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Weighted boundedness for multilinear singular integral operators with non-smooth kernels on Morrey spaces

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Abstract In this paper, we prove some weighted boundedness for the multilinear operators related to some singular integral operators with non-smooth kernels on the generalized Morrey spaces by using a sharp function estimate of the multilinear operators.

Acotaciones ponderadas para operadores integrales singulares multilineales con núcleos no suaves en espacios de Morrey

Resumen. En este artículo se utiliza una función de estimación precisa de operadores multilineales con la que se prueban algunas acotaciones ponderadas para operadores multilineales relacionados con operadores integrales singulares con núcleos no suaves en espacios de Morrey generalizados.

1 Preliminaries and Main Result

As the development of the Calderón-Zygmund singular integral operators, their commutators and multilinear operators have been well studied (see [2–5, 7, 11]). In [14], Hu and Yang proved a variant sharp function estimate for the multilinear singular integral operators. In [18–20], C. Pérez, G. Pradolini and R. Trujillo-González obtained a sharp weighted estimates for the singular integral operators and their commutators. The main purpose of this paper is to study the multilinear singular integral operators with non-smooth kernels as follows.

Definition 1. *A family of operators D_t, $t > 0$ is said to be an* approximation to the identity *if, for every $t > 0$, D_t can be represented by a kernel $a_t(x, y)$ in the following sense:*

$$D_t(f)(x) = \int_{\mathbb{R}^n} a_t(x, y) f(y)\,\mathrm{d}y$$

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for every $f \in L^p(\mathbb{R}^n)$ *with* $p \geq 1$, *and* $a_t(x, y)$ *satisfies:*

$$|a_t(x, y)| \leq h_t(x, y) = Ct^{-n/2}s(|x - y|^2/t),$$

where s *is a positive, bounded and decreasing function satisfying*

$$\lim_{r \to \infty} r^{n+\epsilon}s(r^2) = 0$$

for some $\epsilon > 0$.

Definition 2. *A linear operator* T *is called a singular integral operator with non-smooth kernels if* T *is bounded on* $L^2(\mathbb{R}^n)$ *and associated with a kernel* $K(x, y)$ *such that*

$$T(f)(x) = \int_{\mathbb{R}^n} K(x, y)f(y)\,\mathrm{d}y$$

for every continuous function f *with compact support, and for almost all* x *not in the support of* f.

(1) There exists an "approximation to the identity" $\{B_t,\ t > 0\}$ such that TB_t has the associated kernel $k_t(x, y)$ and there exist c_1, $c_2 > 0$ so that

$$\int_{|x-y|>c_1 t^{1/2}} |K(x, y) - k_t(x, y)|\,\mathrm{d}x \leq c_2 \qquad \text{for all } y \in \mathbb{R}^n.$$

(2) There exists an "approximation to the identity" $\{A_t,\ t > 0\}$ such that A_tT has the associated kernel $K_t(x, y)$ which satisfies

$$|K_t(x, y)| \leq c_4 t^{-n/2} \qquad \text{if } |x - y| \leq c_3 t^{1/2},$$

and

$$|K(x, y) - K_t(x, y)| \leq c_4 t^{\delta/2}|x - y|^{-n-\delta} \qquad \text{if } |x - y| \geq c_3 t^{1/2},$$

for some c_3, $c_4 > 0$, $\delta > 0$.

Let m_j be the positive integers ($j = 1, \ldots, l$), $m_1 + \cdots + m_l = m$ and b_j be the functions on $\mathbb{R}^n$ ($j = 1, \ldots, l$). Set, for $1 \leq j \leq m$,

$$R_{m_j+1}(b_j; x, y) = b_j(x) - \sum_{|\alpha| \leq m_j} \frac{1}{\alpha!} D^\alpha b_j(y)(x - y)^\alpha.$$

The multilinear operator associated to T is defined by

$$T^b(f)(x) = \int_{\mathbb{R}^n} \frac{\prod_{j=1}^{l} R_{m_j+1}(b_j; x, y)}{|x - y|^m} K(x, y)f(y)\,\mathrm{d}y.$$

Note that when $m = 0$, T^b is just the multilinear commutator generated by T and b (see [19, 20]). While when $m > 0$, T^b is a non-trivial generalization of the commutator. It is well known that multilinear operators are of great interest in harmonic analysis and have been widely studied by many authors (see [2–5]). In [12, 15], the boundedness of the singular integral operator with non-smooth kernel are obtained. In [8], the boundedness of the commutator associated to the singular integral operator with non-smooth kernel are obtained. Our works is motivated by these papers.

Now, let us introduce some notations. Throughout this paper, $Q = Q(x, r)$ will denote a cube of $\mathbb{R}^n$ with sides parallel to the axes and center at x and edge is r. For any locally integrable function f, the sharp function of f is defined by

$$f^\#(x) = \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f(y) - f_Q|\,\mathrm{d}y,$$

where, and in what follows, $f_Q = |Q|^{-1}\int_Q f(x)\,\mathrm{d}x$. It is well-known that (see [13])

$$f^{\#}(x) \approx \sup_{Q\ni x}\inf_{c\in\mathbf{C}} \frac{1}{|Q|}\int_Q |f(y)-c|\,\mathrm{d}y$$

and

$$\|b - b_{2^kQ}\|_{\mathrm{BMO}} \le Ck\|b\|_{\mathrm{BMO}} \qquad \text{for } k \ge 1.$$

We say that f belongs to $\mathrm{BMO}(\mathbb{R}^n)$ if $f^{\#}$ belongs to $L^\infty(\mathbb{R}^n)$ and $|f\|_{\mathrm{BMO}} = |f^{\#}\|_{L^\infty}$. Let M be the Hardy-Littlewood maximal operator defined by

$$M(f)(x) = \sup_{Q\ni x}|Q|^{-1}\int_Q |f(y)|\,\mathrm{d}y,$$

and we write that $M_p(f) = (M(f^p))^{1/p}$ for $0 < p < \infty$. The sharp maximal function $M_A(f)$ associated with the "approximation to the identity" $\{A_t,\, t>0\}$ is defined by

$$M_A^{\#}(f)(x) = \sup_{x\in Q}\frac{1}{|Q|}\int_Q |f(y) - A_{t_Q}(f)(y)|\,\mathrm{d}y,$$

where $t_Q = l(Q)^2$ and $l(Q)$ denotes the side length of Q. We denote the Muckenhoupt weights by A_1, that is (see [13]):

$$A_1 = \{0 < w \in L^1_{\mathrm{loc}}(\mathbb{R}^n) : M(w)(x) \le Cw(x),\ a.e.\}.$$

Throughout this paper, φ will denote a positive, increasing function on $\mathbb{R}^+$ for which there exists a constant $D > 0$ such that

$$\varphi(2t) \le D\varphi(t) \qquad \text{for } t \ge 0.$$

Let w be a weight function on $\mathbb{R}^n$ (that is w is a non-negative locally integrable function) and f be a locally integrable function on $\mathbb{R}^n$. Set, for $1 \le p < \infty$,

$$\|f\|_{L^{p,\varphi}(w)} = \sup_{\substack{x\in\mathbb{R}^n,\\ d>0}}\left(\frac{1}{\varphi(d)}\int_{Q(x,d)}|f(y)|^p w(y)\,\mathrm{d}y\right)^{1/p}.$$

The generalized weighted Morrey spaces are defined by

$$L^{p,\varphi}(\mathbb{R}^n, w) = \{\, f \in L^1_{\mathrm{loc}}(\mathbb{R}^n) : \|f\|_{L^{p,\varphi}(w)} < \infty \,\}.$$

If $\varphi(d) = d^\delta$, $\delta > 0$, then $L^{p,\varphi}(\mathbb{R}^n, w) = L^{p,\delta}(\mathbb{R}^n, w)$, which is the classical Morrey space (see [1, 16, 17]). As the Morrey space may be considered as an extension of the Lebesgue space (the Morrey space $L^{p,\lambda}$ becomes the Lebesgue space L^p when $\lambda = 0$), it is natural and important to study the boundedness of the multilinear singular integral operator on the Morrey spaces $L^{p,\lambda}$ with $\lambda > 0$ (see [1, 9, 10]). The purpose of this paper is twofold. First, we establish a sharp inequality for the multilinear singular integral operator T^b with non-smooth kernels, and second, we prove the boundedness for the multilinear operator on the generalized weighted Morrey spaces by using the sharp inequality.

We shall prove the following theorem.

Theorem 1. *Suppose T is the singular integral operator with non-smooth kernels as Definition 2. Let $1 < p < \infty$, $0 < D < 2^n$, $w \in A_1$, $D^\alpha b_j \in \mathrm{BMO}(\mathbb{R}^n)$ for all α with $|\alpha| = m_j$ and $j = 1,\dots,l$. Then T^b is bounded on $L^{p,\varphi}(\mathbb{R}^n, w)$, that is*

$$\|T^b(f)\|_{L^{p,\varphi}(w)} \le C\prod_{j=1}^{l}\left(\sum_{|\alpha_j|=m_j}\|D^{\alpha_j}b_j\|_{\mathrm{BMO}}\right)\|f\|_{L^{p,\varphi}(w)}.$$

2 Proof of Theorem

To prove the theorem, we need the following lemmas.

Lemma 1 ([2]). *Let b be a function on $\mathbb{R}^n$ and $D^\alpha b \in L^q(\mathbb{R}^n)$ for all α with $|\alpha| = m$ and some $q > n$. Then, for $x \neq y$,*

$$|R_m(b; x, y)| \leq C|x-y|^m \sum_{|\alpha|=m} \left(\frac{1}{|\tilde{Q}(x,y)|} \int_{\tilde{Q}(x,y)} |D^\alpha b(z)|^q \, \mathrm{d}z \right)^{1/q},$$

where $\tilde{Q}$ is the cube centered at x and having side length $5\sqrt{n}\,|x-y|$.

Lemma 2 ([12, 15]). *Let T be the singular integral operator with non-smooth kernels as Definition 2. Then, for every $f \in L^p(\mathbb{R}^n)$, $1 < p < \infty$,*

$$\|T(f)\|_{L^p} \leq C\|f\|_{L^p}.$$

Lemma 3 ([8]). *Let $\{A_t,\, t > 0\}$ be an "approximation to the identity" and $b \in \mathrm{BMO}(\mathbb{R}^n)$. Then, for every locally integrable function f, $1 < r < \infty$ and $\tilde{x} \in \mathbb{R}^n$,*

$$\sup_{Q \ni \tilde{x}} \frac{1}{|Q|} \int_Q |A_{t_Q}((b - b_Q)f)(y)| \, \mathrm{d}y \leq C\|b\|_{\mathrm{BMO}} M_r(f)(\tilde{x}),$$

where $t_Q = l(Q)^2$ and $l(Q)$ denotes the side length of Q.

Lemma 4 ([12, 15]). *Let $\{A_t,\, t > 0\}$ be an "approximation to the identity". For any $\gamma > 0$, there exists a constant $C > 0$ independent of γ such that*

$$|\{\, x \in \mathbb{R}^n : M(f)(x) > D\lambda, M_A^\#(f)(x) \leq \gamma\lambda \,\}| \leq C\gamma |\{\, x \in \mathbb{R}^n : M(f)(x) > \lambda \,\}|$$

for $\lambda > 0$, where D is a fixed constant which only depends on n. Thus, for $w \in A_1$ and $f \in L^p(\mathbb{R}^n)$, $1 < p < \infty$,

$$\|M(f)\|_{L^p(w)} \leq C\|M_A^\#(f)\|_{L^p(w)}.$$

Lemma 5. *Let $\{A_t,\, t > 0\}$ be an "approximation to the identity", $1 < p < \infty$, $0 < D < 2^n$ and $w \in A_1$. Then, for $f \in L^{p,\varphi}(\mathbb{R}^n, w)$,*

(a) $\|M(f)\|_{L^{p,\varphi}(w)} \leq C\|M_A^\#(f)\|_{L^{p,\varphi}(w)}$;

(b) $|M_q(f)\|_{L^{p,\varphi}(w)} \leq C\|f\|_{L^{p,\varphi}(w)}$ *for* $1 < q < p$.

Proof. (a) Let $f \in L^{p,\varphi}(\mathbb{R}^n, w)$. For a cube $Q = Q(x, d) \subset \mathbb{R}^n$, note that $M(w\chi_Q) \in A_1$ (see [8]) and by Lemma 4, we get

$$\begin{aligned}
\int_Q |M(f)(y)|^p w(y) \, \mathrm{d}y &\leq \int_{\mathbb{R}^n} |M(f)(y)|^p M(w\chi_Q)(y) \, \mathrm{d}y \\
&\leq C \int_{\mathbb{R}^n} |M_A^\#(f)(y)|^p M(w\chi_Q)(y) \, \mathrm{d}y \\
&\leq C \left[\int_Q |M_A^\#(f)(y)|^p M(w)(y) \, \mathrm{d}y + \sum_{k=0}^\infty \int_{2^{k+1}Q \setminus 2^k Q} |M_A^\#(f)(y)|^p \left(\sup_{Q \ni y} \frac{1}{|Q|} \int_Q w(z) \, \mathrm{d}z \right) \mathrm{d}y \right] \\
&\leq C \left[\int_Q |M_A^\#(f)(y)|^p M(w)(y) \, \mathrm{d}y + \sum_{k=0}^\infty \int_{2^{k+1}Q \setminus 2^k Q} |M_A^\#(f)(y)|^p \left(\frac{1}{|2^{k+1}Q|} \int_B w(z) \, \mathrm{d}z \right) \mathrm{d}y \right]
\end{aligned}$$

$$\leq C\left[\int_Q |M_A^{\#}(f)(y)|^p w(y)\,\mathrm{d}y + \sum_{k=0}^{\infty}\int_{2^{k+1}Q}|M_A^{\#}(f)(y)|^p \frac{w(y)}{2^{nk}}\,\mathrm{d}y\right]$$
$$\leq C\|M_A^{\#}(f)\|_{L^{p,\varphi}(w)}^p \sum_{k=0}^{\infty} 2^{-nk}\varphi(2^{k+1}d)$$
$$\leq C\|M_A^{\#}(f)\|_{L^{p,\varphi}(w)}^p \sum_{k=0}^{\infty}(2^{-n}D)^k\varphi(d)$$
$$\leq C\|M_A^{\#}(f)\|_{L^{p,\varphi}(w)}^p\varphi(d),$$

thus

$$\|M(f)\|_{L^{p,\varphi}(w)} \leq C\|M_A^{\#}(f)\|_{L^{p,\varphi}(w)}.$$

A similar argument as in the proof of (a) will give the proof of (b), we omit the details. ■

Key Lemma. *Suppose T is the singular integral operator with non-smooth kernel as Definition 2. Let $D^\alpha b_j \in \mathrm{BMO}(\mathbb{R}^n)$ for all α with $|\alpha| = m_j$ and $j = 1,\dots,l$. Then there exists a constant $C > 0$ such that for any $f \in C_0^\infty(\mathbb{R}^n)$, $1 < s < \infty$ and $\tilde{x} \in \mathbb{R}^n$,*

$$M_A^{\#}(T^b(f))(\tilde{x}) \leq C\prod_{j=1}^{l}\left(\sum_{|\alpha_j|=m_j}\|D^{\alpha_j}b_j\|_{\mathrm{BMO}}\right)M_s(f)(\tilde{x}).$$

PROOF. It suffices to prove for $f \in C_0^\infty(\mathbb{R}^n)$ and some constant C_0, the following inequality holds:

$$\frac{1}{|Q|}\int_Q \left|T^b(f)(x) - A_{t_Q}T^b(f)(x)\right|\mathrm{d}x \leq C\prod_{j=1}^{l}\left(\sum_{|\alpha_j|=m_j}\|D^{\alpha_j}b_j\|_{\mathrm{BMO}}\right)M_s(f)(\tilde{x}).$$

Without loss of generality, we may assume $l = 2$. Fix a cube $Q = Q(x_0, d)$ and $\tilde{x} \in Q$. Let $\tilde{Q} = 5\sqrt{n}\,Q$ and $\tilde{b}_j(x) = b_j(x) - \sum_{|\alpha|=m_j}\frac{1}{\alpha!}(D^\alpha b_j)_{\tilde{Q}}x^\alpha$. Then $R_{m_j}(b_j;x,y) = R_{m_j}(\tilde{b}_j;x,y)$ and $D^\alpha\tilde{b}_j = D^\alpha b_j - (D^\alpha b_j)_{\tilde{Q}}$ for $|\alpha| = m_j$. We write, for $f = f\chi_{\tilde{Q}} + f\chi_{\mathbb{R}^n\setminus\tilde{Q}} = f_1 + f_2$,

$$\begin{aligned}
T^b(f)(x) &= \int_{\mathbb{R}^n}\frac{\prod_{j=1}^2 R_{m_j+1}(\tilde{b}_j;x,y)}{|x-y|^m}K(x,y)f(y)\,\mathrm{d}y\\
&= \int_{\mathbb{R}^n}\frac{\prod_{j=1}^2 R_{m_j}(\tilde{b}_j;x,y)}{|x-y|^m}K(x,y)f_1(y)\,\mathrm{d}y\\
&\quad - \sum_{|\alpha_1|=m_1}\frac{1}{\alpha_1!}\int_{\mathbb{R}^n}\frac{R_{m_2}(\tilde{b}_2;x,y)(x-y)^{\alpha_1}D^{\alpha_1}\tilde{b}_1(y)}{|x-y|^m}K(x,y)f_1(y)\,\mathrm{d}y\\
&\quad - \sum_{|\alpha_2|=m_2}\frac{1}{\alpha_2!}\int_{\mathbb{R}^n}\frac{R_{m_1}(\tilde{b}_1;x,y)(x-y)^{\alpha_2}D^{\alpha_2}\tilde{b}_2(y)}{|x-y|^m}K(x,y)f_1(y)\,\mathrm{d}y\\
&\quad + \sum_{\substack{|\alpha_1|=m_1,\\|\alpha_2|=m_2}}\frac{1}{\alpha_1!\alpha_2!}\int_{\mathbb{R}^n}\frac{(x-y)^{\alpha_1+\alpha_2}D^{\alpha_1}\tilde{b}_1(y)D^{\alpha_2}\tilde{b}_2(y)}{|x-y|^m}K(x,y)f_1(y)\,\mathrm{d}y\\
&\quad + \int_{\mathbb{R}^n}\frac{\prod_{j=1}^2 R_{m_j+1}(\tilde{b}_j;x,y)}{|x-y|^m}K(x,y)f_2(y)\,\mathrm{d}y\\
&= T\left(\frac{\prod_{j=1}^2 R_{m_j}(\tilde{b}_j;x,\cdot)}{|x-\cdot|^m}f_1\right)(x) - T\left(\sum_{|\alpha_1|=m_1}\frac{1}{\alpha_1!}\frac{R_{m_2}(\tilde{b}_2;x,\cdot)(x-\cdot)^{\alpha_1}D^{\alpha_1}\tilde{b}_1}{|x-\cdot|^m}f_1\right)(x)
\end{aligned}$$

$$
\begin{aligned}
&- T\left(\sum_{|\alpha_2|=m_2} \frac{1}{\alpha_2!} \frac{R_{m_1}(\tilde{b}_1;x,\cdot)(x-\cdot)^{\alpha_2} D^{\alpha_2}\tilde{b}_2}{|x-\cdot|^m} f_1\right)(x) \\
&+ T\left(\sum_{\substack{|\alpha_1|=m_1,\\ |\alpha_2|=m_2}} \frac{1}{\alpha_1!\alpha_2!} \frac{(x-\cdot)^{\alpha_1+\alpha_2} D^{\alpha_1}\tilde{b}_1 D^{\alpha_2}\tilde{b}_2}{|x-\cdot|^m} f_1\right)(x) \\
&+ T\left(\frac{\prod_{j=1}^2 R_{m_j+1}(\tilde{b}_j;x,\cdot)}{|x-\cdot|^m} f_2\right)(x)
\end{aligned}
$$

and

$$
\begin{aligned}
A_{t_Q} T^b(f)(x) &= \int_{\mathbb{R}^n} \frac{\prod_{j=1}^2 R_{m_j}(\tilde{b}_j;x,y)}{|x-y|^m} K_t(x,y) f_1(y)\,\mathrm{d}y \\
&\quad - \sum_{|\alpha_1|=m_1} \frac{1}{\alpha_1!} \int_{\mathbb{R}^n} \frac{R_{m_2}(\tilde{b}_2;x,y)(x-y)^{\alpha_1} D^{\alpha_1}\tilde{b}_1(y)}{|x-y|^m} K_t(x,y) f_1(y)\,\mathrm{d}y \\
&\quad - \sum_{|\alpha_2|=m_2} \frac{1}{\alpha_2!} \int_{\mathbb{R}^n} \frac{R_{m_1}(\tilde{b}_1;x,y)(x-y)^{\alpha_2} D^{\alpha_2}\tilde{b}_2(y)}{|x-y|^m} K_t(x,y) f_1(y)\,\mathrm{d}y \\
&\quad + \sum_{\substack{|\alpha_1|=m_1,\\ |\alpha_2|=m_2}} \frac{1}{\alpha_1!\alpha_2!} \int_{\mathbb{R}^n} \frac{(x-y)^{\alpha_1+\alpha_2} D^{\alpha_1}\tilde{b}_1(y) D^{\alpha_2}\tilde{b}_2(y)}{|x-y|^m} K_t(x,y) f_1(y)\,\mathrm{d}y \\
&\quad + \int_{\mathbb{R}^n} \frac{\prod_{j=1}^2 R_{m_j+1}(\tilde{b}_j;x,y)}{|x-y|^m} K_t(x,y) f_2(y)\,\mathrm{d}y \\
&= A_{t_Q} T\left(\frac{\prod_{j=1}^2 R_{m_j}(\tilde{b}_j;x,\cdot)}{|x-\cdot|^m} f_1\right)(x) \\
&\quad - A_{t_Q} T\left(\sum_{|\alpha_1|=m_1} \frac{1}{\alpha_1!} \frac{R_{m_2}(\tilde{b}_2;x,\cdot)(x-\cdot)^{\alpha_1} D^{\alpha_1}\tilde{b}_1}{|x-\cdot|^m} f_1\right)(x) \\
&\quad - A_{t_Q} T\left(\sum_{|\alpha_2|=m_2} \frac{1}{\alpha_2!} \frac{R_{m_1}(\tilde{b}_1;x,\cdot)(x-\cdot)^{\alpha_2} D^{\alpha_2}\tilde{b}_2}{|x-\cdot|^m} f_1\right)(x) \\
&\quad + A_{t_Q} T\left(\sum_{\substack{|\alpha_1|=m_1,\\ |\alpha_2|=m_2}} \frac{1}{\alpha_1!\alpha_2!} \frac{(x-\cdot)^{\alpha_1+\alpha_2} D^{\alpha_1}\tilde{b}_1 D^{\alpha_2}\tilde{b}_2}{|x-\cdot|^m} f_1\right)(x) \\
&\quad + A_{t_Q} T\left(\frac{\prod_{j=1}^2 R_{m_j+1}(\tilde{b}_j;x,\cdot)}{|x-\cdot|^m} f_2\right)(x).
\end{aligned}
$$

Then

$$
\begin{aligned}
&\frac{1}{|Q|} \int_Q \left|T^b(f)(x) - A_{t_Q} T^b(f)(x)\right| \mathrm{d}x \\
&\quad \le \frac{1}{|Q|} \int_Q \left| T\left(\frac{\prod_{j=1}^2 R_{m_j}(\tilde{b}_j;x,\cdot)}{|x-\cdot|^m} f_1\right)(x)\right| \mathrm{d}x
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{1}{|Q|}\int_Q\left|T\left(\sum_{|\alpha_1|=m_1}\frac{1}{\alpha_1!}\frac{R_{m_2}(\tilde{b}_2;x,\cdot)(x-\cdot)^{\alpha_1}D^{\alpha_1}\tilde{b}_1}{|x-\cdot|^m}f_1\right)(x)\right|\mathrm{d}x\\
&+\frac{1}{|Q|}\int_Q\left|T\left(\sum_{|\alpha_2|=m_2}\frac{1}{\alpha_2!}\frac{R_{m_1}(\tilde{b}_1;x,\cdot)(x-\cdot)^{\alpha_2}D^{\alpha_2}\tilde{b}_2}{|x-\cdot|^m}f_1\right)(x)\right|\mathrm{d}x\\
&+\frac{1}{|Q|}\int_Q\left|T\left(\sum_{\substack{|\alpha_1|=m_1,\\|\alpha_2|=m_2}}\frac{1}{\alpha_1!\alpha_2!}\frac{(x-\cdot)^{\alpha_1+\alpha_2}D^{\alpha_1}\tilde{b}_1D^{\alpha_2}\tilde{b}_2}{|x-\cdot|^m}f_1\right)(x)\right|\mathrm{d}x\\
&+\frac{1}{|Q|}\int_Q\left|A_{t_Q}T\left(\frac{\prod_{j=1}^2R_{m_j}(\tilde{b}_j;x,\cdot)}{|x-\cdot|^m}f_1\right)(x)\right|\mathrm{d}x\\
&+\frac{1}{|Q|}\int_Q\left|A_{t_Q}T\left(\sum_{|\alpha_1|=m_1}\frac{1}{\alpha_1!}\frac{R_{m_2}(\tilde{b}_2;x,\cdot)(x-\cdot)^{\alpha_1}D^{\alpha_1}\tilde{b}_1}{|x-\cdot|^m}f_1\right)(x)\right|\mathrm{d}x\\
&+\frac{1}{|Q|}\int_Q\left|A_{t_Q}T\left(\sum_{|\alpha_2|=m_2}\frac{1}{\alpha_2!}\frac{R_{m_1}(\tilde{b}_1;x,\cdot)(x-\cdot)^{\alpha_2}D^{\alpha_2}\tilde{b}_2}{|x-\cdot|^m}f_1\right)(x)\right|\mathrm{d}x\\
&+\frac{1}{|Q|}\int_Q\left|A_{t_Q}T\left(\sum_{\substack{|\alpha_1|=m_1,\\|\alpha_2|=m_2}}\frac{1}{\alpha_1!\alpha_2!}\frac{(x-\cdot)^{\alpha_1+\alpha_2}D^{\alpha_1}\tilde{b}_1D^{\alpha_2}\tilde{b}_2}{|x-\cdot|^m}f_1\right)(x)\right|\mathrm{d}x\\
&+\frac{1}{|Q|}\int_Q\left|(T-A_{t_Q}T)\left(\frac{\prod_{j=1}^2R_{m_j+1}(\tilde{b}_j;x,\cdot)}{|x-\cdot|^m}f_2\right)(x)\right|\mathrm{d}x\\
&:=I_1+I_2+I_3+I_4+I_5+I_6+I_7+I_8+I_9.
\end{aligned}
$$

Now, let us estimate I_1, I_2, I_3, I_4, I_5, I_6, I_7, I_8 and I_9, respectively. First, for $x\in Q$ and $y\in\tilde{Q}$, by Lemma 1, we get

$$
R_m(\tilde{b}_j;x,y)\le C|x-y|^m\sum_{|\alpha_j|=m}\|D^{\alpha_j}b_j\|_{\mathrm{BMO}}.
$$

Thus, by the the L^s-boundedness of T (Lemma 2), we obtain

$$
\begin{aligned}
I_1&\le C\prod_{j=1}^2\left(\sum_{|\alpha_j|=m_j}\|D^{\alpha_j}b_j\|_{\mathrm{BMO}}\right)\frac{1}{|Q|}\int_Q|T(f_1)(x)|\mathrm{d}x\\
&\le C\prod_{j=1}^2\left(\sum_{|\alpha_j|=m_j}\|D^{\alpha_j}b_j\|_{\mathrm{BMO}}\right)\left(\frac{1}{|Q|}\int_{\mathbb{R}^n}|T(f_1)(x)|^s\mathrm{d}x\right)^{1/s}\\
&\le C\prod_{j=1}^2\left(\sum_{|\alpha_j|=m_j}\|D^{\alpha_j}b_j\|_{\mathrm{BMO}}\right)\left(\frac{1}{|Q|}\int_{\mathbb{R}^n}|f_1(x)|^s\mathrm{d}x\right)^{1/s}\\
&\le C\prod_{j=1}^2\left(\sum_{|\alpha_j|=m_j}\|D^{\alpha_j}b_j\|_{\mathrm{BMO}}\right)\left(\frac{1}{|\tilde{Q}|}\int_{\tilde{Q}}|f(x)|^s\,\mathrm{d}x\right)^{1/s}
\end{aligned}
$$

$$\leq C\prod_{j=1}^{2}\left(\sum_{|\alpha_j|=m_j}\|D^{\alpha_j}b_j\|_{\mathrm{BMO}}\right)M_s(f)(\tilde{x}).$$

For I_2, denoting $s=pq$ for $1<p<\infty$, $q>1$ and $1/q+1/q'=1$, we have, by Lemma 2 and Hölder's inequality,

$$\begin{aligned}
I_2 &\leq C\sum_{|\alpha_2|=m_2}\|D^{\alpha_2}b_2\|_{\mathrm{BMO}}\sum_{|\alpha_1|=m_1}\frac{1}{|Q|}\int_Q|T(D^{\alpha_1}\tilde{b}_1f_1)(x)|\,\mathrm{d}x\\
&\leq C\sum_{|\alpha_2|=m_2}\|D^{\alpha_2}b_2\|_{\mathrm{BMO}}\sum_{|\alpha_1|=m_1}\left(\frac{1}{|Q|}\int_{\mathbb{R}^n}|T(D^{\alpha_1}\tilde{b}_1f_1)(x)|^p\,\mathrm{d}x\right)^{1/p}\\
&\leq C\sum_{|\alpha_2|=m_2}\|D^{\alpha_2}b_2\|_{\mathrm{BMO}}\sum_{|\alpha_1|=m_1}\left(\frac{1}{|Q|}\int_{\mathbb{R}^n}|D^{\alpha_1}\tilde{b}_1(x)f_1(x)|^p\,\mathrm{d}x\right)^{1/p}\\
&\leq C\sum_{|\alpha_2|=m_2}\|D^{\alpha_2}b_2\|_{\mathrm{BMO}}\sum_{|\alpha_1|=m_1}\left(\frac{1}{|\tilde{Q}|}\int_{\tilde{Q}}|D^{\alpha_1}b_1(x)-(D^{\alpha}b_j)_{\tilde{Q}}|^{pq'}\,\mathrm{d}x\right)^{1/pq'}\left(\frac{1}{|\tilde{Q}|}\int_{\tilde{Q}}|f(x)|^{pq}\,\mathrm{d}x\right)^{1/pq}\\
&\leq C\prod_{j=1}^{2}\left(\sum_{|\alpha|=m_j}\|D^{\alpha}b_j\|_{\mathrm{BMO}}\right)M_s(f)(\tilde{x}).
\end{aligned}$$

For I_3, similar to the proof of I_2, we get

$$I_3\leq C\prod_{j=1}^{2}\left(\sum_{|\alpha|=m_j}\|D^{\alpha}b_j\|_{\mathrm{BMO}}\right)M_s(f)(\tilde{x}).$$

Similarly, for I_4, denoting $s=pq_3$ for $1<p<\infty$, q_1, q_2, $q_3>1$ and $1/q_1+1/q_2+1/q_3=1$, we obtain

$$\begin{aligned}
I_4 &\leq C\sum_{\substack{|\alpha_1|=m_1,\\|\alpha_2|=m_2}}\left(\frac{1}{|Q|}\int_{\mathbb{R}^n}|T(D^{\alpha_1}\tilde{b}_1\,D^{\alpha_2}\tilde{b}_2f_1)(x)|^p\,\mathrm{d}x\right)^{1/p}\\
&\leq C\sum_{\substack{|\alpha_1|=m_1,\\|\alpha_2|=m_2}}\left(\frac{1}{|Q|}\int_{\mathbb{R}^n}|D^{\alpha_1}\tilde{b}_1(x)\,D^{\alpha_2}\tilde{b}_2(x)f_1(x)|^p\,\mathrm{d}x\right)^{1/p}\\
&\leq C\sum_{\substack{|\alpha_1|=m_1,\\|\alpha_2|=m_2}}\prod_{j=1}^{2}\left(\frac{1}{|\tilde{Q}|}\int_{\tilde{Q}}|D^{\alpha_j}\tilde{b}_j(x)|^{pq_j}\,\mathrm{d}x\right)^{1/pq_j}\left(\frac{1}{|\tilde{Q}|}\int_{\tilde{Q}}|f(x)|^{pq_3}\,\mathrm{d}x\right)^{1/pq_3}\\
&\leq C\prod_{j=1}^{2}\left(\sum_{|\alpha|=m_j}\|D^{\alpha}b_j\|_{\mathrm{BMO}}\right)M_s(f)(\tilde{x}).
\end{aligned}$$

For I_5, I_6, I_7 and I_8, by Lemma 3 and similar to the proof of I_1, I_2, I_3 and I_4, we get

$$\begin{aligned}
I_5+I_6+I_7+I_8 &\leq C\prod_{j=1}^{2}\left(\sum_{|\alpha_j|=m_j}\|D^{\alpha_j}b_j\|_{\mathrm{BMO}}\right)\frac{1}{|Q|}\int_{\mathbb{R}^n}|A_{t_Q}T(f_1)(x)|\,\mathrm{d}x\\
&\quad+C\sum_{|\alpha_2|=m_2}\|D^{\alpha_2}b_2\|_{\mathrm{BMO}}\sum_{|\alpha_1|=m_1}\frac{1}{|Q|}\int_{\mathbb{R}^n}|A_{t_Q}T(D^{\alpha_1}\tilde{b}_1f_1)(x)|\,\mathrm{d}x
\end{aligned}$$

$$
\begin{aligned}
&+C\sum_{|\alpha_1|=m_1}\|D^{\alpha_1}b_1\|_{\mathrm{BMO}}\sum_{|\alpha_2|=m_2}\frac{1}{|Q|}\int_{\mathbb{R}^n}|A_{t_Q}T(D^{\alpha_2}\tilde{b}_2f_1)(x)|\,\mathrm{d}x\\
&+C\sum_{\substack{|\alpha_1|=m_1,\\|\alpha_2|=m_2}}\frac{1}{|Q|}\int_{\mathbb{R}^n}|A_{t_Q}T(D^{\alpha_1}\tilde{b}_1\,D^{\alpha_2}\tilde{b}_2f_1)(x)|\,\mathrm{d}x\\
\le{}& C\prod_{j=1}^{2}\left(\sum_{|\alpha|=m_j}\|D^{\alpha}b_j\|_{\mathrm{BMO}}\right)M_s(f)(\tilde{x}).
\end{aligned}
$$

For I_9, we write

$$
\begin{aligned}
&(T-A_{t_Q}T)\left(\frac{\prod_{j=1}^2 R_{m_j+1}(\tilde{b}_j;x,\cdot)}{|x-\cdot|^m}f_2\right)\\
&=\int_{\mathbb{R}^n}\frac{\prod_{j=1}^2 R_{m_j+1}(\tilde{b}_j;x,y)}{|x-y|^m}(K(x,y)-K_t(x,y))f_2(y)\,\mathrm{d}y\\
&=\int_{\mathbb{R}^n}\frac{\prod_{j=1}^2 R_{m_j}(\tilde{b}_j;x,y)}{|x-y|^m}(K(x,y)-K_t(x,y))f_2(y)\,\mathrm{d}y\\
&\quad-\sum_{|\alpha_1|=m_1}\frac{1}{\alpha_1!}\int_{\mathbb{R}^n}\frac{D^{\alpha_1}\tilde{b}_1(y)(x-y)^{\alpha_1}R_{m_2}(\tilde{b}_2;x,y)}{|x-y|^m}(K(x,y)-K_t(x,y))f_2(y)\,\mathrm{d}y\\
&\quad-\sum_{|\alpha_2|=m_2}\frac{1}{\alpha_2!}\int_{\mathbb{R}^n}\frac{D^{\alpha_2}\tilde{b}_2(y)(x-y)^{\alpha_2}R_{m_1}(\tilde{b}_1;x,y)}{|x-y|^m}(K(x,y)-K_t(x,y))f_2(y)\,\mathrm{d}y\\
&\quad+\sum_{\substack{|\alpha_1|=m_1,\\|\alpha_2|=m_2}}\frac{1}{\alpha_1!\alpha_2!}\int_{\mathbb{R}^n}\frac{D^{\alpha_1}\tilde{b}_1(y)\,D^{\alpha_2}\tilde{b}_2(y)(x-y)^{\alpha_1+\alpha_2}}{|x-y|^m}(K(x,y)-K_t(x,y))f_2(y)\,\mathrm{d}y\\
&=I_9^{(1)}+I_9^{(2)}+I_9^{(3)}+I_9^{(4)}.
\end{aligned}
$$

By Lemma 1 and the following inequality (see [13])

$$
|b_{Q_1}-b_{Q_2}|\le C\log(|Q_2|/|Q_1|)\,\|b\|_{\mathrm{BMO}}\qquad\text{for } Q_1\subset Q_2,
$$

we know that, for $x\in Q$ and $y\in 2^{k+1}\tilde{Q}\setminus 2^k\tilde{Q}$,

$$
\begin{aligned}
|R_m(\tilde{b};x,y)|&\le C|x-y|^m\sum_{|\alpha|=m}(\|D^{\alpha}b\|_{\mathrm{BMO}}+|(D^{\alpha}b)_{\tilde{Q}(x,y)}-(D^{\alpha}b)_{\tilde{Q}}|)\\
&\le Ck|x-y|^m\sum_{|\alpha|=m}\|D^{\alpha}b\|_{\mathrm{BMO}}.
\end{aligned}
$$

Note that $|x-y|\ge d=t^{1/2}$ and $|x-y|\sim|x_0-y|$ for $x\in Q$ and $y\in\mathbb{R}^n\setminus\tilde{Q}$. By the conditions on K and K_t, we obtain

$$
\begin{aligned}
|I_9^{(1)}|&=\sum_{k=0}^{\infty}\int_{2^{k+1}\tilde{Q}\setminus 2^k\tilde{Q}}\frac{\prod_{j=1}^2|R_{m_j}(\tilde{b}_j;x,y)|}{|x-y|^m}|K(x,y)-K_t(x,y)|\,|f(y)|\,\mathrm{d}y\\
&\le C\prod_{j=1}^{2}\left(\sum_{|\alpha|=m_j}\|D^{\alpha}b_j\|_{\mathrm{BMO}}\right)\sum_{k=0}^{\infty}\int_{2^{k+1}\tilde{Q}\setminus 2^k\tilde{Q}}k^2\frac{d^{\delta}}{|x_0-y|^{n+\delta}}|f(y)|\,\mathrm{d}y
\end{aligned}
$$

$$\leq C\prod_{j=1}^{2}\left(\sum_{|\alpha|=m_j}\|D^{\alpha}b_j\|_{\mathrm{BMO}}\right)\sum_{k=1}^{\infty}k^2 2^{-\delta k}\frac{1}{|2^k\tilde{Q}|}\int_{2^k\tilde{Q}}|f(y)|\,\mathrm{d}y$$
$$\leq C\prod_{j=1}^{2}\left(\sum_{|\alpha|=m_j}\|D^{\alpha}b_j\|_{\mathrm{BMO}}\right)M_s(f)(\tilde{x}).$$

For $I_9^{(2)}$, we get

$$|I_9^{(2)}|\leq C\left(\sum_{|\alpha|=m_2}\|D^{\alpha}b_2\|_{\mathrm{BMO}}\right)\sum_{|\alpha|=m_1}\sum_{k=0}^{\infty}\int_{2^{k+1}\tilde{Q}\backslash 2^k\tilde{Q}}k\frac{d^{\delta}}{|x_0-y|^{n+\delta}}|D^{\alpha_1}\tilde{b}_1(y)|\,|f(y)|\,\mathrm{d}y$$
$$\leq C\left(\sum_{|\alpha|=m_2}\|D^{\alpha}b_2\|_{\mathrm{BMO}}\right)\sum_{|\alpha_1|=m_1}\sum_{k=1}^{\infty}k2^{-\delta k}\left(\frac{1}{|2^k\tilde{Q}|}\int_{2^k\tilde{Q}}|D^{\alpha_1}\tilde{b}_1(y)|^{r'}\,\mathrm{d}y\right)^{1/r'}$$
$$\times\left(\frac{1}{|2^k\tilde{Q}|}\int_{2^k\tilde{Q}}|f(y)|^s\,\mathrm{d}y\right)^{1/s}$$
$$\leq C\prod_{j=1}^{2}\left(\sum_{|\alpha|=m_j}\|D^{\alpha}b_j\|_{\mathrm{BMO}}\right)M_s(f)(\tilde{x}).$$

Similarly,

$$|I_9^{(3)}|\leq C\prod_{j=1}^{2}\left(\sum_{|\alpha|=m_j}\|D^{\alpha}b_j\|_{\mathrm{BMO}}\right)M_s(f)(\tilde{x}).$$

For $I_9^{(4)}$, taking $q_1, q_2>1$ such that $1/s+1/q_1+1/q_2=1$, then

$$|I_9^{(4)}|\leq C\sum_{\substack{|\alpha_1|=m_1,\\ |\alpha_2|=m_2}}\sum_{k=0}^{\infty}\int_{2^{k+1}\tilde{Q}\backslash 2^k\tilde{Q}}\frac{d^{\delta}}{|x_0-y|^{n+\delta}}|D^{\alpha_1}\tilde{b}_1(y)|\,|D^{\alpha_2}\tilde{b}_2(y)|\,|f(y)|\,\mathrm{d}y$$
$$\leq C\sum_{\substack{|\alpha_1|=m_1,\\ |\alpha_2|=m_2}}\sum_{k=1}^{\infty}\left(\frac{1}{|2^k\tilde{Q}|}\int_{2^k\tilde{Q}}|D^{\alpha_1}\tilde{b}_1(y)|^{q_1}\,\mathrm{d}y\right)^{1/q_1}\left(\frac{1}{|2^k\tilde{Q}|}\int_{2^k\tilde{Q}}|D^{\alpha_2}\tilde{b}_2(y)|^{q_2}\,\mathrm{d}y\right)^{1/q_2}$$
$$\times 2^{-\delta k}\left(\frac{1}{|2^k\tilde{Q}|}\int_{2^k\tilde{Q}}|f(y)|^s\,\mathrm{d}y\right)^{1/s}$$
$$\leq C\prod_{j=1}^{2}\left(\sum_{|\alpha|=m_j}\|D^{\alpha}b_j\|_{\mathrm{BMO}}\right)M_s(f)(\tilde{x}).$$

Thus

$$I_9\leq C\prod_{j=1}^{2}\left(\sum_{|\alpha|=m_j}\|D^{\alpha}b_j\|_{\mathrm{BMO}}\right)M_s(f)(\tilde{x}).$$

This completes the proof of Key Lemma. ■

PROOF OF THEOREM 1. Taking $1<s<p$ in Key Lemma and by Lemma 5, we obtain

$$\|T^b(f)\|_{L^{p,\varphi}(w)}\leq\|M(T^b(f))\|_{L^{p,\varphi}(w)}$$

$$\leq C\|M_A^{\#}(T^b(f))\|_{L^{p,\varphi}(w)}$$
$$\leq C\prod_{j=1}^{2}\left(\sum_{|\alpha|=m_j}\|D^\alpha b_j\|_{\mathrm{BMO}}\right)\|M_s(f)\|_{L^{p,\varphi}(w)}$$
$$\leq C\prod_{j=1}^{2}\left(\sum_{|\alpha|=m_j}\|D^\alpha b_j\|_{\mathrm{BMO}}\right)\|f\|_{L^{p,\varphi}(w)}.$$

This finishes the proof. ■

3 Applications

In this section we shall apply Theorem 1 of the paper to the holomorphic functional calculus of linear elliptic operators. First, we review some definitions regarding the holomorphic functional calculus (see [12, 15]). Given $0 \leq \theta < \pi$. Define

$$S_\theta = \{\, z \in C : |\arg(z)| \leq \theta \,\} \bigcup \{0\}$$

and its interior by S_θ^0. Set $\tilde{S}_\theta = S_\theta \setminus \{0\}$. A closed operator L on some Banach space E is said to be of type θ if its spectrum $\sigma(L) \subset S_\theta$ and for every $\nu \in (\theta, \pi]$, there exists a constant C_ν such that

$$|\eta|\|(\eta I - L)^{-1}\| \leq C_\nu, \qquad \eta \notin \tilde{S}_\theta.$$

For $\nu \in (0, \pi]$, let

$$H_\infty(S_\mu^0) = \{\, f\colon S_\theta^0 \longrightarrow C :\ f \text{ is holomorphic and } \|f\|_{L^\infty} < \infty \,\},$$

where $\|f\|_{L^\infty} = \sup\{\, |f(z)| : z \in S_\mu^0 \,\}$. Set

$$\Psi(S_\mu^0) = \left\{ g \in H_\infty(S_\mu^0) : \exists s > 0, \exists c > 0 \text{ such that } |g(z)| \leq c\frac{|z|^s}{1+|z|^{2s}} \right\}.$$

If L is of type θ and $g \in H_\infty(S_\mu^0)$, we define $g(L) \in L(E)$ by

$$g(L) = -(2\pi i)^{-1}\int_\Gamma (\eta I - L)^{-1} g(\eta)\,\mathrm{d}\eta,$$

where Γ is the contour $\{\, \xi = r\mathrm{e}^{\pm i\phi} : r \geq 0 \,\}$ parameterized clockwise around S_θ with $\theta < \phi < \mu$. If, in addition, L is one-one and has dense range, then, for $f \in H_\infty(S_\mu^0)$,

$$f(L) = [h(L)]^{-1}(fh)(L),$$

where $h(z) = z(1+z)^{-2}$. L is said to have a bounded holomorphic functional calculus on the sector S_μ, if

$$\|g(L)\| \leq N\|g\|_{L^\infty}$$

for some $N > 0$ and for all $g \in H_\infty(S_\mu^0)$.

Now, let L be a linear operator on $L^2(\mathbb{R}^n)$ with $\theta < \pi/2$ so that $(-L)$ generates a holomorphic semigroup e^{-zL}, $0 \leq |\arg(z)| < \pi/2 - \theta$. Applying Theorem 6 of [12] and Theorem 1, we get

Theorem 2. *Assume the following conditions are satisfied:*

(i) *The holomorphic semigroup* e^{-zL}, $0 \le |\arg(z)| < \pi/2 - \theta$ *is represented by the kernels* $a_z(x, y)$ *which satisfy, for all* $\nu > \theta$, *an upper bound*

$$|a_z(x, y)| \le c_\nu h_{|z|}(x, y)$$

for $x, y \in \mathbb{R}^n$, *and* $0 \le |\arg(z)| < \pi/2 - \theta$, *where* $h_t(x, y) = Ct^{-n/2} s(|x - y|^2/t)$ *and* s *is a positive, bounded and decreasing function satisfying*

$$\lim_{r \to \infty} r^{n+\epsilon} s(r^2) = 0.$$

(ii) *The operator* L *has a bounded holomorphic functional calculus in* $L^2(\mathbb{R}^n)$, *that is, for all* $\nu > \theta$ *and* $g \in H_\infty(S^0_\mu)$, *the operator* $g(L)$ *satisfies*

$$\|g(L)(f)\|_{L^2} \le c_\nu \|g\|_{L^\infty} \|f\|_{L^2}.$$

Then, for $0 < D < 2^n$, $w \in A_1$, $D^\alpha b_j \in \mathrm{BMO}(\mathbb{R}^n)$ *for all* α *with* $|\alpha| = m_j$ *and* $j = 1, \ldots, l$, *the multilinear operator* $g(L)^b$ *associated to* $g(L)$ *is bounded on* $L^{p,\varphi}(\mathbb{R}^n, w)$, *that is*

$$\|g(L)^b(f)\|_{L^{p,\varphi}(w)} \le C \prod_{j=1}^{l} \left(\sum_{|\alpha_j| = m_j} \|D^{\alpha_j} b_j\|_{\mathrm{BMO}} \right) \|f\|_{L^{p,\varphi}(w)}.$$

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