mathbf{I}_x$ is mapped to $\mathbf{I}_z$ because parallel transport is a diffeomorphism.  ∎

**Remark 1** *A similar statement also holds for the linear Cartan connection $\nabla^c$ because it is a g-compatible connection. For the linear Berwald connection $\nabla^b$ the result is not true for general Finsler structure, because it is not g-compatible. However, in the case of Berwald structure Proposition 4 holds for $\nabla^b$ because both Cartan and Berwald connections coincide.*

Let us consider $\pi_v^* \mathbf{\Gamma M}$ a fiber over $v \in \mathbf{N}$ and the tensor space over $x$, the fiber $\mathbf{\Gamma}_x \mathbf{M}$. For each $S_x \in \mathbf{\Gamma}_x \mathbf{M}$ and $v \in \pi^{-1}(z)$, $z \in \mathbf{U}$ we consider the isomorphisms

$$\pi_2|_v \colon \pi_v^* \mathbf{\Gamma M} \longrightarrow \mathbf{\Gamma}_z \mathbf{M}, \qquad S_v \longrightarrow S_z$$
$$\pi_v^* \colon \mathbf{\Gamma}_z \mathbf{M} \longrightarrow \pi_v^* \mathbf{\Gamma M}, \qquad S_z \longrightarrow \pi_v^* S_z.$$

**Definition 4** *Let $(\mathbf{M}, F)$ be a Finsler structure, $\pi(u) = x$ and $f \in \mathcal{F}\mathbf{M}$. Then $\pi^* f \in \mathcal{F}(\pi^* \mathbf{TM})$ is defined by*

$$\pi_u^* f = f(x), \qquad \forall u \in \mathbf{I}_x \subset \pi^{-1}(x) \subset \mathbf{N}.$$

Let us denote the horizontal lifted operator in the following way:

$$\iota \colon \mathbf{TM} \longrightarrow \mathbf{TN}, \quad X = X^i \frac{\partial}{\partial x^i}|_x \longrightarrow \iota(X) = X^i \frac{\delta}{\delta x^i}|_u := X^i \delta_i, \qquad u \in \mathbf{I}_x \subset \pi^{-1}(x) \subset \mathbf{N},$$

and the horizontal lift, defined by the non-linear connection $N_j^i$,

$$\iota \colon \mathbf{TM} \longrightarrow \mathbf{TN}, \qquad \iota(X) = \tilde{X}, \mid \tilde{X} \in \mathcal{H},$$

such that if $\rho \colon \mathbf{TN} \to \mathbf{N}$ is the canonical projection, $\pi \cdot \rho(\iota(X)) = X$ for $X \neq 0$.

**Definition 5** *Consider a family of operators $A_w := \{A_w \colon \pi_w^* \mathbf{TM} \longrightarrow \pi_w^* \mathbf{TM}\}$ with $w \in \pi^{-1}(x) \subset \mathbf{N}$. The average of this family is another operator $A_x \colon \mathbf{T}_x \mathbf{M} \to \mathbf{T}_x \mathbf{M}$ with $x \in \mathbf{M}$ given by the action on arbitrary sections $S \in \Gamma \mathbf{TM}$ by the point-wise formula*

$$\langle A_w \rangle := \langle \pi_2|_u A \, \pi_u^* \rangle_u S_x = \frac{1}{\mathrm{vol}(\mathbf{I}_x)} \left( \int_{\mathbf{I}_x} \pi_2|_u A_u \pi_u^* \, \mathrm{dvol} \right) S_x, \qquad u \in \pi^{-1}(x), \quad S_x \in \mathbf{\Gamma}_x \mathbf{M}. \quad (6)$$

*The volume form on $\mathrm{dvol}$ is the standard volume form induced from the indicatrix $\mathbf{I}_x$ from the Riemannian volume of the Riemannian structure $(\mathbf{T}_x \mathbf{M} - \{0\}, g_x)$.*

**Proposition 6 (Averaged connection of a Finsler connection [2])** *Let $(\mathbf{M}, F)$ be a Finsler structure and let us consider a linear connection $\nabla$ on $\pi^* \mathbf{TM}$. Then, there is defined on $\mathbf{M}$ a linear covariant derivative along $X$, $\tilde{\nabla}_X$ characterized by the conditions:*

1. *$\forall X \in \mathbf{T}_x \mathbf{M}$ and $Y \in \Gamma \mathbf{TM}$, the covariant derivative of $Y$ in the direction $X$ is given by the following average operation:*

   $$\tilde{\nabla}_X Y = \langle \nabla \rangle_X Y := \langle \pi_2|_u \nabla_{\iota_u(X)} \pi_u^* Y \rangle_u, \qquad u \in \mathbf{I}_x \subset \pi^{-1}(x) \subset \mathbf{N},$$

2. *For every smooth function $f \in \mathcal{F}(\mathbf{M})$ the covariant derivative is given by the following average:*

   $$\tilde{\nabla}_X f = \langle \nabla \rangle_X f := \langle \pi_2|_u \nabla_{\iota_u(X)} \pi_u^* f \rangle_u, \qquad u \in \mathbf{I}_x \subset \pi^{-1}(x) \subset \mathbf{N}.$$

PROOF. There is a complete proof in reference [2, section 4] of this fact. It consists on checking that effectively $\langle \nabla \rangle$ is a covariant derivative. Here we provide a different argument. This argument also holds for different averages, like the one used in [9] or the one used more recently in [5].

The argument follows in the following way. Consider a convex sum of linear connections $t_1 \nabla_1 + \cdots + t_p \nabla_p$ such that $t_1 + \cdots + t_p = 1$; the connections are linear connections on $\mathbf{M}$. It is well known that

$t_1\nabla_1 + \cdots + t_p\nabla_p$ is also a linear connection. Now, consider a compact manifold $\Sigma_x \subset \pi^{-1}(X) \subset \mathbf{N}$ and a set of connections on $\mathbf{M}$, all of them labelled by points on $\Sigma$, so there is a map $\Theta\colon \mathbf{M} \to \mathrm{Mod}(\mathbf{TM})$ such that $\int_\Sigma \Theta = 1$ and that $\Theta \geq 0$. Then, using a limit argument of the convex sum of linear connections on $\mathbf{M}$, we have that the averaged of the family of connections $\{\nabla_u\}$ defines also a linear connection on $\mathbf{M}$. To apply to our case this argument, we only need to specify that $\Sigma_x = \mathbf{I}_x$ and that $\Theta(u) = \pi_2|_u \nabla_{\iota_u} \pi^*$, where the right hand side must be understood for fixed $u \in \mathbf{I}_x$ and as acting on sections of $\Gamma\mathbf{M}$. ∎

Let $\nabla$ be a linear connection on $\pi^*\mathbf{TM}$. Then the generalized torsion operator acting on the vector fields $X, Y \in \mathbf{TM}$ is

$$\mathrm{Tor}_u(\nabla)\colon \mathbf{T}_x\mathbf{M} \times \Gamma\mathbf{TM} \longrightarrow \pi_u^*\mathbf{TM}$$

$$\mathrm{Tor}_u(\nabla)(X,Y) = \nabla_{\iota_u(X)}\pi_u^*Y - \nabla_{\iota_u(Y)}\pi_u^*X - \pi_u^*[X,Y], \qquad \forall u \in \mathbf{N}.$$

Since this definition is point-wised, we can define globally the $\mathrm{Tor}(\nabla)$ as the family of maps defined as before.

**Proposition 7** *Let $(\mathbf{M},F)$ be a Finsler structure and let us define a linear connection $\nabla$ with $\mathrm{Tor}(\nabla) = 0$. Then the torsion $\mathrm{Tor}(\tilde{\nabla})$ of the average connection is zero.*

PROOF. As with the proposition before, there is a proof in [2, section 4]; it is just a calculation. However, one can see that the proof is rather direct from the definition of torsion and from the fact that convex sum of linear connections define a linear connection. ∎

# 4 A rigidity property of Berwald Spaces

We start considering a generalization of some well known properties of linear connections over $\mathbf{M}$ ([4, section 5.4]) to linear connections defined on the bundle $\pi^*\mathbf{TM} \to \mathbf{N}$.

Given two linear connections $\nabla_1$ and $\nabla_2$ on the bundle $\pi^*\mathbf{TM} \to \mathbf{N}$, the difference operator

$$B\colon \mathbf{HN} \otimes \pi^*\Gamma\mathbf{TM} \longrightarrow \pi^*\Gamma\mathbf{TM}$$

$$B(\iota_u(X), \pi_u^*Y) = {}^1\nabla_{\iota_u(X)}\pi_u^*Y - {}^2\nabla_{\iota_u(X)}\pi^*Y, \qquad \forall u \in \mathbf{N}, \quad X, Y \in \Gamma\mathbf{TM}$$

is an homomorphism that holds the Leibnitz rule. It is essential in this definition that we have to our disposition a non-linear connection to define the horizontal lift $\iota_u X$.

The symmetric and skew-symmetric parts $S$ and $A$ of B are defined in the following way

$$S_u\colon \mathbf{T}_x\mathbf{M} \times \Gamma\mathbf{TM} \longrightarrow \pi_u^*\mathbf{TM}$$

$$S_u(X,Y) := \frac{1}{2}\big(B(\iota_u X, \pi_u^*Y) + B(\iota_u Y, \pi_u^*X)\big). \qquad \forall u \in \pi^{-1}(x), \quad X \in \mathbf{T}_x\mathbf{M}, \quad Y \in \Gamma\mathbf{TM}.$$

The antisymmetric part $A$ is defined in a similar way,

$$A_u\colon \mathbf{T}_x\mathbf{M} \times \Gamma\mathbf{TM} \longrightarrow \pi_u^*\mathbf{TM}$$

$$A_u(X,Y) := \frac{1}{2}\big(B(\iota_u X, \pi^*Y) - B(\iota_u Y, \pi^*X)\big), \qquad \forall u \in \pi^{-1}(x), \quad X \in \mathbf{T}_x\mathbf{M}, \quad Y \in \Gamma\mathbf{TM}.$$

As for the torsion, one can define the symmetric and skew-symmetric parts $S$ and $A$ as a family of operators, because the above definitions are point-wise.

Consider to vector fields $X$ and $Y$ on $\mathbf{M}$ such that $[X,Y] = 0$. Then, the following relation holds:

$$2A_u(X,Y) = \nabla_{1(\iota_u(X))}\pi_u^*Y - \nabla_{2(\iota_u(X))}\pi_u^*Y - (\nabla_{1(\iota_u(Y))}\pi_u^*X - \nabla_{2(\iota_u(Y))}\pi_u^*X)$$
$$= \mathrm{Tor}_u(\nabla_1)(X,Y) - \mathrm{Tor}_u(\nabla_2)(X,Y).$$

Since this relation holds point-wise for all $u \in \pi^{-1}(x) \subset \mathbf{N}$ we can write

$$2A(X,Y) = \mathrm{Tor}(\nabla_1)(X,Y) - \mathrm{Tor}(\nabla_2)(X,Y).$$

**Definition 6** *Let $\nabla$ be a linear connection on the vector bundle $\pi^*\mathbf{TM} \longrightarrow \mathbf{N}$ with connection coefficients $\Gamma^i_{jk}$. The geodesics of $\nabla$ are the solutions of the differential equations*

$$\frac{\mathrm{d}^2 x^i}{\mathrm{d}s^2} + \Gamma^i_{jk}\left(x, \frac{\mathrm{d}x}{\mathrm{d}s}\right) \frac{\mathrm{d}x^j}{\mathrm{d}s}\frac{\mathrm{d}x^k}{\mathrm{d}s} = 0, \qquad i, j, k = 1, \ldots, n,$$

*where $\Gamma^i_{jk}$ are the connection coefficients of $\nabla$.*

This differential equation can be written as

$$\nabla_{\iota_u(X)}\pi^*_u X = 0, \qquad u = \frac{\mathrm{d}x}{\mathrm{d}t}$$

being $X$ the unit tangent vector to the solution in the given point. In order to check equation (6) one uses local coordinates.

The following propositions are direct generalizations of the analogous results for affine connections over $\mathbf{M}$ ([4]).

**Proposition 8** *Let us consider two linear connections $\nabla_1$ and $\nabla_2$ on the vector bundle $\pi^*\mathbf{TM} \to \mathbf{N}$ such that the covariant derivative along vertical directions are zero. Then the following conditions are equivalent:*

1. *The connections $\nabla_1$ and $\nabla_2$ have the same geodesic curves in $\mathbf{M}$.*

2. *$B(X, X) = 0$, where $B = \nabla_1 - \nabla_2$.*

3. *$S = 0$.*

4. *$B = A$.*

The proof follows the lines of reference [3, pp. 64–65] and it is omitted here. However we should mention that the equivalence of the first statement and the other requires that the covariant derivative of sections along vertical directions must be zero. This condition allows to define geodesics in the way we did, being independent of the derivative of sections of $\pi^*\mathbf{TM}$ along vertical directions in $\mathbf{TN}$ and in this sense being independent of type of lift, as soon as we have a complete horizontal lift.

**Proposition 9** *Let $\nabla_1$ and $\nabla_2$ be linear connections on the vector bundle $\pi^*\mathbf{TM} \to \mathbf{N}$ such that they have null covariant derivative in vertical directions. Then $\nabla_1 = \nabla_2$ if and only if they have the same parameterized geodesics and $\mathrm{Tor}(\nabla_1) = \mathrm{Tor}(\nabla_2)$.*

PROOF. If $\nabla_1$ and $\nabla_2$ have the same geodesics, they have the same symmetric part (the geodesic flow determines the symmetric part of a connection). If they have the same torsion, then $A = 0$. ∎

Let us consider the bundles $\pi^*\mathbf{TM} \to \mathbf{N}$ and the tangent bundle $\mathbf{TM} \to \mathbf{M}$ endowed with a linear connection $\nabla$. The horizontal lift of $\nabla$ (or pull-back connection, ([8, p. 57])) is a connection on $\pi^*\mathbf{TM} \to \mathbf{N}$ defined by

$$(\pi^*\nabla)_{\iota(X)}\pi^*S = \pi^*(\nabla_X S), \qquad \tilde{X} \in \mathbf{TM}.$$

One can show, writing the geodesic equation in local coordinates, that the parameterized geodesics of both connections $\pi^*\nabla$ and $\nabla$ are the same,

$$(\pi^*\nabla)_{\iota_u(X)}\pi^*_u X = 0 \Longleftrightarrow \nabla_X X = 0,$$

because the possibly non-zero connection coefficients are the same:

$$\nabla_{\partial_j}\partial_k = \Gamma^i_{jk}\partial_i \Rightarrow \pi^*\nabla_{\delta_j}\pi^*\partial_k = \pi^*(\Gamma^i_{jk}\partial_i) = (\Gamma^i_{jk}\pi^*\partial_i).$$

**Proposition 10** *Let $\nabla^{\mathrm{ch}}$ be the Chern connection of a Finsler structure $(\mathbf{M}, F)$, $\nabla^b$ the linear Berwald connection and consider the average connection $\langle \nabla^{\mathrm{ch}} \rangle$. Then*

1. *The structure is Berwald if and only if $\pi^* \langle \nabla^{\mathrm{ch}} \rangle = \nabla^{\mathrm{ch}}$.*

2. *If $\pi^* \langle \nabla^b \rangle = \nabla^b$, the structure is Berwald.*

PROOF.    If $\pi^* \langle \nabla^{\mathrm{ch}} \rangle = \nabla^{\mathrm{ch}}$, since the induced horizontal connection $\pi^* \langle \nabla^{\mathrm{ch}} \rangle$ has the same coefficients that $\langle \nabla^{\mathrm{ch}} \rangle$ and they live on $\mathbf{M}$, the structure $(\mathbf{M}, F)$ is Berwald.

Let us suppose that the structure is Berwald. Then $\pi^* \langle \nabla^{\mathrm{ch}} \rangle = \pi^* \langle 1 \rangle \nabla^{\mathrm{ch}} = \nabla^{\mathrm{ch}}$. This relation is checked writing the action of the average covariant derivative on arbitrary vector sections.

An alternative proof of is the following. We know that $\mathrm{Tor}(\nabla^{\mathrm{ch}}) = \mathrm{Tor}(\langle \nabla^{\mathrm{ch}} \rangle) = 0$. On the other hand, the parameterized geodesics of $\pi^* \langle \nabla^{\mathrm{ch}} \rangle$ are the same than the geodesics of $\langle \nabla^{\mathrm{ch}} \rangle$. But if the space is Berwald, the geodesic equation of $\langle \nabla^{\mathrm{ch}} \rangle$ are the same than the geodesic equation of $\nabla^{\mathrm{ch}}$. From this fact it follows $\pi^* \langle \nabla^{\mathrm{ch}} \rangle = \nabla^{\mathrm{ch}}$.

To proof the second statement we follow a similar reasoning. If $\pi^* \langle \nabla^b \rangle = \nabla^b$, the Berwald connection lives on $\mathbf{M}$ and therefore the structure is Berwald.    ■

**Proposition 11** *Let $(\mathbf{M}, F)$ be a Finsler structure. Then there is an affine equivalent Riemannian structure $(\mathbf{M}, h)$ if and only if the structure is Berwald.*

PROOF.    If there is an affine equivalence Riemannian structure $h$ such that its Levi-Civita connection $\nabla^h$ has the same parameterized geodesics as the linear Berwald connection $\nabla^b$ and both connection have also null torsion, then both connections are the same ([4, section 5.4]) and since the connection coefficients $^h\Gamma^i_{ij}$ live in $\mathbf{M}$, the structure is Berwald.  Conversely, if $(\mathbf{M}, F)$ is Berwald, its Berwald connection is metrizable ([9]).    ■.

Recall that for Berwald spaces $\nabla^b = \nabla^{\mathrm{ch}}$. Then,

**Proposition 12** *Let $(\mathbf{M}, F)$ be a Berwald structure. Then any Riemannian $h$ on $\mathbf{M}$ such that $\nabla^b \pi^* h = 0$ then $\nabla^h$ leaves invariant the indicatrix under horizontal parallel transport.*

PROOF.    If the Riemannian structure is conserved by the Berwald connection, $\nabla^b \pi^* h = 0$. This implies that $\langle \nabla^b \rangle h = 0$. In addition, $\langle \nabla^b \rangle$ is torsion free. Therefore, $\langle \nabla^b \rangle = \nabla^h$. If $\nabla^b$ leaves invariant the indicatrix, also $\pi^* \langle \nabla^b \rangle = \nabla^h$ leaves invariant the structure.    ■

There is a converse of this result,

**Proposition 13** *Let $(\mathbf{M}, F)$ be a Finsler structure. Then if there is a Riemannian metric $h$ that leaves invariant the indicatrix under the parallel transport pull-back of its Levi-Civita connection $\pi^* \nabla^h$, the structure is Berwald.*

PROOF.    Let us consider such Riemannian metric $h$ and the associated Levi-Civita connection $\nabla^h$. The induced connection $\pi^* \nabla^h$ is torsion free and its connection coefficients in natural coordinates live on $\mathbf{M}$. In addition, the averaged connection $\langle \pi^* \nabla^h \rangle$ coincides with $\nabla^h$, so $\pi^* \nabla^h = \pi^* \langle \pi^* \nabla^h \rangle = \nabla^b$, the last equality because $\pi^* \langle \pi^* \nabla^h \rangle$ leaves invariant the indicatrix and it is torsion-free. Therefore the result follows because the connection $\pi^* \langle \pi^* \nabla^h \rangle$ has coefficients living on $\mathbf{M}$.    ■

## 5   A corollary on pure Landsberg spaces

Let us consider a metric $h$ such that its parallel Riemannian transport leaves invariant the indicatrix of the Finsler metric $F$, following proposition 5. Then, let us consider the interpolating set of metrics

$$F_t(x,y) = (1-t)F(x,y) + t\sqrt{h(x)_{ij}y^i y^j}, \qquad i,j = 1,\ldots,n, \quad t \in [0,1]$$

and their indicatrix,

$$\mathbf{I}_x(t) := \{F_t(x,y) = 1,\, y \in \mathbf{T}_x\mathbf{M}\}.$$

Since the metric $F$ is Berwald, all the above interpolating metrics define indicatrix that are invariant under the Levi-Civita connection of $h$.

Let us consider the hypothesis that each of these indicatrix defines a submanifold of $\mathbf{T}_x\mathbf{M}$ of co-dimension 1 and that they are non-intersecting. Therefore the union of indicatrix $\{\mathbf{I}_x(t), \in [0,1]\}$ defines a submanifold of $\mathbf{T}_x\mathbf{M}$ of codimension 0 that is invariant under the holonomy of the metric $h$. This conditions are interesting for us because it help to provided a necessary criteria for pure Landsberg spaces,

**Definition 7** *A Finsler structure* $(\mathbf{M},F)$ *is a Landsberg space if the $hv$-curvature $P$ is such that $\dot{A}_{ijk} = P^n_{ijk} = 0$, where the vector field is defined as $e_n = y/F(y)$. A pure Landsberg space is such that it is Landsberg and it is not Berwald or locally Minkowski.*

This definition that we take of Landsberg space is a bit unusual, although can be obtained from the standard characterizations straightforward. In particular, Landsberg space is such that ([1, section 3.4]).

$$0 = \dot{A}_{ikl} = -l^j\, P_{jikl} = \tilde{l}_j\, P^j_{ikl} := P^n_{ikl}.$$

**Theorem 2** *Let* $(\mathbf{M},F)$ *be a Landsberg space and suppose that the averaged connection* $\langle\nabla^{\mathrm{ch}}\rangle$ *does not leave invariant any compact submanifolds* $\mathbf{I}_x(t) \subset \mathbf{T}_x\mathbf{M}$ *of codimension zero. Then the structure* $(\mathbf{M},F)$ *is a pure Landsberg space.*

PROOF.   Suppose that the Landsberg space is Berwald. Then we know from a theorem of Szabo that this linear Berwald connection is metrizable ([9]). Then, there is a Riemannian connection $\nabla^h$ that is identified with the average connection $\langle\nabla^{\mathrm{ch}}\rangle$ and this is in contradiction with the hypothesis of the theorem because $\pi^*\nabla^h = \pi^*\langle\nabla^{\mathrm{ch}}\rangle = \nabla^h$ leaves invariant the set of indicatrix $\mathbf{I}_x(t)$, $\forall t \in [0,1]$ as we show before, the union defining a submanifold of codimension zero of $\mathbf{T}_x\mathbf{M}$.   ■

In this theorem, the hypothesis of Landsberg metric $F$ can be substituted by a general Finsler metric. Therefore, Theorem 2 is essentially a criterion for not being Berwald.

**Application of the Theorem 2 in dimension 2.**   Let us consider the set of possible holonomy groups of affine free-torsion connections ([6]). Then we look for the holonomy groups that can leave invariant a compact, foliated manifold of dimension 2. The possible holonomy groups for averaged connection of pure Landsberg spaces should be excluded from this list. In particular, Riemannian holonomies are excluded. Since the torsion of the averaged connection is zero, the only candidates for the holonomy of the averaged connection in dimension 2 are of the form $T_{\mathbf{R}}\cdot\mathrm{SL}(2,\mathbf{R})$ for real representations, where $T_{\mathbf{R}}$ denotes any connected Lie subgroup of $\mathbf{R}$. The second possibility is the whole general group $\mathrm{GL}(2,\mathbf{R})$. From this family of groups, $\mathrm{SL}(2,\mathbf{R})$ and $\mathrm{GL}(2,\mathbf{R})$ are the candidate that can supply the additional Landsberg condition,

**Corollary 1** *Let* $(\mathbf{M},F)$ *be a two-dimensional Finsler structure such that the average connection is* $\langle\nabla^{\mathrm{ch}}\rangle$. *Then if the space is pure Landsberg, the holonomy group of* $\langle\nabla^{\mathrm{ch}}\rangle$ *is* $\mathrm{SL}(2,\mathbf{R})$ *or* $\mathrm{GL}(2,\mathbf{R})$.

This result provides a strategy to solve the problem of the existence of pure Landsberg spaces in dimension 2. We hope that future research could reveal the existence of pure Landsberg spaces, following the direction of Corollary 2.10 (see reference [3] for a suggestion of realization of this strategy).

A generalization of this strategy to higher dimensions can also be fruitful, but additional techniques are required, due to the growth of the possible holonomies.

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**Ricardo Gallego Torrome**
Department of Physics,
Lancaster University,
Lancaster, LA1 4YB, &
The Cockcroft Institute, UK,
United Kingdom
rgallegot@gmx.de

**Fernando Etayo**
Departamento de Matemáticas, Estadística y Computación,
Facultad de Ciencias, Universidad de Cantabria,
Avda. de los Castros, s/n,
39071 Santander,
Spain
etayof@unican.es