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Mathematical Analysis

REVISTA DE LA REAL ACADEMIA DE CIENCIAS
EXACTAS, FISICAS Y NATURALES.
SERIE A: MATEMATICAS
Madrid (España / Spain)

RACSAM **104** (1), 2010, 1–3. DOI:10.5052/RACSAM.2010.01

A remark on a theorem of Howard

Ignacio Monterde, Vicente Montesinos and Manuel Valdivia

Abstract We prove that Eberlein's theorem holds for the Mackey* topology in the dual X^* of a Banach space. This improves a result of Howard. We prove, too, that in general the space X^* endowed with the Mackey* topology is not angelic.

Una extensión de un teorema de Howard

Resumen. Probamos que el Teorema de Eberlein se verifica para la topología de Mackey en el dual X^* de un espacio de Banach X. Se extiende así un resultado de Howard. Probamos, también, que, en general, el espacio X^* con la topología de Mackey no es angélico.

A subset A of a topological space is called *(relatively) countably compact* if every sequence in A has a cluster point in A (a cluster point). It is called (relatively) sequentially compact if every sequence in A has a subsequence that converges to a point of A (that converges). The following result gives a kind of "Šmulyan type" theorem for the dual of a Banach space X endowed with the dual Mackey topology $\mu(X^*, X)$, i.e., the topology on X^* of the uniform convergence on the family of all absolutely convex and weakly compact subsets of X.

Theorem 1 (Howard [3]) *Let X be a Banach space. If $A \subset X^*$ is $\mu(X^*, X)$-relatively sequentially compact, then A is $\mu(X^*, X)$-relatively compact. The converse does not hold true.*

In this note we prove an strengthening of the previous result, an "Eberlein type" theorem for the same dual Mackey topology in X^*.

The notation used is standard, and we refer to [1] for all non-defined concepts. Let $\langle E, F\rangle$ be a dual pair. The topology on E of the pointwise convergence on all points in F is denoted $w(E, F)$. If E is a locally convex space, E^* denotes its topological dual. As it is usual, the weak topology $w(X, X^*)$ of a Banach space X is denoted by w, and the weak* topology of the dual X^* by w^*. The *absolutely convex hull* (i.e., the convex and balanced hull) of a set A of a normed space is denoted $\Gamma(A)$. Given a disk B (i.e.,

Submitted by José Bonet
Received: October 29, 2009 *Accepted*: November 6, 2009
Keywords: Weak compactness, Eberlein-Šmulyan theorem, angelic spaces, Mackey topology
Mathematics Subject Classifications: Primary 46B50, Secondary: 46A04

a bounded absolutely convex set) in a locally convex space E, the space E_B generated by B is the linear hull of B endowed with the Minkowski functional $\|\cdot\|_B$ of B. It is a normed space. If E_B is a Banach space, we say that B is a *Banach disk*.

The following is the proposed improvement of Howard's result.

Theorem 2 *Let X be a Banach space. Then, every $\mu(X^*, X)$-relatively countably compact subset of X^* is $\mu(X^*, X)$-relatively compact.*

PROOF. The space $\big(X^*, \mu(X^*, X)\big)$ is complete. Hence, in order to see that a $\mu(X^*, X)$-relatively countably compact subset M of X^* is $\mu(X^*, X)$-relatively compact it is enough to show that M is $\mu(X^*, X)$-precompact. Assume not. Then we can find a weakly compact absolutely convex subset K of X and a sequence (x_n^*) in M with no subsequence Cauchy for the topology of the uniform convergence on K. By the Davis-Figiel-Johnson-Pełczyński factorization theorem (see, e.g., [1, Theorem 11.17]) we may assume, without loss of generality, that K generates a reflexive Banach space X_K. The set M can be seen naturally as a subset of $(X_K)^*$. Let $T\colon X_K \to X$ be the canonical injection. The transposed mapping $T^*\colon X^* \to (X_K)^*$ is $\mu(X^*, X)$-$\mu\big((X_K)^*, X_K\big)$-continuous. Since X_K is reflexive, $T^*(M)$ is a relatively compact subset of the Banach space $(X_K)^*$. However, $\{T^*x_n^*\}$ has no Cauchy subsequence, a contradiction. ∎

Let E be a Fréchet locally convex space, and let K be a weakly compact absolutely convex subset of E. Let $\{U_n\}_{n=1}^{\infty}$ be a fundamental sequence of closed absolutely convex neighborhoods of 0 in E. Put $C_n := 2^n K + U_n$ for $n \in \mathbb{N}$. Then each C_n is a closed absolutely convex subset of E; as C_n is absorbing, its Minkowski functional on E is a seminorm $\|\cdot\|_n$. Put $L := \ell_2\big((E, \|\cdot\|_1), (E, \|\cdot\|_2), \dots\big)$ and equip L with the seminorm

$$\|(x_n)\| := \left(\sum_{n=1}^{\infty} \|x_n\|_n^2\right)^{1/2}, \qquad \text{for } (x_n) \in L.$$

Put

$$C := \{\, x \in E;\ \sum_{n=1}^{\infty} \|x\|_n^2 \le 1 \,\}.$$

Then, obviously, $C \subset C_n$ for all $n \in \mathbb{N}$. Since, for each $n \in \mathbb{N}$ we have that $2^n K$ is a w-compact subset of E, it is clear that C is itself w-compact (and absolutely convex) in E. It is simple to prove that $K \subset C$.

Let $T\colon E_C \to L$ be defined by $T(x) := (x, x, \dots)$ for $x \in E$. Then T is an isometry into, when E_C is endowed with the norm defined by the Minkowski functional of C. Since $T(C)$ is bounded, it follows that on $T(C)$ the topologies $w\big(E_C, (E_C)^*\big)$ and $w(E, E^*)$ coincide. In particular, E_C is a reflexive Banach space and K is a w-compact subset of E_C.

Hence, with the same proof of Theorem 2, we have the following extension.

Theorem 3 *Let E be a Fréchet locally convex space. Then, every $\mu(E^*, E)$-relatively countably compact subset of E^* is $\mu(E^*, E)$-relatively compact.*

Let E be a locally convex space. We define a family $\mathcal{K}$ of subsets of E in the following way: $K \in \mathcal{K}$ if and only if K is a weakly compact absolutely convex subset of E such that there exists a Banach disk B in E with $K \subset B$ and K is weakly compact in E_B. Let $\nu(E^*, E)$ be the topology on E^* of the uniform convergence on all elements of $\mathcal{K}$. Then, by using similar arguments to those in the proof of Theorems 2 and 3 we get the following result.

Theorem 4 *Let E be a locally convex. If $\big(E^*, \nu(E^*, E)\big)$ is locally complete and M is a $\nu(E^*, E)$-relatively countably compact subset of E^*, then M is $\mu(E^*, E)$-relatively compact.*

Recall that a topological space $(T, \mathcal{T})$ is called *angelic* if every relatively countably compact subset S is relatively compact and, moreover, every point in $\overline{S}$ is the limit of a sequence in S. In angelic spaces, the classes of (relatively) countably compact, (relatively) sequentially compact and (relatively) compact sets all coincide (see, e.g., [2]).

In view of Theorem 2, it is natural to ask whether, for a Banach space X, the space $\big(X^*, \mu(X^*, X)\big)$ is always angelic. This is not the case, as the following example shows:

Example 1 *Let $X := \ell_1(\Gamma)$, where Γ is an uncountable set. Set*

$$E := \bigcup_{N \in \mathcal{N}} \overline{N}^{w^*} \quad \big(\subset \ell_\infty(\Gamma)\big),$$

where $\mathcal{N}$ is the family of all countable subsets of $c_0(\Gamma)$. E is a vector subspace of X^* ($= \ell_\infty(\Gamma)$). Let $S := E \cap B_{\ell_\infty(\Gamma)}$. We shall prove that, in $\big(X^*, \mu(X^*, X)\big)$, *the set S is sequentially compact but not* compact *(although it is, by Theorem 2, relatively compact), hence* the space $\big(X^*, \mu(X^*, X)\big)$ is not angelic *(and, according to Remark 1, the topological space $\big(B_{X^*}, w(X^*, X)\big)$ is not angelic either). Indeed, $\ell_1(\Gamma)$ has the Schur property, hence the class of w-compact and the class of $\|\cdot\|$-compact subsets of $\ell_1(\Gamma)$ coincide. A simple consequence is that, on bounded subsets of $\ell_\infty(\Gamma)$, the $\mu(X^*, X)$-topology coincides with the topology of the coordinatewise convergence (and hence with the w^*-topology). Now, every element in E, being in the w^*-closure of a countable subset of $c_0(\Gamma)$, has countable support, and so all elements in a sequence (x_n^*) in S have a common countable support. A diagonal argument gives a coordinatewise convergent (hence $\mu(X^*, X)$-convergent) subsequence, and its limit is still in S. It follows that S is $\mu(X^*, X)$-sequentially compact. It is not $\mu(X^*, X)$-closed, since its $\mu(X^*, X)$-closure (i.e., its closure in the w^*-topology) is B_{X^*}.*

Remark 1 *A consequence of the so called "angelic lemma" (see, e.g.,* [2, 3.1]*), is that, if X is a Banach space and $\big(B_{X^*}, w(X^*, X)\big)$ is angelic, then $\big(B_{X^*}, \mu(X^*, X)\big)$ is angelic, too, so the classes of $\mu(X^*, X)$-(relatively) countably compact, $\mu(X^*, X)$-(relatively) sequentially compact, and $\mu(X^*, X)$-(relatively) compact subsets of X^* all coincide.*

Acknowledgement. Supported in part by Proyecto MTM2008-03211 (V. Montesinos and M. Valdivia). Ayuda Movilidad GR001/2009/00165130 de la Generalitat Valenciana (España) (V. Montesinos). Ayuda Movilidad PR2009-0267 del Ministerio de Educación y Ciencia (España) (V. Montesinos).

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Ignacio Monterde
Spain
nachomonterde@gmail.com

Vicente Montesinos
Inst. de Matemática Pura y Aplicada,
Universidad Politécnica de Valencia,
Camino de Vera, s/n
46022 Valencia,
Spain
vmontesinos@mat.upv.es

Manuel Valdivia
Dep. de Análisis Matemático
Universidad de Valencia,
Avda. Doctor Moliner, 50
46100 Burjassot (Valencia),
Spain