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Nonlinear evolution equations on locally closed graphs

Mihai Necula, Marius Popescu and Ioan I. Vrabie

Abstract Let X be a real Banach space, let $A\colon D(A) \subseteq X \rightsquigarrow X$ be an m-dissipative operator, let I a nonempty, bounded interval and let $K\colon I \rightsquigarrow \overline{D(A)}$ be a given multi-valued function. By using the concept of A-*quasi-tangent set* introduced by Cârjă, Necula, Vrabie [8] and [9] and using a tangency condition expressed in the terms of this concept, we establish a necessary and sufficient condition for C^0-viability referring to nonlinear evolution inclusions of the form $u'(t) \in Au(t) + F(t, u(t))$, where F is a multi-function defined on the graph of K. As an application, we deduce a comparison result for a class of fully nonlinear evolution inclusions driven by multi-valued perturbations of subdifferentials.

Ecuaciones de evolución no lineales en grafos localmente cerrados.

Resumen. Sea X un espacio de Banach real, sea $A\colon D(A) \subseteq X \rightsquigarrow X$ un operador m-disipativo, sea I un intervalo acotado no vacío y sea $K\colon I \rightsquigarrow \overline{D(A)}$ una función multivaluada. Utilizando el concepto de conjunto A-casi-tangente introducido por Cârjă, Necula, Vrabie [8] y [9] y utilizando condiciones de tangencia expresadas en términos de este concepto, establecemos una condición necesaria y suficiente de C^0-viabilidad para inclusiones de evolución no lineales de la forma $u'(t) \in Au(t) + F(t, u(t))$, donde F es una multi-función definida en el grafo de K. Como aplicación, se deduce un resultado de comparación para una clase de inclusiones de evolución no lineales completas asociadas a perturbaciones multi-valuadas de subdiferenciales.

1 Introduction

Let X be a real Banach space and let $A\colon D(A) \subseteq X \rightsquigarrow X$ be an m-dissipative operator, which means that $-A$ is m-accretive, generating the nonlinear semigroup of contractions $\{\, S(t)\colon \overline{D(A)} \to \overline{D(A)};\ t \geq 0 \,\}$. Let $K\colon I \rightsquigarrow \overline{D(A)}$ and $F\colon \mathcal{K} \rightsquigarrow X$ be two multi-functions with nonempty values, where $I \subseteq \mathbb{R}$ is a nonempty and open from the right interval and $\mathcal{K} := \operatorname{graph}(K)$.

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Our aim here is to prove some new necessary and sufficient conditions in order that $\mathcal{K}$ be viable with respect to $A+F$. To be more precise, let us consider the Cauchy Problem

$$\begin{cases} u'(t) \in Au(t) + F(t, u(t)) \\ u(\tau) = \xi. \end{cases} \tag{1}$$

The next concept is widely known, after the pioneering work of Crandall, Liggett [12], under the name of *mild solution* and, within the frame here considered, coincides with the one of *integral solution* introduced by Benilan [2].

Definition 1 *Let $A\colon D(A) \subseteq X \rightsquigarrow X$ be m-dissipative and let $f \in L^1(\tau, T; X)$. A C^0*-solution, *or DS*-limit solution, *of the equation*

$$u'(t) \in Au(t) + f(t) \tag{2}$$

is a function u in $C([\tau, T]; X)$ satisfying: for each $\tau < c < T$ and $\varepsilon > 0$ there exist

(i) $\tau = t_0 < t_1 < \cdots < c \le t_n < T, \quad t_k - t_{k-1} \le \varepsilon \qquad$ *for* $k = 1, 2, \dots, n$*;*

(ii) $f_1, \dots, f_n \in X \quad$ *with* $\quad \displaystyle\sum_{k=1}^{n} \int_{t_{k-1}}^{t_k} \|f(t) - f_k\| \, \mathrm{d}t \le \varepsilon$*;*

(iii) $v_0, \dots, v_n \in X \quad$ *satisfying:*
$\dfrac{v_k - v_{k-1}}{t_k - t_{k-1}} \in Av_k + f_k \qquad$ *for* $k = 1, 2, \dots, n \quad$ *and such that*
$\|u(t) - v_k\| \le \varepsilon \quad$ *for* $t \in [t_{k-1}, t_k),\ k = 1, 2, \dots, n$*.*

Definition 2 *A function $u\colon [\tau, T] \to X$ is a C^0*-solution *of (1) if $u(\tau) = \xi$, $u(t) \in K(t)$ for each $t \in [\tau, T]$, and there exists an a.e. selection $f \in L^1(\tau, T; X)$ of $t \mapsto F(t, u(t))$, i.e., $f(t) \in F(t, u(t))$ a.e. for $t \in [\tau, T]$, such that u is a C^0-solution on $[\tau, T]$ of the equation (2) in the usual sense.*

The next two classical results will prove useful in that follows.

Theorem 1 *Let X be a Banach space and let $A\colon D(A) \subseteq X \rightsquigarrow X$ be m-dissipative. Then, for each $\xi \in \overline{D(A)}$ and $f \in L^1(\tau, T; X)$, there exists a unique C^0-solution $u\colon [\tau, T] \to \overline{D(A)}$, of (2), which satisfies $u(\tau) = \xi$.*

PROOF. See Lakshmikantham-Leela [14, Theorem 3.6.1, p. 116]. ■

In order to exhibit the dependence of the C^0-solution u of (2) on τ, ξ and f, we denote it by $u = u(\,\cdot\,, \tau, \xi, f)$. Throughout, $[x, y]_+$ denotes the right directional derivative of the norm calculated at x in the direction y, i.e.,

$$[x, y]_+ = \lim_{h \downarrow 0} \frac{1}{h} \left(\|x + hy\| - \|x\| \right).$$

Similarly,

$$(x, y)_+ = \lim_{h \downarrow 0} \frac{1}{2h} \left(\|x + hy\|^2 - \|x\|^2 \right).$$

Theorem 2 *Let X be a Banach space, let $A\colon D(A) \subseteq X \rightsquigarrow X$ be m-dissipative, let $\xi, \eta \in \overline{D(A)}$, f, $g \in L^1(\tau, T; X)$ and let $\widetilde{u} = u(\,\cdot\,, \tau, \xi, f)$ and $\widetilde{v} = u(\,\cdot\,, \tau, \eta, g)$. We have*

$$\|\widetilde{u}(t) - \widetilde{v}(t)\| \le \|\xi - \eta\| + \int_\tau^t [\widetilde{u}(s) - \widetilde{v}(s), f(s) - g(s)]_+ \, \mathrm{d}s$$

and

$$\|\widetilde{u}(t)-\widetilde{v}(t)\|^2 \le \|\xi-\eta\|^2 + 2\int_\tau^t (\widetilde{u}(s)-\widetilde{v}(s), f(s)-g(s))_+ \,\mathrm{d}s,$$

for each $t\in[\tau,T]$*. Moreover, we have the following evolution property*

$$u(t,a,\xi,f) = u(t,\nu,u(\nu,a,\xi,f),f|_{[\nu,\nu+\delta]}), \tag{3}$$

for $\tau\le a\le\nu\le t\le\nu+\delta$*.*

PROOF. See Vrabie [25, Section 1.7]. ■

Since for each x, $y\in X$, $[x,y]_+\le\|y\|$, from Theorem 2, we deduce

Corollary 1 *Let* X *be a Banach space, let* $A\colon D(A)\subseteq X\rightsquigarrow X$ *be* m*-dissipative, let* ξ*,* $\eta\in\overline{D(A)}$*,* f*,* $g\in L^1(\tau,T;X)$ *and let* $\widetilde{u}=u(\,\cdot\,,\tau,\xi,f)$ *and* $\widetilde{v}=u(\,\cdot\,,\tau,\eta,g)$*. We have*

$$\|\widetilde{u}(t)-\widetilde{v}(t)\|\le\|\widetilde{u}(s)-\widetilde{v}(s)\|+\int_s^t\|f(\theta)-g(\theta)\|\,\mathrm{d}\theta$$

for each $\tau\le s\le t\le T$*.*

Definition 3 *We say that the graph,* $\mathcal{K}$*, of* $K\colon I\rightsquigarrow\overline{D(A)}$*, is* C^0-viable *with respect to* $A+F$*, where* $F\colon\mathcal{K}\rightsquigarrow X$*, if for each* $(\tau,\xi)\in\mathcal{K}$*, there exists* $T>\tau$*, such that* $[\tau,T]\subseteq I$ *and* (1) *has at least one* C^0*-solution* $u\colon[\tau,T]\to X$*. If* $T\in(\tau,\sup I)$ *can be taken arbitrary, we say that* $\mathcal{K}$ *is* globally C^0-viable *with respect to* $A+F$*.*

A short review of the main contributions to the viability theory for evolution inclusions is given below. Roughly speaking, the S-viability of a set, K, with respect to the right-hand side of an evolution inclusion means that for each $\xi\in K$ there exists at least one S-solution u of the evolution inclusion in question satisfying $u(\tau)=\xi$ and $u(t)\in K$ for each $t\in[\tau,T]$. Here S means the sense in which the term solution should be understood, sense which has to be made very precise case by case.

Throughout, if $u\in X$, $B\subseteq X$ and $C\subseteq X$, we denote by

$$\mathrm{dist}\,(u,C)=\inf_{v\in C}\|u-v\|,\qquad \mathrm{dist}\,(B,C)=\inf_{\substack{v\in B\\ w\in C}}\|v-w\|\qquad\text{and}\qquad \|B\|=\sup_{v\in B}\|v\|$$

Emerged from its classical ordinary differential equations counterpart initiated by Perron [22] in the one-dimensional case and extended by Nagumo [16] to the arbitrary but finite dimensions, the viability theory for ordinary differential inclusions of the form $u'(t)\in F(u(t))$ started with the paper of Bebernes, Shuur [1] where they have shown that, whenever $F\colon K\rightsquigarrow X$ is upper semi-continuous (u.s.c.) with nonempty, convex, closed and bounded values, and $K\subseteq X$ is locally closed, a necessary and sufficient condition in order that K be absolutely continuous-viable with respect to F is

$$F(\xi)\cap\mathcal{T}_K(\xi)\neq\emptyset \tag{4}$$

for each $\xi\in K$, where $\mathcal{T}_K(\xi)$ denotes the contingent cone to K at $\xi\in K$. We recall that $\mathcal{T}_K(\xi)$ consists of all vectors $\eta\in X$ which are tangent to K at $\xi\in K$ in the sense of Bouligand [5] and Severi [23], i.e.,

$$\liminf_{h\downarrow 0}\frac{1}{h}\,\mathrm{dist}(\xi+h\eta;\ K)=0.$$

Clearly, (4) is nothing but a simple extension of the well-known tangency condition: $f(\xi)\in\mathcal{T}_K(\xi)$ for each $\xi\in K$, used by Nagumo [16] in the single-valued and autonomous case, i.e. $F(\xi)=\{f(\xi)\}$. Extensions of the main result of Bebernes-Schuur [1], to multi-functions defined on graphs, in finite dimensional spaces, were obtained subsequently by Methlouthi [15], for u.s.c. F, and by Bothe [3], for almost u.s.c. F. For similar results in infinite dimensional spaces, see Bothe [4], as well as Cârjă, Necula, Vrabie [9].

Recently, Cârjă, Necula, Vrabie [10] considered the multi-valued perturbed case $u'(t) \in Au(t) + F(u(t))$ with A the infinitesimal generator of a C_0-semigroup and $F\colon K \rightsquigarrow X$. In order to cover more general situations, from the viewpoints of both necessary and sufficient conditions of mild-viability, they introduced the concept of *A-quasi-tangent set to a given set K at a given point $\xi \in K$* by saying that a nonempty and bounded subset E in X is *A-quasi-tangent to K at $\xi \in K$* if

$$\liminf_{h \downarrow 0} \frac{1}{h} \operatorname{dist}\left(S(h)\xi + \int_0^h S(h-s)\mathcal{F}_E \,\mathrm{d}s;\ K \right) = 0,$$

where

$$\mathcal{F}_E = \left\{ f \in L^1_{\mathrm{loc}}(\mathbb{R}; X);\ f(s) \in E \text{ a. e. for } s \in \mathbb{R} \right\}. \tag{5}$$

We notice that this concept is strictly more general than that one of A-tangent vector used by Pavel [20] in the single-valued case, and by Pavel, Vrabie [21] in the multi-valued case. Subsequently, inspired by the notion of A-tangent vector, with A nonlinear, used by Vrabie [24], they have extended this concept to the case in which A is nonlinear and have proved some necessary and sufficient conditions of C^0-viability. See Cârjă, Necula, Vrabie [8] and [11].

By imposing a tangency condition expressed in terms of A-quasi-tangent sets, in this paper, we extend the main result in Necula, Popescu, Vrabie [19] to the fully nonlinear case, by proving a sufficient, and even a necessary and sufficient condition for C^0-viability referring to nonlinear evolution inclusions driven by multi-valued nonautonomous and t-discontinuous perturbations defined on graphs. The advantage of using A-quasi-tangent sets instead of A-tangent vectors consists in that, in infinite dimensions, the "multi-valued tangency condition" turns out to be not only sufficient for C^0-viability, but necessary as well.

Finally, it should be noticed that there are two main difficulties to overcome in this context. The first one consists in finding a suitable definition of the approximate solutions, in the general case here considered, i.e., that one of a multi-function defined on a non-cylindric domain, multi-function which may fail to be u.s.c. with respect to the t-variable. The second main difficult point here is to construct a sequence of approximate solutions which, under some additional fairly natural assumptions, has at least one convergent subsequence.

The paper is divided into 7 sections, the second and the third ones being merely concerned with the definition of both tangency concepts and special classes of multi-functions to be used in the sequel. The fourth section contains the main necessary condition of C^0-viability, while in the fifth section we state the main mild-viability sufficient condition and prove the existence of approximation solutions. In section 6 we prove the main sufficient condition for mild-viability, while the last section 7, as an application, we include a comparison result referring to a class of nonlinear evolution inclusions governed by multi-valued perturbations of subdifferentials.

2 Tangency concepts

Let X be a real Banach space, $I \subseteq \mathbb{R}$ a nonempty and open from the right interval, let $K\colon I \rightsquigarrow \overline{D(A)}$ be a multi-function with nonempty values and let $\mathcal{K}$ be the graph of K, i.e. $\mathcal{K} := \{ (\tau, \xi) \in I \times X;\ \tau \in I,\ \xi \in K(\tau) \}$. Let $(\tau, \xi) \in \mathcal{K}$, let $\eta \in X$ and let $E \in \mathcal{B}(X)$, where $\mathcal{B}(X)$ denotes the class of all nonempty and bounded subsets in X.

Definition 4 *We say that*

(i) *the vector η is* A-tangent *to $\mathcal{K}$ at (τ, ξ) if*

$$\liminf_{h \downarrow 0} \frac{1}{h} \operatorname{dist}\big(u(\tau+h, \tau, \xi, \eta);\ K(\tau+h)\big) = 0,$$

where $u(\,\cdot\,,\tau,\xi,\eta)$ *denotes the unique* C^0-solution *of the Cauchy problem*

$$\begin{cases} v'(t) \in Av(t) + \eta \\ v(\tau) = \xi, \end{cases}$$

on $[\tau, +\infty)$*. See Definition* 1.

(ii) *the set* E *is* A-tangent *to* $\mathcal{K}$ *at* (τ,ξ) *if*

$$\liminf_{h\downarrow 0} \frac{1}{h} \operatorname{dist}\big(u(\tau+h,\tau,\xi,E);\ K(\tau+h)\big) = 0,$$

where $u(\tau+h,\tau,\xi,E) = \{\, u(\tau+h,\tau,\xi,\eta);\ \eta \in E\,\}$.

(iii) E *is* A-quasi-tangent *to* $\mathcal{K}$ *at* (τ,ξ) *if*

$$\liminf_{h\downarrow 0} \frac{1}{h} \operatorname{dist}\big(u(\tau+h,\tau,\xi,\mathcal{F}_E);\ K(\tau+h)\big) = 0,$$

where $\mathcal{F}_E$ *is defined by* (5)*, and* $u(\tau+h,\tau,\xi,\mathcal{F}_E) = \{\, u(\tau+h,\tau,\xi,f);\ f \in \mathcal{F}_E\,\}$.

Throughout, we denote by:

(i) $\mathcal{T}^A_{\mathcal{K}}(\tau,\xi)$ the set of all A-tangent vectors to $\mathcal{K}$ at (τ,ξ);

(ii) $\mathcal{TS}^A_{\mathcal{K}}(\tau,\xi)$ the set of all A-tangent sets to $\mathcal{K}$ at (τ,ξ);

(iii) $\mathcal{QTS}^A_{\mathcal{K}}(\tau,\xi)$ the set of all A-quasi-tangent sets to $\mathcal{K}$ at (τ,ξ).

Identifying vectors with singletons, and constants with locally integrable functions, we deduce

$$\mathcal{T}^A_{\mathcal{K}}(\tau,\xi) \subseteq \mathcal{TS}^A_{\mathcal{K}}(\tau,\xi) \subseteq \mathcal{QTS}^A_{\mathcal{K}}(\tau,\xi),$$

and it may happen, even in the simplest case $A \equiv 0$, that both inclusions to be strict. See Example 2.4.1, p. 36 in Cârjă, Necula, Vrabie [9].

3 Special classes of multi-functions

Throughout, $\mathcal{K}$ is endowed with the metric, d, defined by

$$d\big((\tau,\xi),(\theta,\mu)\big) = \max\{\, |\tau-\theta|, \|\xi-\mu\|\,\},$$

for all (τ,ξ), $(\theta,\mu) \in \mathcal{K}$. Furthermore, whenever we will use the term *strongly-weakly u.s.c.* we will mean that the domain of the multi-function in question is equipped with the strong topology, while the range is equipped with the weak topology. The term *u.s.c.* refers to the case in which both domain and range are endowed with the strong, i.e. norm, topology. Finally, in all that follows, λ stands for the Lebesgue measure.

Definition 5 *The multi-function* $F\colon \mathcal{K} \rightsquigarrow X$ *is called (*strongly-weakly*)* almost u.s.c. *if for each* $\varepsilon > 0$ *there exists an open set* $\mathcal{O}_\varepsilon \subseteq I$ *such that* $\lambda(\mathcal{O}_\varepsilon) \le \varepsilon$ *and* $F_{|[(I\setminus\mathcal{O}_\varepsilon)\times X]\cap\mathcal{K}}$ *is (strongly-weakly) u.s.c.*

Definition 6 *We say that* $F\colon \mathcal{K} \rightsquigarrow X$ *is* essentially locally bounded *if, for each* $(\tau,\xi) \in \mathcal{K}$*, there exist a negligible set* $N_1 \subseteq I$*,* $\rho > 0$*, and* $\ell_1 \in L^\infty_{\mathrm{loc}}(I;\mathbb{R})$ *such that for all* $(t,u) \in (I \setminus N_1) \times D(\xi,\rho)) \cap \mathcal{K}$*, we have*

$$\|F(t,u)\| \le \ell_1(t).$$

If we relax the condition $\ell_1 \in L^\infty_{\mathrm{loc}}(I;\mathbb{R})$ *to* $\ell_1 \in L^1_{\mathrm{loc}}(I;\mathbb{R})$*, we say that* F *is* locally integrally bounded.

Remark 1 *If $\overline{D(A)}$ is separable, we can choose N_1 in Definition 6 independent of $(\tau,\xi)\in\mathcal{K}$ and, in this case, for each $(\tau,\xi)\in[(I\setminus N_1)\times X]\cap\mathcal{K}$, $F(\tau,\xi)$ is bounded.*

Excepting the case when $K\colon I\rightsquigarrow X$ is constant, i.e., $K(t)\equiv C$, when $\mathcal{K}=I\times C$ is a cylindrical domain, one may happen that there would be no multi-function $F\colon\mathcal{K}\rightsquigarrow X$ such that $\mathcal{K}$ be C^0-viable with respect to $A+F$. See Example 2.1 in Necula, Popescu, Vrabie [18].

So, in order to get necessary and even necessary and sufficient conditions for the viability of a non-cylindrical graph with respect to a given multi-functions, one has to consider merely a special class of graphs. This class of graphs, we are going to define precisely below, was considered for the first time by Necula [17].

Definition 7 *Let $K\colon I\rightsquigarrow\overline{D(A)}$ be a multi-function. The graph, $\mathcal{K}$, of K is said to be A-C^0-*viable by itself *if for each $(\tau,\xi)\in\mathcal{K}$, there exist $T>\tau$, $\rho>0$ and $\ell_2\in L^1_{\rm loc}(I;\mathbb{R})$, so that for each $(\widetilde{\tau},\widetilde{\xi})\in([\tau,T)\times S(\xi,\rho))\cap\mathcal{K}$, there exist $\widetilde{T}\in(\widetilde{\tau},T]$ and a pair of functions, $(g,v)\in L^1\big([\widetilde{\tau},\widetilde{T}];X\big)\times C([\widetilde{\tau},\widetilde{T}];X\big)$, satisfying:*

(v_1) $v(t)=u(t,\widetilde{\tau},\widetilde{\xi},g)$ *for each $t\in[\widetilde{\tau},\widetilde{T}]$;*

(v_2) $(t,v(t))\in([\widetilde{\tau},\widetilde{T}]\times S(\xi,\rho))\cap\mathcal{K}$ *for each $t\in[\widetilde{\tau},\widetilde{T}]$;*

(v_3) $\|g(s)\|\le\ell_2(s)$ *a.e. for $s\in[\widetilde{\tau},\widetilde{T}]$.*

By a simple solution issuing from $(\widetilde{\tau},\widetilde{\xi})\in([\tau,T)\times S(\xi,\rho))\cap\mathcal{K}$ *we mean a pair (g,v) satisfying (v_1)–(v_3).*

Remark 2 *In other words, the graph, $\mathcal{K}$, of $K\colon I\rightsquigarrow\overline{D(A)}$ is A-C^0-viable by itself if and only if, for each $(\tau,\xi)\in\mathcal{K}$, there exist $T>\tau$, $\rho>0$ and $\ell_2\in L^1_{\rm loc}(I;\mathbb{R})$, so that $\big([\tau,T)\times S(\xi,\rho)\big)\cap\mathcal{K}$ is C^0-viable with respect to $A+G$, where the multi-function $G\colon([\tau,T)\times X)\cap\mathcal{K}\rightsquigarrow X$ is defined by*

$$G(t,\xi):=\{\,v\in X;\ \|v\|\le\ell_2(t)\,\},$$

for each $(t,\xi)\in([\tau,T)\times X)\cap\mathcal{K}$.

Remark 3 (i) *If $K\colon I\rightsquigarrow\overline{D(A)}$ is constant and $S(t)K\subseteq K$ for each $t\ge 0$, then $\mathcal{K}$ is A-C^0-viable by itself.*

(ii) *If $\mathcal{K}$ is C^0-viable with respect to $A+F$, where $F\colon\mathcal{K}\rightsquigarrow X$ is some locally essentially bounded multi-function then, for each $(\tau,\xi)\in\mathcal{K}$, the function G, defined as in Remark 2, with $\ell_2=\ell_1$, where ℓ_1 is given by Definition 6, satisfies the conditions in Remark 6, and thus $\mathcal{K}$ is A-C^0-viable by itself.*

4 Necessary conditions for viability

Throughout, λ denotes the Lebesgue measure on $\mathbb{R}$. First we recall

Definition 8 *An m-dissipative operator $A\colon D(A)\rightsquigarrow X$ is* of compact type *if for each sequence $(f_n,u_n)_n$ in $L^1(\tau,T;X)\times C([\tau,T];X)$ with u_n a C^0-solution of the problem $u_n'(t)\in Au_n(t)+f_n(t)$ on $[\tau,T]$ for $n=1,2,\dots$, $\lim_n f_n=f$ weakly in $L^1(\tau,T;X)$ and $\lim_n u_n=u$ strongly in $C([\tau,T];X)$, it follows that u is a C^0-solution of the problem $u'(t)\in Au(t)+f(t)$ on $[\tau,T]$.*

A typical example of m-dissipative nonlinear operator of compact type is given by $\Delta\beta$ in $L^1(\Omega)$ with Dirichlet boundary conditions. See Diaz, Vrabie [13] and Cârjă, Necula, Vrabie [9], Theorem 1.7.9, p. 22.

The hypotheses we will use in the sequel are listed below.

(A_1) $A\colon D(A) \subseteq X \rightsquigarrow X$ is m-dissipative and $\{\, S(t)\colon \overline{D(A)} \to \overline{D(A)};\ t \geq 0 \,\}$ is the nonlinear semigroup of contractions generated by A;

(A_2) $A\colon D(A) \subseteq X \rightsquigarrow X$ is m-dissipative and $\{\, S(t)\colon \overline{D(A)} \to \overline{D(A)};\ t \geq 0 \,\}$ is compact, i.e., $S(t)$ is compact for each $t > 0$;

(A_3) $A\colon D(A) \subseteq X \rightsquigarrow X$ is m-dissipative and of compact type;

(F_1) the graph $\mathcal{K}$ is A-C^0-viable by itself;

(F_2) F has nonempty and closed values;

(F_3) $F\colon \mathcal{K} \rightsquigarrow X$ is almost u.s.c.;

(F_4) $F\colon \mathcal{K} \rightsquigarrow X$ is essentially locally bounded;

(F_5) $F\colon \mathcal{K} \rightsquigarrow X$ is almost strongly-weakly u.s.c.;

(F_6) there exists a set $N \subseteq I$, with $\lambda(N) = 0$, and such that for each $(\tau, \xi) \in [(I \setminus N) \times X] \cap \mathcal{K}$, we have $F(\tau, \xi) \in \mathfrak{QTS}^A_{\mathcal{K}}(\tau, \xi)$.

Theorem 3 *Let $\overline{D(A)}$ be separable. If (A_1), and (F_3) are satisfied, F is nonempty-valued and locally integrally bounded, and $\mathcal{K}$ is C^0-viable with respect to $A + F$, then both (F_1) and (F_6) hold.*

PROOF. In view of (ii) in Remark 3, it remains to check out only (F_6). Since $\overline{D(A)}$ is separable, and F is locally integrally bounded, from Remark 1, it follows that there exist a finite or countable set Γ, $(\tau_i, \xi_i)_{i \in \Gamma} \subseteq \mathcal{K}$, $(\delta_i)_{i \in \Gamma} \subseteq (0, \infty)$, $(\rho_i)_{i \in \Gamma} \subseteq (0, \infty)$, $(\ell_i)_{i \in \Gamma} \subseteq L^1_{\rm loc}(I; \mathbb{R})$ and a negligible set $N \subseteq I$ such that $\mathcal{K} \subseteq \cup_{i \in \Gamma} (\tau_i - \delta_i, \tau_i + \delta_i) \times S(\xi_i, \rho_i)$ and, for each $i \in \Gamma$ and each $(t, u) \in (((\tau_i - \delta_i, \tau_i + \delta_i) \setminus N) \times S(\xi_i, \rho_i)) \cap \mathcal{K}$, we have $\|F(t, u)\| \leq \ell_i(t)$.

From (F_3) it follows that for each $n \geq 1$ it exists $I_n \subset I$ such that $\lambda(I \setminus I_n) < 1/n$ and F is u.s.c. on $(I_n \times X) \cap \mathcal{K}$.

Let $E_n \subset I_n$ the set of all density points of I_n which are also Lebesgue points for ℓ_i, for all $i \in \Gamma$. Let $E = (\cup_{n \geq 1} E_n) \cap (I \setminus N)$. Obviously, $\lambda(I \setminus E) = 0$.

Let $\tau \in E$ and $\xi \in K(\tau)$. We will show that $F(\tau, \xi) \in \mathfrak{QTS}^A_{\mathcal{K}}(\tau, \xi)$.

Let $u\colon [\tau, T] \to \overline{D(A)}$ be a solution of (1). Hence there exists $f \in L^1(\tau, T; X)$ such that $f(s) \in F(s, u(s))$ a.e. $s \in [\tau, T]$ and $u = u(\,\cdot\,, \tau, \xi, f)$.

Since $\tau \in E$, there exists $n_0 \in \mathbb{N}$ such that $\tau \in E_{n_0}$. Analogously, since $\mathcal{K} \subseteq \cup_{i \in \Gamma} (\tau_i - \delta_i, \tau_i + \delta_i) \times S(\xi_i, \rho_i)$, there exists $i_0 \in \Gamma$ such that $(\tau, \xi) \in (\tau_{i_0} - \delta_{i_0}, \tau_{i_0} + \delta_{i_0}) \times S(\xi_{i_0}, \rho_{i_0})$. Let $\varepsilon > 0$ be arbitrary but fixed and let $0 < \delta < \delta_{i_0}$ be such that

$$f(s) \in F(s, u(s)) \subset F(\tau, \xi) + D(0, \varepsilon),$$

a.e. for $s \in [\tau, \tau + \delta] \cap E_{n_0}$ and $u(s) \in S(\xi_{i_0}, \rho_{i_0})$ for all $s \in [\tau, \tau + \delta]$.

Let $\eta \in F(\tau, \xi)$ be fixed and

$$\widetilde{f}(s) = \begin{cases} f(s) & \text{for } s \in [\tau, \tau + \delta] \cap E_{n_0} \\ \eta & \text{for } s \in [\tau, \tau + \delta] \setminus E_{n_0}. \end{cases}.$$

Hence $\widetilde{f}(s) \in F(\tau, \xi) + D(0, \varepsilon)$ a.e. for $s \in [\tau, \tau + \delta]$.

Let $\overline{f}\colon [\tau, \tau + \delta] \to X$ countably valued such that $\|\overline{f}(s) - \widetilde{f}(s)\| < \varepsilon$ a.e. for $s \in [\tau, \tau + \delta]$. So, we have

$$\overline{f}(s) \in F(\tau, \xi) + D(0, 2\varepsilon)$$

a.e. for $s \in [\tau, \tau + \delta]$.

Then, there exist two countably valued functions $g\colon[\tau,\tau+\delta]\to F(\tau,\xi)$ and $r\colon[\tau,\tau+\delta]\to D(0,2\varepsilon)$ such that

$$\overline{f}(s)=g(s)+r(s)$$

a.e. for $s\in[\tau,\tau+\delta]$. Hence $g,r\in L^1(\tau,\tau+\delta;X)$.

Since $u(\tau+h)\in K(\tau+h)$, $\|g(s)-\widetilde{f}(s)\|\le 3\varepsilon$ a.e. for $s\in[\tau,\tau+\delta]$, using Corollary 1, we deduce that for each $0<h<\delta$

$$\begin{aligned}\frac{1}{h}\operatorname{dist}\big(u(\tau+h,\tau,\xi,\mathcal{F}_{F(\tau,\xi)}),\ K(\tau+h)\big)&\le\frac{1}{h}\|u(\tau+h,\tau,\xi,g)-u(\tau+h,\tau,\xi,f)\|\\ &\le\frac{1}{h}\int_\tau^{\tau+h}\|g(s)-f(s)\|\,\mathrm{d}s\\ &\le\frac{1}{h}\int_\tau^{\tau+h}\|g(s)-\widetilde{f}(s)\|\,\mathrm{d}s+\frac{1}{h}\int_\tau^{\tau+h}\|\widetilde{f}(s)-f(s)\|\,\mathrm{d}s\\ &\le 3\varepsilon+\frac{1}{h}\int_{[\tau,\tau+h]\setminus E_{n_0}}\|f(s)-\eta\|\,\mathrm{d}s\\ &\le 3\varepsilon+\frac{1}{h}\int_{[\tau,\tau+h]\setminus E_{n_0}}\|\ell_{i_0}(s)\|\,\mathrm{d}s+\frac{1}{h}\int_{[\tau,\tau+h]\setminus E_{n_0}}\|\eta\|\,\mathrm{d}s\\ &\le 3\varepsilon+\frac{1}{h}\int_\tau^{\tau+h}\|\ell_{i_0}(s)-\ell_{i_0}(\tau)\|\,\mathrm{d}s+(\|\ell_{i_0}(\tau)\|+\|\eta\|)\left(1-\frac{\lambda([\tau,\tau+h]\cap E_{n_0})}{\lambda([\tau,\tau+h])}\right).\end{aligned}$$

Passing to $\limsup$ in the inequality above and recalling that τ is both a density point and a Lebesgue point for ℓ_{i_0}, we get

$$\limsup_{h\downarrow 0}\frac{1}{h}\operatorname{dist}\big(u(\tau+h,\tau,\xi,\mathcal{F}_{F(\tau,\xi)}),\ K(\tau+h)\big)\le 3\varepsilon.$$

As $\varepsilon>0$ was arbitrary, we deduce (F_6). ∎

In fact, we have proved a stronger result, i.e.,

Theorem 4 *Let $\overline{D(A)}$ be separable. If (A_1), and (F_3) are satisfied, F is nonempty-valued and locally integrally bounded, and $\mathcal{K}$ is C^0-viable with respect to $A+F$, then (F_1) holds and there exists a set $N\subseteq I$, with $\lambda(N)=0$, and such that for each $(\tau,\xi)\in((I\setminus N)\times X)\cap\mathcal{K}$, we have*

$$\lim_{h\downarrow 0}\frac{1}{h}\operatorname{dist}\big(u(\tau+h,\tau,\xi,\mathcal{F}_{F(\tau,\xi)});\ K(\tau+h)\big)=0.$$

5 Sufficient conditions for viability

Definition 9 *We say that the graph $\mathcal{K}$ is:*

(i) locally closed from the left *if for each $(\tau,\xi)\in\mathcal{K}$ there exist $T>\tau$ and $\rho>0$ such that, for each $(\tau_n,\xi_n)\in([\tau,T]\times D(\xi,\rho))\cap\mathcal{K}$, with $(\tau_n)_n$ nondecreasing, $\lim_n\tau_n=\widetilde{\tau}$ and $\lim_n\xi_n=\widetilde{\xi}$, we have $(\widetilde{\tau},\widetilde{\xi})\in\mathcal{K}$;*

(ii) closed from the left *if for each $(\tau_n,\xi_n)\in\mathcal{K}$, with $(\tau_n)_n$ nondecreasing, $\lim_n\tau_n=\widetilde{\tau}$ and $\lim_n\xi_n=\widetilde{\xi}$, we have $(\widetilde{\tau},\widetilde{\xi})\in\mathcal{K}$.*

Theorem 5 *Let $\mathcal{K}$ be locally closed from the left and let $F\colon\mathcal{K}\rightsquigarrow X$ be nonempty, convex and weakly compact valued. If (A_1), (A_2), (A_3), (F_1), (F_2), (F_4) and (F_5) are satisfied, then a sufficient condition in order that $\mathcal{K}$ be C^0-viable with respect to $A+F$ is (F_6). If, instead of (F_5), the stronger condition (F_3) is satisfied, then (F_6) is also necessary in order that $\mathcal{K}$ be C^0-viable with respect to $A+F$.*

The necessity follows from Theorem 3 by observing that (A_2) implies the separability of $\overline{D(A)}$. This separability result is a straightforward extension of its linear counterpart in Vrabie [26, Theorem 6.2.2, p. 136]. The sufficiency, which is by far the most interesting part of Theorem 5, will be proved later.

From Theorem 5, by a slight extension of Brezis-Browder Ordering Principle [7], i.e. Theorem 2.1.1, p. 30 in Cârjă, Necula, Vrabie [9], we deduce the global C^0-viability result below.

Theorem 6 *Let $\mathcal{K}$ be closed from the left and let $F\colon \mathcal{K} \rightsquigarrow X$ be nonempty, convex and weakly compact valued. If (A_1), (A_2), (A_3), (F_1), (F_2), (F_4) and (F_5) are satisfied, then a sufficient condition in order that $\mathcal{K}$ be globally C^0-viable with respect to $A+F$ is (F_6). If, instead of (F_5), the stronger condition (F_3) is satisfied, then (F_6) is also necessary.*

The next lemma is the main step through the proof of Theorem 5.

Lemma 1 *Let X be a real Banach space, $A\colon D(A) \subseteq X \rightsquigarrow X$ an m-dissipative operator, I a nonempty and open from the right interval, $K\colon I \rightsquigarrow \overline{D(A)}$ a multi-function with locally closed from the left graph and $F\colon \mathcal{K} \rightsquigarrow X$ a nonempty-valued, locally essentially bounded multi-function. Let us assume that (A_1), (F_1) (F_2), (F_4), and (F_6) are satisfied. Let $(\tau,\xi) \in \mathcal{K}$ and let $Z = N_1 \cup N$, where N_1 and N are the negligible sets in (F_4) and in (F_6).*

Let $\rho > 0$ and $\widetilde{T} > \tau$ be such that $([\tau,\widetilde{T}] \times D(\xi,\rho)) \cap \mathcal{K}$ is closed from the left, F is a.e. bounded by $\ell_1 \in L^\infty_{\rm loc}(I;\mathbb{R})$ on $([\tau,\widetilde{T}] \times D(\xi,\rho)) \cap \mathcal{K}$ —see Definition 6 and let $\ell_2 \in L^1_{\rm loc}(I;\mathbb{R})$ be given by Definition 7.

Then, for each $\varepsilon \in (0,1)$ and each open set $\mathcal{O} \subseteq I$, with $Z \subseteq \mathcal{O}$, there exist $T \in (\tau,\widetilde{T}]$ and three functions: $\alpha\colon [\tau,T] \to [\tau,T]$ nondecreasing and right continuous, $f\colon [\tau,T] \to X$ measurable and $v\colon [\tau,T] \to X$ continuous satisfying:

(i) $t-\varepsilon \le \alpha(t) \le t$ *for all* $t \in [\tau,T]$, $\alpha(T) = T$;

(ii) *for each* $t \in [\tau,T]$ *for which* $\alpha(t) \in \mathcal{O}$ *it follows that* $[\alpha(t),t] \subseteq \mathcal{O}$;

(iii) $v(\alpha(t)) \in D(\xi,\rho) \cap K(\alpha(t))$ *for all* $t \in [\tau,T]$;

(iv) $f(t) \in F\big(\alpha(t), v(\alpha(t))\big)$ *for each* $t \in [\tau,T] \setminus \mathcal{O}$;

(v) $\|f(t)\| \le \ell(t)$ *a.e. for* $t \in [\tau,T]$, *with* $\ell(t) = \max\{\ell_1(t),\ell_2(t)\}$, *where* $\ell_1 \in L^\infty_{\rm loc}(I;\mathbb{R})$ *is as in Definition 6 and* $\ell_2 \in L^1_{\rm loc}(I;\mathbb{R})$ *as in Definition 7;*

(vi) $v(\tau) = \xi$ *and* $\|v(t) - u(t,\alpha(s),v(\alpha(s)),f)\| \le (t-\alpha(s))\varepsilon$ *for all* $t, s \in [\tau,T]$, $\tau \le s \le t \le T$;

(vii) $\|v(t) - v(\alpha(t))\| \le \varepsilon$ *for all* $t \in [\tau,T]$;

(viii) $\displaystyle\sup_{t\in[\tau,T]} \|S(t-\tau)\xi - \xi\| + \int_\tau^T \ell(s)\,{\rm d}s + T - \tau \le \rho.$

Definition 10 *Let $(\tau,\xi) \in \mathcal{K}$, $\varepsilon \in (0,1)$ and $\mathcal{O} \subseteq I$ a nonempty and open set with $Z \subseteq \mathcal{O}$. A triplet (α,f,v) satisfying* (i)–(viii) *is called an* $(\varepsilon,\mathcal{O})$-approximate C^0-solution *of* (1).

We can now proceed to the proof of Lemma 1.

Proof of Lemma 1 Let $(\tau,\xi) \in \mathcal{K}$ be arbitrary and choose $\rho > 0$ and $T > \tau$ such that

$$\big([\tau,T] \times D(\xi,\rho)\big) \cap \mathcal{K}$$

is closed from the left. This is always possible because $\mathcal{K}$ is locally closed. Next, diminishing $T > \tau$ if necessary, we may assume that (viii) holds.

We first prove that the conclusion of Lemma 1 remains true if we replace T as above with a possible smaller number $\tau + \delta$ with $\delta \in (0, T-\tau]$ which, at this stage, is allowed to depend on $\varepsilon \in (0,1)$. Then, by using the extension of Brezis-Browder Ordering Principle [7], i.e. Theorem 2.1.1, p. 30 in Cârjă, Necula, Vrabie [9], we will prove that we can take $\tau + \delta = T$ independent of ε.

Let $\varepsilon \in (0,1)$ be arbitrary. We distinguish between the following two complementary cases.

Case 1. If $\tau \in \mathcal{O}$, we take $\alpha\colon [\tau, \tau+\delta] \to [\tau, \tau+\delta]$ defined by $\alpha(t) = \tau$ for $t \in [\tau, \tau+\delta)$, $\alpha(\tau+\delta) = \tau+\delta$. In order to define f and v, let us recall that there exists a simple solution (g, v) issuing from (τ, ξ), defined on $[\tau, \tau+\delta]$. Let (g, v) be such a simple solution, and let us define $f(s) = g(s)$ a.e. for $s \in [\tau, \tau+\delta]$. Obviously (i) and (iii)–(vi) are satisfied and, taking into account that $\mathcal{O}$ is open and v is continuous, diminishing δ if necessary, we conclude that (ii) and (vii) are satisfied too.

Case 2. If $\tau \notin \mathcal{O}$ then $\tau \notin Z$, then $F(\tau, \xi) \in \mathcal{QTS}^A_K(\tau, \xi)$, and therefore there exist $f \in \mathcal{F}_{F(\tau,\xi)}$, $\delta \in (0, \varepsilon)$ and $p \in X$ with $\|p\| \le \varepsilon$ such that

$$u(\tau+\delta, \tau, \xi, f) + \delta p \in K(\tau+\delta).$$

We recall that $\mathcal{F}_{F(\tau,\xi)} = \{\, f \in L^1(\mathbb{R}_+; X);\ f(s) \in F(\tau, \xi) \text{ a.e. for } s \in \mathbb{R}_+ \,\}$. With f as above, let us define $\alpha\colon [\tau, \tau+\delta] \to [\tau, \tau+\delta]$ and $v\colon [\tau, \tau+\delta] \to X$ by $\alpha(t) = \tau$ for $t \in [\tau, \tau+\delta)$, $\alpha(\tau+\delta) = \tau+\delta$, and respectively by

$$v(t) = u(t, \tau, \xi, f) + (t-\tau)p$$

for each $t \in [\tau, \tau+\delta]$.

Let us observe that the functions α, f and v satisfy (i)–(v) with $T = \tau+\delta$. Clearly, $v(\tau) = \xi$. Moreover, since $\|p\| \le \varepsilon$, we deduce

$$\|v(t) - u(t, \alpha(s), v(\alpha(s)), f)\| = \|v(t) - u(t, \tau, v(\tau), f)\| = (t-\tau)\|p\| \le (t-\alpha(s))\varepsilon$$

for all $\tau \le s \le t \le \tau+\delta$. Thus (vi) is also satisfied. Next, diminishing $\delta > 0$ and redefining α if necessary, we get

$$\begin{aligned}
\|v(t) - v(\alpha(t))\| &= \|v(t) - v(\tau)\| \\
&\le \|u(t, \tau, \xi, f) - \xi\| + (t-\tau)\|p\| \\
&\le \|S(t-\tau)\xi - \xi\| + \int_\tau^t \|f(s)\|\,\mathrm{d}s + (t-\tau)\varepsilon \\
&\le \sup_{t\in[\tau,\tau+\delta]} \|S(t-\tau)\xi - \xi\| + \int_\tau^{\tau+\delta} \ell(s)\,\mathrm{d}s + \delta \le \varepsilon
\end{aligned}$$

for each $t \in [\tau, \tau+\delta)$, and thus (vii) is also satisfied.

Next, we will show that there exists at least one triplet (α, f, v) satisfying (i)–(viii) on $[\tau, T]$. We shall use the extension of Brezis-Browder Ordering Principle [7], i.e. Theorem 2.1.1, p. 30 in Cârjă, Necula, Vrabie [9], as follows. Let $\mathcal{S}$ be the set of all triplets (α, f, v), defined on $[\tau, \mu]$, with $\tau < \mu \le T$ and satisfying (i)–(viii) with μ instead of T. This set is clearly nonempty, as we have already proved. On $\mathcal{S}$ we introduce a partial order $\preceq$ as follows. We say that

$$(\alpha_1, f_1, v_1) \preceq (\alpha_2, f_2, v_2)$$

if $\mu_1 \le \mu_2$ and $\alpha_1(s) = \alpha_2(s)$, $f_1(s) = f_2(s)$ and $v_1(s) = v_2(s)$ for each $s \in [\tau, \mu_1]$.

Let us define the function $\mathcal{N}\colon \mathcal{S} \to \mathbb{R}$ by

$$\mathcal{N}(\alpha, f, v) = \mu.$$

It is clear that $\mathcal{N}$ is increasing on $\mathcal{S}$. Let us now take an increasing sequence

$$((\alpha_j, f_j, v_j))_j$$

in $\mathcal{S}$ and let us show that it is bounded from above in $\mathcal{S}$. We define an upper bound as follows. First, set

$$\mu^* = \sup\{\, \mu_j;\ j \in \mathbb{N} \,\}.$$

If $\mu^* = \mu_j$ for some $j \in \mathbb{N}$, (α_j, f_j, v_j) is clearly an upper bound. If $\mu_j < \mu^*$ for each $j \in \mathbb{N}$, let us define

$$\alpha(t) = \alpha_j(t), \qquad f(t) = f_j(t), \qquad v(t) = v_j(t)$$

for $j \in \mathbb{N}$ and every $t \in [\tau, \mu_j]$. To extend α, f and v to $t = \mu^*$, we proceed as follows.

First, we extend f at μ^* by setting $f(\mu^*) = \eta$, where $\eta \in X$ is arbitrary but fixed.

Second, by (iv) and (v), it follows that $f \in L^1(\tau, \mu^*; X)$ and therefore, for each $j \in \mathbb{N}$, the function $u(\,\cdot\,, \mu_j, v(\mu_j), f)\colon [\mu_j, \mu^*] \to \overline{D(A)}$ is continuous.

To extend v to μ^*, it suffices to show that there exists $\lim_{t\uparrow\mu^*} v(t)$. To this aim, let us observe that, in view of (vi), we have

$$\begin{aligned}\|v(t) - v(\widetilde{t})\| &\le \|v(t) - u(t, \mu_j, v(\mu_j), f)\| \\ &\qquad + \|u(t, \mu_j, v(\mu_j), f) - u(\widetilde{t}, \mu_j, v(\mu_j), f)\| + \|u(\widetilde{t}, \mu_j, v(\mu_j), f) - v(\widetilde{t})\| \\ &\le (t - \mu_j)\varepsilon + \|u(t, \mu_j, v(\mu_j), f) - u(\widetilde{t}, \mu_j, v(\mu_j), f)\| + (\widetilde{t} - \mu_j)\varepsilon \\ &\le (\mu^* - \mu_j)\varepsilon + \|u(t, \mu_j, v(\mu_j), f) - u(\widetilde{t}, \mu_j, v(\mu_j), f)\| + (\mu^* - \mu_j)\varepsilon,\end{aligned}$$

for each $j \in \mathbb{N}$, each $t \ge \mu_j$ and each $\widetilde{t} \ge \mu_j$. Since $\lim_j \mu_j = \mu^*$ and $u(\,\cdot\,, \mu_j, v(\mu_j))$ is continuous at $t = \mu^*$, we conclude that $t \mapsto v(t)$ satisfies the Cauchy necessary and sufficient condition for the existence of the limit at $t = \mu^*$. Indeed, let $\varepsilon' > 0$ be arbitrary and let us fix $j \in \mathbb{N}$ such that $(\mu^* - \mu_j)\varepsilon \le \varepsilon'/3$. Next, let us fix $\delta(\varepsilon') > 0$ such that, for each $t, \widetilde{t} \in [\mu_j, \mu^*)$ with $\mu^* - t \le \delta(\varepsilon')$ and $\mu^* - \widetilde{t} \le \delta(\varepsilon')$, we have $\|u(t, \mu_j, v(\mu_j), f) - u(\widetilde{t}, \mu_j, v(\mu_j), f)\| \le \varepsilon'/3$. Thus, for each t and $\widetilde{t}$ as above, we have $\|v(t) - v(\widetilde{t})\| \le \varepsilon'$, as claimed. So, we can extend v, by continuity, to the whole interval $[\tau, \mu^*]$. Finally, we define $\alpha(\mu^*) = \mu^*$.

Since $v(\mu_m) \in D(\xi, \rho) \cap K(\mu_m)$, for each $m \in \mathbb{N}$, and the latter is closed from the left, we deduce that $v(\mu^*) \in D(\xi, \rho) \cap K(\mu^*)$. At this point, let us observe that, if necessary, i.e., if $\mu^* \notin \mathcal{O}$, we have to redefine $f(\mu^*) = \eta$ by choosing $\eta \in F(\mu^*, v(\mu^*))$, in order that (iv) be satisfied. This is always possible because f is supposed to be in $L^1(\tau, \mu^*; X)$. Hence, (α, f, v) satisfies (i)–(iv). Next, we may easily verify that (α, f, v) satisfies (v)–(viii) and so, it is an upper bound for $((\alpha_j, f_j, v_j))_j$. Consequently the set $\mathcal{S}$ endowed with the partial order $\preceq$ and the function $\mathcal{N}$ satisfy the hypotheses of the extension of Brezis-Browder Ordering Principle [7], i.e. Theorem 2.1.1, p. 30 in Cârjă, Necula, Vrabie [9]. Accordingly, there exists at least one $\mathcal{N}$-maximal element $(\alpha_\nu, f_\nu, v_\nu)$ in $\mathcal{S}$, i.e., an element such that, if $(\alpha_\nu, f_\nu, v_\nu) \preceq (\alpha_\sigma, f_\sigma, v_\sigma)$ then $\nu = \sigma$.

We next show that $\nu = T$, where T satisfies (viii). We prove this by contraposition, i.e., we show that an element $(\alpha_\nu, f_\nu, v_\nu)$ in $\mathcal{S}$ with $\nu < T$ is not $\mathcal{N}$-maximal. So, let us assume that $\nu < T$ and let $\xi_\nu = v_\nu(\nu) = v_\nu(\alpha_\nu(\nu))$ which, by (iii), belongs to $D(\xi, \rho) \cap K(\nu)$. In view of (v) and (vi), we have

$$\begin{aligned}\|\xi_\nu - \xi\| &\le \|S(\nu - \tau)\xi - \xi\| + \|u(\nu, \tau, \xi, f_\nu) - S(\nu)\xi\| + \|v_\nu(\nu) - u(\nu, \tau, \xi, f_\nu)\| \\ &\le \|S(\nu - \tau)\xi - \xi\| + \int_\tau^\nu \|f_\nu(s)\|\,\mathrm{d}s + (\nu - \tau)\varepsilon \\ &\le \sup_{0 \le t \le \nu - \tau} \|S(t)\xi - \xi\| + \int_\tau^\nu \ell(s)\,\mathrm{d}s + (\nu - \tau)\varepsilon.\end{aligned}$$

Recalling that $\nu < T$ and $\varepsilon < 1$, from (viii), we get

$$\|\xi_\nu - \xi\| < \rho. \tag{6}$$

At this point we act as at the beginning of the proof with ν instead of τ and with ξ_ν instead of ξ. So, distinguish between the following two complementary cases.

Case 1. If $\nu \in \mathcal{O}$, we take $\alpha\colon [\tau, \nu + \delta] \to [\tau, \nu + \delta]$ defined by

$$\alpha_{\nu+\delta}(t) = \begin{cases} \alpha_\nu(t) & \text{if } t \in [\tau, \nu] \\ \nu & \text{if } t \in (\nu, \nu + \delta) \\ \nu + \delta & \text{if } t = \nu + \delta, \end{cases}$$

In order to define $f_{\nu+\delta}$ and $v_{\nu+\delta}$, let us recall that there exists a simple solution (g, v) issuing from (ν, ξ_ν), defined on $[\nu, \nu+\delta]$. Let (g, v) be such a simple solution, and let us define

$$f_{\nu+\delta}(t) = \begin{cases} f_\nu(t) & \text{if } t \in [\tau, \nu] \\ g(t) & \text{if } t \in (\nu, \nu+\delta], \end{cases}$$

and

$$v_{\nu+\delta}(t) = \begin{cases} v_\nu(t) & \text{if } t \in [\tau, \nu] \\ v(t) & \text{if } t \in (\nu, \nu+\delta]. \end{cases}$$

One may easily see that (i) and (iii)–(vi) are satisfied and, taking into account that $\mathcal{O}$ is open and v is continuous, diminishing δ if necessary, we conclude that (ii), (vii) and (viii) are satisfied too.

Case 2. If $\nu \notin \mathcal{O}$ then $\nu \notin Z$, then $F(\nu, \xi) \in \mathcal{QTS}^A_K(\nu, \xi)$. So, from (6), we infer that there exist $f \in \mathcal{F}_{F(\nu,\xi_\nu)}$, $\delta \in (0, \varepsilon]$ with $\nu + \delta \leq T$ and $p \in X$ satisfying $\|p\| \leq \varepsilon$, such that

$$u(\nu+\delta, \nu, \xi_\nu, f) + \delta p \in D(\xi, \rho) \cap K(\nu+\delta).$$

Let us define $\alpha_{\nu+\delta} \colon [\tau, \nu+\delta] \to [\tau, \nu+\delta]$, $f_{\nu+\delta} \colon [\tau, \nu+\delta] \to X$ and $v_{\nu+\delta} \colon [\tau, \nu+\delta] \to X$ by

$$\alpha_{\nu+\delta}(t) = \begin{cases} \alpha_\nu(t) & \text{if } t \in [\tau, \nu] \\ \nu & \text{if } t \in (\nu, \nu+\delta) \\ \nu+\delta & \text{if } t = \nu+\delta, \end{cases}$$

$$f_{\nu+\delta}(t) = \begin{cases} f_\nu(t) & \text{if } t \in [\tau, \nu] \\ f(t) & \text{if } t \in (\nu, \nu+\delta], \end{cases}$$

and

$$v_{\nu+\delta}(t) = \begin{cases} v_\nu(t) & \text{if } t \in [\tau, \nu] \\ u(t, \nu, \xi_\nu, f_{\nu+\delta}) + (t-\nu)p & \text{if } t \in (\nu, \nu+\delta]. \end{cases}$$

Since $v_{\nu+\delta}(\nu+\delta) \in D(\xi, \rho) \cap K(\nu+\delta)$, $(\alpha_{\nu+\delta} f_{\nu+\delta}, v_{\nu+\delta})$, it follows that satisfies (i)–(v), with T replaced by $\nu+\delta$.

To check (vi) we consider the complementary cases: $s \leq t \leq \nu$, $\nu < s \leq t$ and $s \leq \nu \leq t$.

Clearly (vi) holds for each t, s satisfying $s \leq t \leq \nu$. If $\nu < s \leq t$, we have

$$\begin{aligned} \|v_{\nu+\delta}(t) - &u(t, \alpha_{\nu+\delta}(s), v_{\nu+\delta}(\alpha_{\nu+\delta}(s)), f_{\nu+\delta})\| \\ &= \|u(t, \nu, \xi_\nu, f_{\nu+\delta}) + (t-\nu)p - u(t, \nu, \xi_\nu, f_{\nu+\delta})\| \\ &\leq (t-\nu)\varepsilon = (t - \alpha_{\nu+\delta}(s))\varepsilon. \end{aligned}$$

Let now $s < \nu \leq t$, and let us observe that, by virtue of the evolution property (3) and of (vi) (which is valid on both $[\tau, \nu]$ and $[\nu, \nu+\delta]$), we have

$$\begin{aligned} &v_{\nu+\delta}(t) - u(t, \alpha_{\nu+\delta}(s), v_{\nu+\delta}(\alpha_{\nu+\delta}(s)), f_{\nu+\delta}) \\ &\quad = u(t, \nu, v_{\nu+\delta}(\nu), f_{\nu+\delta}) + (t-\nu)p - u(t, \nu, u(\nu, \alpha_{\nu+\delta}(s), v_{\nu+\delta}(\alpha_{\nu+\delta}(s)), f_{\nu+\delta}), f_{\nu+\delta}). \end{aligned}$$

Therefore

$$\begin{aligned} \|v_{\nu+\delta}(t) - &u(t, \alpha_{\nu+\delta}(s), v_{\nu+\delta}(\alpha_{\nu+\delta}(s)), f_{\nu+\delta})\| \\ &\leq \|v_{\nu+\delta}(\nu) - u(\nu, \alpha_{\nu+\delta}(s), v_{\nu+\delta}(\alpha_{\nu+\delta}(s)), f_{\nu+\delta})\| + (t-\nu)\|p\| \\ &\leq (\nu - \alpha_{\nu+\delta}(s))\varepsilon + (t-\nu)\varepsilon \\ &= (t - \alpha_{\nu+\delta}(s))\varepsilon, \end{aligned}$$

which proves (vi).

Similarly, diminishing δ if necessary and redefining the functions α, f and v, we deduce that (vii) and (viii) are satisfied. So $(\alpha_{\nu+\delta}, f_{\nu+\delta}, v_{\nu+\delta}) \in \mathcal{S}$,

$$(\alpha_\nu, f_\nu, v_\nu) \preceq (\alpha_{\nu+\delta}, f_{\nu+\delta}, v_{\nu+\delta})$$

and $\nu < \nu + \delta$. Hence $(\alpha_\nu, f_\nu, v_\nu)$ is not $\mathcal{N}$-maximal, and this completes the proof of Lemma 1. ■

Remark 4 *Under the general hypotheses of Lemma 1, for each $\gamma > \tau$, we can diminish both $\rho > 0$ and $T > \tau$, such that $T < \gamma$, $\rho < \gamma - \tau$ and all the conditions (i)–(viii) in Lemma 1 be satisfied.*

6 Proof of Theorem 5

PROOF OF THEOREM 5 Since the necessity follows from Theorem 3, we will confine ourselves only to the proof of the sufficiency.

Let $Z \subseteq \mathbb{R}$ be a negligible set including the negligible set N_1 appearing in Definition 6 and the negligible set N in (F_6). Let $\rho > 0$ and $T > \tau$ and ℓ be as in Lemma 1. Let $\varepsilon_n \in (0,1)$, with $\varepsilon_n \downarrow 0$, let $(\mathcal{O}_n)_{n\geq 1} \subseteq \mathbb{R}$ be a sequence of open sets such that:

(a) $Z \subseteq \mathcal{O}_n$ for each $n \in \mathbb{N}$, $n \geq 1$;

(b) $\mathcal{O}_{n+1} \subseteq \mathcal{O}_n$ and $\lambda([\tau, T] \cap \mathcal{O}_n) \leq \varepsilon_n$ for each $n \in \mathbb{N}$, $n \geq 1$;

(c) $F_{|[(I \setminus \mathcal{O}_n) \times D(\xi,\rho)] \cap \mathcal{K}}$ is strongly-weakly u.s.c., for each $n \in \mathbb{N}$, $n \geq 1$.

Let $((a_n, f_n, u_n))_n$ be a sequence of $(\varepsilon_n, \mathcal{O}_n)$-approximate solutions of (1), sequence given by Lemma 1. Clearly

$$\lim_n a_n(s) = s$$

uniformly for $s \in [\tau, T)$.

In view of (vi) in Lemma 1, we have

$$u_n(t) = u(t, \tau, \xi, f_n) + w_n(t) \tag{7}$$

for each $n \in \mathbb{N}$ and $t \in [\tau, T]$, where $\lim_n w_n(t) = 0$ uniformly for $t \in [\tau, T]$. We will show that, on a subsequence at least, $(u_n)_n$ is uniformly convergent on $[\tau, T]$ to some function u which will turn out to be a C^0-solution for the problem (1).

To do this, it suffices to show that the sequence $(u(\,\cdot\,, \tau, \xi, f_n))_n$ is uniformly convergent on $[\tau, T]$ to some function u.

Since $\|f_n(t)\| \leq \ell(t)$ for each $n \in \mathbb{N}$ and a.e. for $t \in [\tau, T]$, and the semigroup generated by A is compact, by virtue of Baras' Theorem 2.3.3, p. 47, in Vrabie [25], we conclude that $(u_n)_n$ has at least one uniformly convergent subsequence to some function u. But $a_n(t) \uparrow t$ and $\lim_n u_n(a_n(t)) = u(t)$, uniformly for $t \in [\tau, T)$, and hence, for each $k \geq 1$, the set

$$C_k = \overline{\{\,(a_n(t), u_n(a_n(t))); n \geq k,\ t \in [\tau, T)\,\}}$$

is compact. Since F is strongly-weakly u.s.c. and has weakly compact values, by Lemma 2.6.1, p. 47, in Cârjă, Necula, Vrabie [9], it follows that, for each $k \geq 1$, the set

$$B_k := \overline{\operatorname{conv}} \left(\bigcup_{n \geq k} \bigcup_{t \in [\tau,T] \setminus \mathcal{O}_k} F(a_n(t), u_n(a_n(t))) \right)$$

is weakly compact. Using again the fact that $\|f_n(t)\| \le \ell(t)$ for each $n \in \mathbb{N}$ and a.e. for $t \in [\tau, T]$, where $\ell \in L^1(\tau, T; \mathbb{R})$, recalling that B_k is weakly compact and $\lim_k \lambda(\mathcal{O}_k) = 0$, by Diestel's Theorem 1.3.8, p. 10, in Cârjă, Necula, Vrabie [9], it follows that, on a subsequence at least, $\lim_n f_n = f$ weakly in $L^1(\tau, T; X)$. By (ii) in Lemma 1, for each $k \ge 1$ there exists $n(k) \in \mathbb{N}$ so that, for each $n \ge n(k) \ge k$, we have $a_n(s) \in [\tau, T] \setminus \mathcal{O}_k$ a.e. for $s \in [\tau, T] \setminus \mathcal{O}_k$. As $\lim_n u_n(t) = u(t)$ uniformly for $t \in [\tau, T]$, $\lim_n f_n = f$ weakly in $L^1(\tau, T; X)$, $f_n(s) \in F(a_n(s), u_n(a_n(s)))$ a.e. for $s \in [\tau, T] \setminus \mathcal{O}_k$, and $F_{|[(I \setminus \mathcal{O}_k) \times D(\xi, \rho)] \cap \mathcal{K}}$ is strongly-weakly u.s.c., from Theorem 3.1.2, p. 88, in Vrabie [25], we conclude that $f(s) \in F(s, u(s))$ for each $k \ge 1$ and a.e. for $s \in [\tau, T] \setminus \mathcal{O}_k$. Since $\lim_k \lambda(\mathcal{O}_k) = 0$, we deduce that

$$f(s) \in F(s, u(s)) \tag{8}$$

a.e. for $s \in [\tau, T]$.

Finally, taking into account that A is of compact type —see Definition 8— and passing to the limit both sides in (7), for $n \to \infty$, we get

$$u(t) = u(t, \tau, \xi, f),$$

for each $t \in [\tau, T]$. Since, by (i), (iii), (vi) and (vii) in Lemma 1, we have $u_n(a_n(t)) \in K(a_n(t))$, $u_n(T) \in K(T)$, $a_n(t) \uparrow t$, as $n \to \infty$, uniformly for $t \in [\tau, T)$, $\lim_n u_n(a_n(t)) = \lim_n u_n(t) = u(t)$ uniformly for $t \in [\tau, T]$, and $\mathcal{K}$ is locally closed from the left, it follows that $u(t) \in K(t)$ for each $t \in [\tau, T]$. By (8), we conclude that u is a C^0- solution of (1), and this completes the proof. ■

7 A comparison result

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$, let $C \subseteq H$ be a closed convex cone with $C \cap (-C) = \{0\}$, let "$\preceq$" be the partial order on H defined by C, i.e., $x \preceq y$ if and only if $y - x \in C$. Let $\varphi \colon H \to \mathbb{R}_+ \cup \{+\infty\}$ be a proper, l.s.c., convex function and let $\partial\varphi \colon D(\partial\varphi) \subseteq H \rightsquigarrow H$ be the subdifferential of φ. It is known that $-\partial\varphi$ is the infinitesimal generator of a nonlinear semigroup of contractions $\{ S(t) \colon \overline{D(\partial\varphi)} \to \overline{D(\partial\varphi)};\ t \ge 0 \}$. Let $a \colon I \to D(\partial\varphi)$ be a continuous function and let $K \colon I \rightsquigarrow H$ be defined by $K(t) := \{ x \in H;\ a(t) \preceq x \}$ for each $t \in I$. Let $\mathcal{K}$ be the graph of K and $F \colon \mathcal{K} \rightsquigarrow H$ be a given multi-function. We are interested in finding sufficient conditions in order that $\mathcal{K}$ be *strongly-viable* with respect to $-\partial\varphi + F$, i.e., in order that, for each $(\tau, \xi) \in I \times H$ with $a(\tau) \preceq \xi$, to exists at least one *strong-solution* u, on $[\tau, T]$, of the problem

$$\begin{cases} u'(t) \in -\partial\varphi(u(t)) + F(t, u(t)) \\ u(\tau) = \xi \\ a(t) \preceq u(t) \qquad \text{for each } t \in [\tau, T], \end{cases}$$

i.e. a continuous function $u \colon [\tau, T] \to D(\partial\varphi)$ with $u \in W^{1,2}(\tau, T; H)$ and for which there exists $f \in L^2(\tau, T; H)$ such that:

(S_1) $u'(t) \in -\partial\varphi(u(t)) + f(t)$, a.e. for $t \in [\tau, T]$;

(S_2) $f(t) \in F(t, u(t))$, a.e. for $t \in [\tau, T]$;

(S_3) $u(\tau) = \xi$;

(S_4) $a(t) \preceq u(t)$, for each $t \in [\tau, T]$.

For a thorough study of problems of this kind, with F single-valued and independent of u, that is $F(t, u) = \{f(t)\}$, and without the monotonicity constraint (S_4), see Brezis [6].

Definition 11 *We say that a convex function* $\varphi\colon H \to \mathbb{R}_+ \cup \{+\infty\}$ *is of* compact type *if for each* $k > 0$*, the level set*

$$\mathcal{L}_k = \left\{ u \in H;\ \|u\|^2 + \varphi(u) \le k \right\}$$

is relatively compact in the norm topology of H.

Remark 5 *If* $\varphi\colon H \to \mathbb{R}_+ \cup \{+\infty\}$ *is a proper, l.s.c., convex function of compact type and* $\partial\varphi$ *is its subdifferential, then* $A = -\partial\varphi$ *generates a compact semigroup —see Vrabie* [25, Proposition 2.2.2, p. 42],— *and is of compact type in the sense of Definition 8 —Vrabie* [25, Corollary 2.3.2, p. 50].

Theorem 7 *Let* $\varphi\colon H \to \mathbb{R}_+ \cup \{+\infty\}$ *be a proper, l.s.c., convex function of compact type with* $\partial\varphi$ *single-valued, let* $a \in W^{1,1}_{\text{loc}}(I;H)$*, with* $a(t) \in D(\partial\varphi)$ *for each* $t \in I$*, let* $C \subseteq \overline{D(\partial\varphi)}$ *be a closed convex cone with* $C \cap (-C) = \{0\}$ *and* $\overline{D(\partial\varphi) \cap C} = C$*, and let* $\mathcal{K}$ *be the graph of the multi-function* $K\colon I \rightsquigarrow H$ *defined by* $K(t) = a(t) + C$ *for* $t \in I$*. Let us assume that* $S(t)C \subseteq C$ *for each* $t \ge 0$*, and* $\mathcal{K}$ *is* $-\partial\varphi$*-*C^0*-viable by itself. Let us further assume that* F *is a nonempty, convex and weakly compact valued multi-function which is essentially locally bounded and almost strongly-weakly u.s.c. Then, a sufficient condition in order that* $\mathcal{K}$ *be* C^0*-viable with respect to* $-\partial\varphi + F$ *is to exists a negligible set* $N \subseteq I$ *such that, for each* $\tau \in I \setminus N$ *and each* $\xi \in \partial C \cap D(\partial\varphi)$*, we have*

$$\operatorname{dist}\big(-\partial\varphi(a(\tau)+\xi) + \partial\varphi(\xi) - a'(\tau) + F(\tau, a(\tau)+\xi);\ C\big) = 0. \tag{9}$$

PROOF. Throughout, $O, O_1, \ldots, O_4$, denote some functions defined on $(0,1)$ with values in H, with $\lim_{h\downarrow 0} O(h) = \lim_{h\downarrow 0} O_1(h) = \cdots = \lim_{h\downarrow 0} O_4(h) = 0$.

First, let us notice that, for every $h \in (0,1)$, $\xi \in D(\partial\varphi)$ and $\eta \in H$, we have

$$\begin{cases} a(\tau+h) = a(\tau) + ha'(\tau) + hO(h) \\ u(\tau+h,\tau,\xi,\eta) = \xi - h\partial\varphi(\xi) + h\eta + hO(h) \\ S(h)\xi = \xi - h\partial\varphi(\xi) + hO(h). \end{cases} \tag{10}$$

To prove that (9) implies the tangency condition

$$F(\tau, a(\tau)+\xi) \in \mathcal{TS}^{-\partial\varphi}_{\mathcal{K}}(\tau, a(\tau)+\xi), \tag{11}$$

for each $\tau \in I \setminus N$ and each $\xi \in C \cap D(\partial\varphi)$, let us observe that, in view of (10), for each $\eta \in F(\tau, a(\tau)+\xi)$, we have

$$\begin{aligned} &\operatorname{dist}\big(u(\tau+h,\tau,a(\tau)+\xi,\eta);\ K(\tau+h)\big) \\ &\quad= \operatorname{dist}\big(u(\tau+h,\tau,a(\tau)+\xi,\eta);\ a(\tau+h)+C\big) \\ &\quad= \operatorname{dist}\big(a(\tau)+\xi+h[-\partial\varphi(a(\tau)+\xi)+\eta]+hO_1(h);\ a(\tau)+ha'(\tau)+hO_2(h)+C\big) \\ &\quad= \operatorname{dist}\big(\xi - S(h)\xi + h[-\partial\varphi(a(\tau)+\xi) - a'(\tau)+\eta] + hO_3(h);\ -S(h)\xi + C\big) \\ &\quad= \operatorname{dist}\big(h[-\partial\varphi(a(\tau)+\xi)+\partial\varphi(\xi)-a'(\tau)+\eta]+hO_4(h);\ -S(h)\xi+C\big). \end{aligned} \tag{12}$$

Since, for each $\xi \in C \cap D(\partial\varphi)$ and each $h > 0$, we have $S(h)C \subseteq C$ and C is a convex cone, it follows that

$$C \subseteq -S(h)\xi + C \quad \text{and} \quad hC = C. \tag{13}$$

Let now $\eta \in F(\tau, a(\tau)+\xi)$ be arbitrary but fixed. From (10), (12) and (13), we get

$$\begin{aligned} &\operatorname{dist}\big(u(\tau+h,\tau,a(\tau)+\xi,F(\tau,a(\tau)+\xi));\ K(\tau+h)\big) \\ &\quad\le \operatorname{dist}\big(u(\tau+h,\tau,a(\tau)+\xi,\eta);\ K(\tau+h)\big) \\ &\quad\le \operatorname{dist}\big(h[-\partial\varphi(a(\tau)+\xi)+\partial\varphi(\xi)-a'(\tau)+\eta+O_4(h)];\ C\big) \end{aligned}$$

$$
\begin{aligned}
&= \operatorname{dist}\big(h[-\partial\varphi(a(\tau)+\xi)+\partial\varphi(\xi)-a'(\tau)+\eta+O_4(h)];\ hC\big)\\
&= h\operatorname{dist}\big(-\partial\varphi(a(\tau)+\xi)+\partial\varphi(\xi)-a'(\tau)+\eta+O_4(h);\ C\big)\\
&\le h\operatorname{dist}\big(-\partial\varphi(a(\tau)+\xi)+\partial\varphi(\xi)-a'(\tau)+\eta;\ C\big)+h\|O_4(h)\|\\
&= h\|O_4(h)\|.
\end{aligned}
$$

Dividing by h and passing to the limit for $h\downarrow 0$, we deduce

$$
\begin{aligned}
\liminf_{h\downarrow 0}\frac{1}{h}&\operatorname{dist}\big(u(\tau+h,\tau,a(\tau)+\xi,F(\tau,a(\tau)+\xi));\ K(\tau+h)\big)\\
&\le \operatorname{dist}\big(-\partial\varphi(a(\tau)+\xi)+\partial\varphi(\xi)-a'(\tau)+\eta;\ C\big)
\end{aligned}
$$

for each $\eta\in F(\tau,a(\tau)+\xi)$. Since for each $\xi\in\partial C\cap D(\partial\varphi)$, we have

$$
\begin{aligned}
\inf_{\eta\in F(\tau,a(\tau)+\xi)}&\operatorname{dist}\big(-\partial\varphi(a(\tau)+\xi)+\partial\varphi(\xi)-a'(\tau)+\eta;\ C\big)\\
&=\operatorname{dist}\big(-\partial\varphi(a(\tau)+\xi)+\partial\varphi(\xi)-a'(\tau)+F(\tau,a(\tau)+\xi);\ C\big)
\end{aligned}
$$

and, by (9), the latter equals 0, we conclude that (11) holds true. If $\xi\in(C\setminus\partial C)\cap D(\partial\varphi)$, the conclusion above follows from the simple remark that, for $h>0$ small enough,

$$
\operatorname{dist}\big(u(\tau+h,\tau,a(\tau)+\xi,\eta);\ K(\tau+h)\big)=\operatorname{dist}\big(u(\tau+h,\tau,a(\tau)+\xi,\eta);\ a(\tau+h)+C\big)=0.
$$

So (11) holds true for each $\xi\in C\cap D(\partial\varphi)$, and thus we are in the hypotheses of Theorem 5 —see also Remark 5. This completes the proof. ∎

Remark 6 *Since $F(\tau,a(\tau)+\xi)$ is convex and weakly compact and C is convex and closed, the condition (9) is equivalent to: for each $\tau\in I\setminus N$ and each $\xi\in\partial C\cap D(\partial\varphi)$, there exists $\eta\in F(\tau,a(\tau)+\xi)$ such that*

$$
-\partial\varphi(a(\tau)+\xi)+\partial\varphi(\xi)-a'(\tau)+\eta\in C.
$$

Remark 7 *In the semi-linear case, i.e. $\partial\varphi=A$ with A linear, we have a sufficient condition better than (9). Namely, if $\partial\varphi$ is linear, in order that $\mathcal{K}$ be C^0-viable with respect to $-\partial\varphi+F$ it suffices to exists a negligible set $N\subseteq I$ such that, for each $\tau\in I\setminus N$ and each $\xi\in\partial C$*

$$
Aa(\tau)-a'(\tau)+F(\tau,a(\tau)+\xi)\in\mathcal{TS}_C^A(\xi).
$$

For details, see Necula, Popescu, Vrabie [19].

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Mihai Necula
Faculty of Mathematics,
"Al. I. Cuza" University,
Iaşi 700506,
Romania
`necula@uaic.ro`

Marius Popescu
Faculty of Sciences,
University ”Dunărea de Jos”,
Galaţi 800201,
Romania
`Marius.Popescu@ugal.ro`

Ioan I. Vrabie
"O. Mayer" Institute of Mathematics,
Romanian Academy,
Iaşi 700506
Romania
`ivrabie@uaic.ro`