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Optimal reinsurance

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Abstract. In this article we present a few of the results obtained on optimal reinsurance, since the pioneer work by Bruno de Finetti in 1940. As literature on the subject increased substantially in the last decade, a particular attention was given to these more recent results.

Reaseguro óptimo

Resumen. Este artículo presenta algunos resultados importanes de reaseguro óptimo, desde el trabajo pionero de Bruno de Finetti en 1940. Ya que la literatura sobre este tema ha aumentado de forma sustancial en la última década, le damos una atención particular a los resultados más recientes.

1 Basics of Reinsurance

1.1 Insurance and Reinsurance

Under an insurance contract, the insurer accepts to pay the policyholder's loss, (or part of it), on the occurrence of an uncertain specified event, and the policyholder accepts to pay the premium. This also happens in reinsurance contracts.

Reinsurance is a form of insurance, with some differences that result from the fact that it is insurance for insurers. Reinsurance contracts are celebrated between a direct insurer and a reinsurer, with the purpose of transferring part of the risks assumed by the insurer in its business. In this way, improved conditions for a better risk management are created. Note that reinsurers have contractual obligations only to direct insurers, not to policyholders.

The problematic risks are those carrying either the possible occurrence of very large individual losses or the possible accumulation of losses from one single event, most of the times because individual risks are not independent. Reinsurance helps insurers to fulfil their solvency requirements and to provide them with additional underwriting capacity to accept individual risks and types of business otherwise unbearable.

As in any other insurance contract, the reinsurer charges a premium to the cedent (the insurer), which is greater than the expected value of the ceded risk. There is then a trade-off between the part of the risk retained by the direct insurer and the premium paid to the reinsurer. Determining the better retention is therefore a very important issue.

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1.2 Reinsurance Forms

Reinsurance arrangements are usually divided into two classes: proportional reinsurance and non-proportional reinsurance. In proportional reinsurance the direct insurer and the reinsurer share premiums and losses at a contractually defined ratio: the reinsurer accepts a fixed share of the liabilities assumed under the original contract, and receives the same proportion of the original premium, minus a commission. In non-proportional reinsurance cessions are no longer linked to the sums insured, but to losses, and there is not a pre-determined ratio to divide premiums and losses between insurer and reinsurer. There is an amount of losses up to which the direct insurer pays —the deductible or net retention— and the reinsurer pays the losses above it. Quota Share and Surplus are proportional reinsurance settlements and Excess of Loss and Stop Loss are non-proportional forms.

Under a Quota Share, a fixed proportion of every risk accepted by the insurer is ceded to the reinsurer. This proportion settles the division rule of premiums and losses between cedent and reinsurer. That is to say: when the insurer takes a risk X_i for a premium $\pi(X_i)$, it will retain a proportion r, $0 < r < 1$, of both the claims X_i and the premium $\pi(X_i)$ and the reinsurer takes a proportion $1 - r$ also of X_i and $\pi(X_i)$. Quota share is a simple form of reinsurance with low administration costs, but does not help to balance the portfolio and provides no good protection against peak risks or the accumulation of losses.

Under Surplus reinsurance, the direct insurer retains every risk up to a certain amount, the retention, and the reinsurer is obliged to accept only the amounts accepted by the insurer above that retention. When the sum insured is below the retention, the insurer retains the entire risk. For each reinsured risk, the ratio between the retained and the ceded amounts determines how the premiums and losses are distributed between them. That is, when the insurer takes a risk X_i for a premium $\pi(X_i)$ and the capital insured and the retention are, respectively, Q_i and M, it will retain a proportion M/Q_i of both the risk X_i and the premium $\pi(X_i)$ and the reinsurer will take a proportion $1 - M/Q_i$ also of X_i and $\pi(X_i)$. Though surplus reinsurance still does not provide an effective protection against the accumulation of losses, it reduces the range of possible retained losses and the relative variability of costs, limiting the highest retained exposures and allowing to adjust the risk.

Excess of Loss (XL) can be contracted under a risk basis or under an occurrence basis. Per risk XL protects against possible large losses produced by one policy, and the reinsurer pays any loss in excess of the deductible. Per event XL protects against an accumulation of individual losses due to a single event and the reinsurer pays when the deductible is exceeded by the aggregate loss from any one occurrence. When the insurer takes a risk X_i and the retention is M, it will retain $\min\{X_i, M\} = X_i \wedge M$ and cedes $\max\{0, X_i - M\} = (X_i - M)_+$ to the reinsurer. Excess of loss is very efficient to stabilize the results of the insurer, since it reduces the exposure on individual risks. It provides also protection against accumulations and catastrophe risks. Because some reinsurers do not accept contracts with low retentions, which may produce numerous claims, the excess of loss cover has often to be organized in layers, increasing the costs.

In Stop Loss covers the reinsurer pays if the aggregate losses for a year in a certain class of business (or the whole business), net of other reinsurance covers, exceed the agreed deductible. It is not relevant whether it is exceeded by one single large loss or an accumulation of small and medium-sized losses. The deductible, now expressed as a function of the aggregate net losses, is settled either as a monetary limit (stop loss) or in terms of a proportion of premium income (excess of loss ratio). When the aggregate losses is Y and the retention is M, the insurer retains $Y \wedge M$ and cedes $(Y - M)_+$. Only stop loss reinsurance can offer protection against both increases in the severity and the frequency of losses. Administration costs are lower, but premium rating may not be an easy task.

Lately, ‘ART - Alternative Risk Transfer’ techniques have appeared, as is the case of life and no life finite risk reinsurance, property/casualty multi-year reinsurance, and multi-risk reinsurance, which are extensions of the conventional types. Another recent proposal is the Adaptative Pivot Smoothing (APS) Reinsurance, designed to reduce the variance of the retained risk without affecting the mean, taking thus account of modern portfolio theory. The proponents of APS, Koller and Dettwyller, declare that traditional forms of reinsurance —proportional and non-proportional— increase the mean burden on the cedent. Considering a risk X_i and a payment function $h(X_i)$, that specifies how much the reinsurer is required to pay of the claim X_i, the authors state that an insurer company would be best advised to take out a reinsurance policy with mean $\mathrm{E}(h(X_i)) = 0$ and standard deviation $\mathrm{D}(h(X_i)) = d\,\mathrm{D}(X_i)$, $0 < d < 1$, and suggest $h(X_i)) = a(X_i - \mathrm{E}(X_i))$, $0 < a < 1$. They also recommend that the premium, “much lower than the premium for traditional reinsurance products” (p. 332) should be calculated by the principle of zero utility.

A more exhaustive synthesis about reinsurance forms is presented in [18].

1.3 Premium Principles

Like any other insurance contract, reinsurance has a price, a premium that the cedent of the risk has to pay to the reinsurer. The premium can be more or less expensive, according to the agreed cover, and some times it is not easily computed. Let, in a general way, represent the part of the claims amount paid by the reinsurer by Z, Z obviously a random variable (r.v.). Then the reinsurance premium must be some functional of the distribution function of Z, say, $\pi(Z)$. The reinsurance premium is of great significance in the reinsurance market. There are multiple references containing an overview of the premium calculation principles, but we will list only those that are more frequently used in the theoretical papers dealing with the problem of optimal reinsurance contracts (see [16]):

Expected value principle	$\pi(Z) = (1+\rho)\mathrm{E}(Z)$
Exponential principle	$\pi(Z) = \frac{1}{\beta}\ln \mathrm{E}\left(\exp(\beta Z)\right)$
Variance principle	$\pi(Z) = \mathrm{E}(Z) + \beta\,\mathrm{Var}(Z)$
Mean value principle	$\pi(Z) = \sqrt{\mathrm{E}(Z^2)} = \sqrt{(\mathrm{E}(Z))^2 + \mathrm{Var}(Z)}$
Standard deviation principle	$\pi(Z) = \mathrm{E}(Z) + \beta\,\mathrm{D}(Z)$
Mixed principle	$\pi(Z) = \mathrm{E}(Z) + \beta_1\mathrm{D}(Z) + \beta_2\mathrm{Var}(Z)$
Modified variance principle	$\pi(Z) = \mathrm{E}(Z) + \beta_1\mathrm{D}(Z) + \beta_2\dfrac{\mathrm{Var}(Z)}{\mathrm{E}(Z)}$
Quadratic utility principle	$\pi(Z) = \mathrm{E}(Z) + e + \sqrt{e^2 - \mathrm{Var}(Z)}$
Zero utility principle	$\pi(Z)$ such that $U(w_0) = \mathrm{E}\left[U(w_0 + \pi(Z) - Z)\right]$,

where ρ, β, β_1, β_2, $e > 0$, $U(w)$ is an utility function of the reinsurer wealth such that $U'(w) > 0$, $U''(w) < 0$, and w_0 is the reinsurer’s initial wealth.

2 Optimal Reinsurance

2.1 Classical results

When designing a reinsurance programme for a risk, there is an attempt to decide optimally on the type of reinsurance and on how much to reinsure. In other words, several questions arise and many factors must be considered: the current and future business models and resultant loss exposures, the financial strength and risk aversion, the market conditions and opportunities. Although the insurer and the reinsurer are both involved, most of the theoretical works on the topic are devoted to the search for the optimal form of reinsurance from the cedent’s perspective and conclude in favour of a particular type of reinsurance, depending on the chosen optimality criteria and the premium principle.

One of the first results to achieve general acknowledgment was obtained by Borch in 1969 [3], proving that stop loss is the optimal form of reinsurance —in the sense that, for a fixed net reinsurance premium, it gives the smallest variance of the net retention. But it is obtained assuming that the loading coefficient on the net premium is not different from that in a conventional quota treaty, which makes Borch to say: “I do not consider this a particularly interesting result…Do we really expect a reinsurer to offer a stop loss contract and a conventional quota treaty with the same loading on the net premium? If the reinsurer is worried about the variance in the portfolio he accepts, he will prefer to sell the quota contract, and we should expect him to demand a higher compensation for the stop loss contract.”

This work was followed by several others with distinct approaches, but the results obtained were still in favour of the stop loss contract. Later on, different contributions appeared in favour of other kinds of reinsurance. In 1977, Beard et al. [2] proved that the quota share arrangement is optimal in the sense that it is the cheapest way to limit the variance of the retained risk, if the reinsurance premium loading increases with the variance of the ceded part. In 1979, assuming the expected value principle ant that the loading coefficient is independent of the reinsurance form, Gerber [14] showed that the excess of loss is optimal when the ceded risk is a function of the individual claims, in the sense that it maximizes the adjustment coefficient R. Remark that to maximize R is to tighten the upper bound for the probability of ruin $\phi(w_0)$, since it is possible to prove that $\phi(w_0) \leq \mathrm{e}^{-Rw_0}$. In 1987, Bowers et al. [4] came to the same conclusion, considering reinsurance based on the individual claims, the premium computed with the expected value principle and the objective of maximizing the expected utility.

Meanwhile, other significant solutions were found, following an approach to the problem that is also very common: to consider that the insurance form is known and compute then the retention level.

In 1940, de Finetti [11] considered a quota share reinsurance of n independent risks and solved the problem of the retentions r_i, $0 \leq r_i \leq 1$, $i = 1, 2, \ldots, n$, that would minimize the retained variance, under the constraint that the expected profit of the cedent would be equal to a constant B. B should be settled in order to keep the ruin probability of the cedent in an adequate level. He derived the solution $r_i = \min\{\mu[(1 - c_i)P_i - \mathrm{E}[S_i]]/\mathrm{Var}(S_i), 1\}$, where c_i is the commission rate, P_i is the gross premium (before expenses and reinsurance) for risk S_i, and μ depends on B. We can see that if a risk is actually reinsured, the retention is directly proportional to the loading and inversely proportional to the variance of the risk.

In 1979, Bühlmann [5] solved the same problem when reinsurance is an excess of loss, S_i being compound-distributed with claim numbers N_i and individual claims distribution G_i. The solution is $M_i = \mu\beta_i - (\mathrm{Var}[N_i] - \mathrm{E}[N_i])/\mathrm{E}[N_i] \int_0^{M_i} (1 - G_i(x))\,\mathrm{d}x$, where the loading β_i is a proportion of the expected ceded risk.

Waters [19] studied the behaviour of the adjustment coefficient as a function of the retention for quota share (r) and for excess of loss (M), in 1983. Under the usual assumptions, he proved that $r < 1$ when the premium is calculated with the variance principle, or the exponential principle. He further proved that, if the aggregate claims are compound Poisson and the reinsurance premium is calculated according to the expected value principle (with loading coefficient β), the optimal retention is attained at the unique point M satisfying $M = R^{-1} \ln(1 + \beta)$, R the adjustment coefficient.

In 1991, Centeno e Simões [10] dealt with the problem of determining the retention limits for mixtures of quota share and excess of loss reinsurance in such a way as to maximize the adjustment coefficient. They proved that the adjustment coefficient is unimodal with the retentions and that the optimal excess of loss reinsurance limits are still of the form $M_i = R^{-1} \ln(1 + \beta_i)$, $i = 1, 2, \ldots, n$, i.e., again the excess of loss retentions are increasing with the loading coefficients β_i.

Further details on classical results can be seen in [8].

2.2 Recent results

The literature on optimal reinsurance increased substantially in the last decade. In this paper we present a non-exhaustive selection of these results. For the sake of the readers, we tried to keep a consistent notation throughout the paper, and apologize to the authors for any errors that might have been created during the process.

Let Y be a non-negative r.v. defined on a probability space $(\Omega, \mathcal{F}, \Pr)$ representing the aggregate claims amount of an insurer in a given period of time. Let $Z(Y) \colon [0, +\infty) \to \mathbb{R}$ be a measurable function of Y, such that $0 \leq Z(Y) \leq Y$ with probability 1 and representing the part of the aggregate claims amount paid by the reinsurer. In the results that follow the authors are interested in determining the function Z in a given space that minimizes a given risk function, possibly under some constraints. The risk function varies from paper to paper as we will see.

We denote by $\pi(Z)$ the premium for the reinsurance arrangement Z.

2.2.1 Insurer's optimal reinsurance strategies (Leslaw Gajek, Dariusz Zagrodny, 2000 [12])

In this work the authors solve the problem:

$$\begin{cases} \text{Minimize}_{Z\in\mathcal{Z}} & \operatorname{Var}(Y - Z(Y)) \\ \text{s. to:} & \pi(Z) = \mathrm{E}Z(Y) + \beta\,\mathrm{D}Z(Y) \le P, \end{cases}$$

where

$$\mathcal{Z} = \{Z\colon\ [0, +\infty[\ \mapsto \mathbb{R} \mid Z \text{ is measurable and } 0 \le Z(y) \le y,\ \forall y \ge 0\}\,. \tag{1}$$

The premium $\pi(Z)$ is calculated by the standard deviation principle with safety loading parameter β, $\beta > 0$. $P > 0$ is the amount of money that the insurer is ready to spend on reinsurance.

In words: Considering the set of all plausible reinsurance arrangements $Z(Y)$, with reinsurance premium (calculated by the standard deviation principle) less than or equal to P, the authors' purpose is to find out the arrangement $Z^*(Y)$ that minimizes the variance of the retained risk, $\operatorname{Var}(Y - Z(Y))$. The standard deviation principle is selected for it takes into account the variability of the reinsurer's share of the risk.

Assuming that $\mathrm{E}Y^2 < \infty$ (i.e. they work on a L^2 space) and $\mathrm{E}Y + \beta\,\mathrm{D}Y > P$, and making use of Gâteaux differentiability and Karush-Kuhn-Tucker theorem, Gajek and Zagrodny prove their main result.

Theorem 1 *Under the given constraint,*

$$Z^*(Y) = \begin{cases} 0 & \text{if } 0 \le Y < M \\ (1-r)(Y-M) & \text{otherwise,} \end{cases}$$

is the optimal reinsurance arrangement, where $M \ge 0$ and $r \in [0,1)$ are numbers such that

$$\mathrm{E}Y - M - \int_{[M,\infty)} (y-M)\,\mathrm{d}F + \frac{r}{\beta}\sqrt{\int_{[M,\infty)} (y-M)^2\,\mathrm{d}F - \left(\int_{[M,\infty)} (y-M)\,\mathrm{d}F\right)^2} = 0$$

$$(1-r)\left[\int_{[M,\infty)} (y-M)\,\mathrm{d}F + \beta\sqrt{\int_{[M,\infty)} (y-M)^2\,\mathrm{d}F - \left(\int_{[M,\infty)} (y-M)\,\mathrm{d}F\right)^2}\,\right] = P,$$

and F is the distribution function of the total claims Y.

Remark 1 *The authors call this rule* change loss reinsurance, *because it is similar to stop loss reinsurance. It is easy to see that when $\beta \to 0$, M is bounded and $r \to 0$, which implies that Z^* tends to a stop loss contract. Additionally, the case $\beta = 0$ corresponds to the pure risk premium and a classical result settles that stop loss is an optimal reinsurance arrangement under the pure risk premium calculation. Therefore, if β is allowed to be equal to 0, the solution obtained in the paper includes this result as a particular case.*

Remark 2 *For a given $\beta > 0$, Z^* can be seen as a combination of the quota share and stop loss reinsurance strategy: if $Y \le M$, the insurer pays the total claim amount, Y; when $Y > M$, it retains $Y - (1-r)(Y-M) = M + r(Y-M)$. That is to say, it pays* $\min\{Y, M + r(Y-M)\}$.

2.2.2 Optimal reinsurance under general risk measures (Leslaw Gajek, Dariusz Zagrodny, 2004 [13])

Consider an insurer interested in purchasing as much risk protection as possible, at a price not exceeding a given limit P. The set of admissible reinsurance arrangements is the class

$$\widehat{\mathcal{Z}}(Z_1, Z_2) = \{Z : [0, +\infty[\ \mapsto \mathbb{R} \mid Z \text{ is measurable and } Z_1(y) \le Z(y) \le Z_2(y),\ y \ge 0\},$$

where the boundary functions Z_1 and $Z_2\colon\ [0,\infty)\to[-\infty,\infty)$ are also measurable functions. It is assumed that the insurer's risk is caused by positive fluctuations of the retained share of the total claim Y, relatively to its expectation.

In order to find an optimal contract, a measurable harm function $\varphi\colon\mathbb{R}\to\mathbb{R}_+$ is introduced, which measures the insurer's loss. The objective is to minimize the expected harm, represented by $\eta(Z)$, the risk measure.

In a more formal way, the authors want to find out the reinsurance arrangement $Z^*(Y)$ that is solution to the following problem:

$$\begin{cases} \text{Minimize}_{Z\in\widehat{\mathcal{Z}}(Z_1,Z_2)} & \eta(Z)=\mathrm{E}\varphi\left(Y-Z(Y)-\mathrm{E}\left(Y-Z\left(Y\right)\right)\right) \\ \text{s. to:} & \pi(Z)=\mathrm{E}Z(Y)+\beta\mathrm{D}Z(Y)\le P. \end{cases}$$

It is assumed that:

(A) $\mathrm{E}Y<\infty$,

(B) $\mathrm{E}Z_1^2(Y)<\infty$ and $\mathrm{E}Z_2^2(Y)<\infty$;

(C) $\mathrm{E}\varphi\left(Y-Z(Y)-\mathrm{E}\left(Y-Z\left(Y\right)\right)\right)<\infty$, $Z\in\widehat{\mathcal{Z}}\left(Z_1,Z_2\right)$.

Moreover, for any function φ satisfying (A)–(C) and a given $Z^*\in\widehat{\mathcal{Z}}\left(Z_1,Z_2\right)$, they define a function supporting η at Z^*, as an integrable function $s^*(\cdot)$ satisfying

$$\int_{[0,\infty]}\Big[\varphi\big(y-Z(y)-\mathrm{E}(Y-Z(Y))\big)-\varphi\big(y-Z^*(y)-\mathrm{E}(Y-Z^*(Y))\big)\Big]\,\mathrm{d}F(y) \\ \ge\int_{[0,\infty]}s^*(y)\Big[-Z(y)-Z^*(y)+\mathrm{E}(Z(Y)-Z^*(Y))\Big]\,\mathrm{d}F(y),$$

where F is the distribution function of the total claim Y.

Considering the Lagrangian function and using the Cauchy-Schwartz inequality, they derive a general sufficient condition for a given contract to be optimal within the class $\widehat{\mathcal{Z}}\left(Z_1,Z_2\right)$.

Theorem 2 *Assume that P, $\beta>0$, $P<\mathrm{E}Y+\beta\,\mathrm{D}Y$ and $\mathrm{D}Y>0$. Let s^* be a function supporting η at $Z^*\in\widehat{\mathcal{Z}}\left(Z_1,Z_2\right)$, such that $\mathrm{D}Z^*(Y)>0$. If s^*, $\lambda>0$ and $Z^*\colon[0,\infty)\to(-\infty,\infty)$ are such that*

(i) *for every $y\ge0$ such that $Z^*(y)=Z_1(y)$,*

$$\lambda-s^*(y)+\mathrm{E}s^*(y)-\lambda\beta\frac{\mathrm{E}Z^*(Y)}{\mathrm{D}Z^*(Y)}+\lambda\beta\frac{Z_1(y)}{\mathrm{D}Z^*(Y)}\ge0;$$

(ii) *for every $y\ge0$ such that $Z^*(y)=Z_2(y)$ and $Z_1(y)<Z_2(y)$,*

$$\lambda-s^*(y)+\mathrm{E}s^*(y)-\lambda\beta\frac{\mathrm{E}Z^*(Y)}{\mathrm{D}Z^*(Y)}+\lambda\beta\frac{Z_2(y)}{\mathrm{D}Z^*(Y)}\ge0;$$

(iii) *for every $y\ge0$ such that $Z_1(y)<Z^*(y)<Z_2(y)$,*

$$\lambda-s^*(y)+\mathrm{E}s^*(y)-\lambda\beta\frac{\mathrm{E}Z^*(Y)}{\mathrm{D}Z^*(Y)}+\lambda\beta\frac{Z^*(y)}{\mathrm{D}Z^*(Y)}\ge0;$$

(iv) *$\pi(Z^*)\le P$ and $\lambda(\pi(Z^*)-P)=0$;*

then Z^* minimizes $\eta(Z)$ within the class $\widehat{\mathcal{Z}}\left(Z_1,Z_2\right)$, under the constraint $\pi(Z)\le P$.

Gajek and Zagrodny derive explicit forms for the optimal contract under different insurer's risk measures. The set of admissible reinsurance arrangements is now the class $\mathcal{Z} \equiv \widehat{\mathcal{Z}}(0, Y)$.

Theorem 3 *Assume that P, $\beta > 0$, $P < \mathrm{E}Y + \beta\,\mathrm{D}Y$ and $\mathrm{D}Y > 0$. Then there are constants M and L such that $0 < M < L < \infty$ and*

$$Z^*(y) = \begin{cases} 0, & \text{for } y \le M \\ y - M, & \text{for } M < y < L \\ L - M, & \text{for } y > L \end{cases}$$

is optimal within the class $\mathcal{Z}$ under the risk measures $\eta_1(Z) = \mathrm{E}|Y - Z(Y) - \mathrm{E}(Y - Z(Y))|$ and $\eta_1^+(Z) = \mathrm{E}(Y - Z(Y) - \mathrm{E}(Y - Z(Y)))^+$. The constants are defined by the following equations:

$$\int_{[0,M]} (M - y)\,\mathrm{d}F(y) = \int_{(L,\infty)} (y - L)\,\mathrm{d}F(y)$$

and

$$P = \mathrm{E}Y - M + \beta\sqrt{(\mathrm{E}Y - M)^2 P(Y \le M) + (L - \mathrm{E}Y)^2 P(Y > L) + \int_{(M,L]} (Y - \mathrm{E}Y)^2\,\mathrm{d}F(y)}.$$

Theorem 4 *Assume that P, $\beta > 0$, $\mathrm{D}Y > 0$ and $P < \mathrm{E}Y + \beta\,\mathrm{D}Y$. Then there are constants M and r such that $M > 0$ and $r \in (0, 1)$ and the change loss contract*

$$Z^{**}(y) = \begin{cases} 0, & \text{for } y \le M \\ (1 - r)(y - M), & \text{for } y > M \end{cases}$$

is optimal within the class $\mathcal{Z}$ under the truncated variance (or semi variance) risk measure $\eta_2^+(Z) = \mathrm{E}[(Y - Z(Y) - \mathrm{E}(Y - Z(Y)))^+]^2$. The constants are solutions of the following equations:

$$P = (1 - r)\left[\int_{(M,\infty)} (y - M)\,\mathrm{d}F(y) + \beta\sqrt{\int_{(M,\infty)} (y - M)^2\,\mathrm{d}F(y) - \left(\int_{(M,\infty)} (y - M)\,\mathrm{d}F(y)\right)^2}\,\right]$$

and

$$\begin{aligned} \int_{[0,M]} (M - y)\,\mathrm{d}F(y) - \int_{[0,\mathcal{Q}(r,M)]} (\mathcal{Q}(r, M) - y)\,\mathrm{d}F(y) \\ = \frac{r}{\beta}\sqrt{\int_{(M,\infty)} (y - M)^2\,\mathrm{d}F(y) - \left(\int_{(M,\infty)} (y - M)\,\mathrm{d}F(y)\right)^2}, \\ \mathcal{Q}(r, M) = \int_{[0,M]} y\,\mathrm{d}F(y) + \int_{(M,\infty)} \big((ry + (1 - r)M\big)\,\mathrm{d}F(y). \end{aligned}$$

Note that $\eta_1(Z)$, $\eta_1^+(Z)$ and $\eta_2^+(Z)$ correspond to $\varphi_1(t) \equiv |t|$, $\varphi_1^+(t) = \max(0, t)$ and $\varphi_2^+(t) = (t^+)^2$, respectively.

2.2.3 Mean-variance Optimal Reinsurance Contracts (Marek Kaluszka, 2004 [16])

The aim of the author is to derive optimal reinsurance rules provided the cedent trades off between reducing both the variance of the retained risks and the expected value of its gains.

Considering a reinsurance treaty arranged on a claim by claim basis with a common compensation function Z, the problem is

$$\begin{cases} \text{Minimize} & \operatorname{Var}\left(\sum_{i=1}^{N}[X_i - Z(X_i)]\right) \\ \text{s. to:} & \mathrm{D}\left(\sum_{i=1}^{N} Z(X_i)\right) \le g\left(P, \mathrm{E}\sum_{i=1}^{N} Z(X_i)\right) \\ & \mathrm{E}\left(\sum_{i=1}^{N}[X_i - Z(X_i)]\right) = m\mathrm{E}N \\ & 0 \le Z(X) \le X, \end{cases}$$

where $X_1, X_2, \ldots$ is the sequence of claims occurring in a time interval, which are assumed to be independent random variables and identically distributed with X, $\operatorname{Var}X_i < \infty$; $Z(X_i)$ is the part of claim X_i that is carried by the reinsurer, to be determined; $\mathrm{D}\big(\sum_{i=1}^{N} Z(X_i)\big)$ is the standard deviation of $\sum_{i=1}^{N} Z(X_i)$ and it is assumed that the reinsurer's premium, say P_{Re}, is defined by $g(P_{\mathrm{Re}}, \mathrm{E}\sum_{i=1}^{N} Z(X_i)) = \mathrm{D}(\sum_{i=1}^{N} Z(X_i))$, $g(x, y)$ a function on $\{\,(x, y) \mid x \ge y,\ y \ge 0\,\}$, increasing in x for each y (a class which includes many usual principles); m is a fixed parameter such as $0 < m < \mathrm{E}X$; in order that the problem has nontrivial solutions, $\mathrm{D}\big(\sum_{i=1}^{N} X_i\big) \le g(P, \mathrm{E}\sum_{i=1}^{N} X_i)$, $P > 0$ the amount of money which the cedent wants to spend on reinsurance.

Using the Cauchy-Schwartz inequality, the following theorem was proved, after some calculations:

Theorem 5 *If* $(\mathrm{E}X - m)^2 \left[\mathrm{E}N \frac{\operatorname{Var}X}{(\mathrm{E}X)^2} + \operatorname{Var}N\right] \le g^2(P, \mathrm{E}N(\mathrm{E}X - m))$, *there exists a real* b *such that* $0 \le b < \sup X = \sup\{b;\ \Pr\{X - b\} > 0\}$ *and*

$$\min\left\{1, \frac{g^2(P, (\mathrm{E}X - m)\mathrm{E}N) - (\mathrm{E}X - m)^2 \operatorname{Var}N}{\mathrm{E}N \operatorname{Var}(X - b)_+}\right\} = \frac{(\mathrm{E}X - m)^2}{(\mathrm{E}(X - b)_+)^2}. \tag{2}$$

Then a solution of the problem is given by $Z^*(X) = \dfrac{\mathrm{E}X - m}{\mathrm{E}(X - b)_+}(X - b)_+$.

Remark 3 *When* $(\mathrm{E}X - m)^2 \left[\mathrm{E}N \frac{\operatorname{Var}X}{(\mathrm{E}X)^2} + \operatorname{Var}N\right] = g^2(P, \mathrm{E}N(\mathrm{E}X - m))$, *the quota share coverage* $Z^*(X) = (1 - \frac{m}{\mathrm{E}X})X$ *is a solution of the problem, since* $b = 0$ *is a solution of* (2). *Moreover, if there is a strictly positive solution* b^* *of the equation in* $\mathrm{E}X - m = \mathrm{E}(X - s)_+$, $s \ge 0$, *such that* $\operatorname{Var}(X - b^*)_+ \mathrm{E}N + (\mathrm{E}X - m)^2 \operatorname{Var}N \le g^2(P, (\mathrm{E}X - m)\mathrm{E}N)$, *then the excess of loss contract* $Z^*(X) = (X - b^*)_+$ *is a solution of the problem.*

Using similar arguments, Kaluszka presents another result for a larger class of admissible reinsurance arrangements, assuming that $0 \le \mathrm{E}Z \le \mathrm{E}X$ instead of $0 \le Z(X) \le X$ with probability 1. The problem is now

$$\begin{cases} \text{Minimize} & \operatorname{Var}\sum_{i=1}^{N}[X_i - Z(X_i)] \\ \text{s. to:} & \mathrm{D}\left(\sum_{i=1}^{N} Z(X_i)\right) \le g\left(P, \mathrm{E}\sum_{i=1}^{N} Z(X_i)\right) \\ & \mathrm{E}(X - Z) = m \\ & 0 \le Z(X) \le X \end{cases}$$

Theorem 6 *Suppose that* $g^2(P,(\mathrm{E}X-m)\mathrm{E}N) > (\mathrm{E}X-m)^2\mathrm{Var}N$. *Then the solution to the problem is given by*

$$Z^{**}(x) = a(x-\mathrm{E}X)+\mathrm{E}X-m,$$

where $a=\min\left(1,\sqrt{\frac{g^2(P,(\mathrm{E}X-m)\mathrm{E}N)-(\mathrm{E}X-m)^2\mathrm{Var}N}{\mathrm{E}N\mathrm{Var}(X)}}\right)$.

Remark 4 *If* $m=\mathrm{E}X$, *then the rule* Z^{**} *is the APS reinsurance arrangement proposed by Koller and Dettuyler* [17].

A third result is derived, under the title "Trade-off between gain and security of cedent", where the author assumes that the cedent is interested in the minimization of a function which depends not only on the variance of his payment but also on his expected gain. The problem is

$$\begin{cases} \text{Minimize} & h\left(\mathrm{E}\sum_{i=1}^{N}[X_i-Z(X_i)],\mathrm{Var}\sum_{i=1}^{N}[X_i-Z(X_i)]\right) \\ \text{s. to:} & \mathrm{D}\left(\sum_{i=1}^{N}Z(X_i)\right)=g\left(P,\mathrm{E}\sum_{i=1}^{N}Z(X_i)\right) \\ & 0\le Z(X)\le X, \end{cases}$$

$h(x,y)$ strictly increasing in y for each x, taking real values. Arguments similar to those used to prove Theorem 5 allow to prove a third theorem, where the auxiliary function

$$\begin{aligned}\varphi(t,b) = h\,&(\mathrm{E}N(\mathrm{E}X-t),\mathrm{E}N[\mathrm{Var}(X-b)_+ - 2t\mathrm{E}(b-X)_+] \\ &+\left[\sqrt{\frac{g^2(P,t\mathrm{E}N)-t^2\mathrm{Var}N}{\mathrm{E}N}}-\mathrm{D}(X-b)_+\right]^2\mathrm{E}N+\,[\mathrm{E}X-t]^2\mathrm{Var}N\big),\end{aligned}$$

$b,t\ge 0$ appears.

Theorem 7 *Assume there exist reals* a *and* b *such that:* $0\le b<\sup X$; $\varphi(a,b)=\min\{\varphi((t,b);0\le t\le \mathrm{E}(X-b)_+\}$; *and* $a^2\left[\mathrm{Var}(X-b)_+/(\mathrm{E}(X-b)_+)^2\right]\mathrm{E}N+a^2\,\mathrm{Var}\,N=g^2(P,a\mathrm{E}N)$. *Then*

$$Z^*(X)=\frac{a}{\mathrm{E}(X-b)_+}(X-b)_+$$

is a solution of the problem.

2.2.4 Optimal reinsurance policy: The adjustment coefficient and the expected utility criteria (Manuel Guerra, Maria de Lourdes Centeno, 2008 [15])

This paper is concerned with the optimal form of reinsurance from the ceding company point of view, when the cedent seeks to maximize the adjustment coefficient of the retained risk. The problem is solved by exploring the relationship between maximizing the adjustment coefficient and maximizing the expected utility of wealth for the exponential utility function.

Under the assumption that the reinsurance premium principle used is a convex functional and that some other quite general conditions are fulfilled, the authors start by proving existence and uniqueness of the solutions and provide a necessary optimal condition. These results are used to find the optimal reinsurance policy when the reinsurance premium calculation principle is the expected value principle or the reinsurance loading is an increasing function of the variance.

Y, is a non-negative r.v. representing the aggregate claims for a given period of time. Aggregate claims over consecutive periods are assumed to be i.i.d..

The set of all possible reinsurance policies is $\mathcal{Z}$ defined by (1).

For each period of time, the premium charged for a reinsurance policy is computed by a real functional $\pi\colon \mathcal{Z} \mapsto [0,+\infty]$, which is assumed to be convex, non-negative, continuous in mean-square sense and such that $\pi(0)=0$. The insurer gross premium per unit of time is c, with $c > E[Y]$, and L_Z is the profit, per unit of time, after acquiring a reinsurance policy $Z(Y)$, i.e.

$$L_Z = c - \pi(Z) - (Y - Z(Y)). \tag{3}$$

It is assumed that Y is a continuous random variable with density function f, that $E\left[Y^2\right] < +\infty$ and that $\Pr\{L_Z < 0\} > 0$ holds for every $Z \in \mathcal{Z}$.

Considering the map $G\colon \mathbb{R} \times \mathcal{Z} \mapsto [0,+\infty]$, defined by

$$G(R,Z) = \int_0^{+\infty} \mathrm{e}^{-RL_Z(y)} f(y)\,\mathrm{d}y, \qquad R \in \mathbb{R},\ Z \in \mathcal{Z},$$

the adjustment coefficient of the retained risk for a particular reinsurance policy, $Z \in \mathcal{Z}$, which is denoted R_Z is defined as the strictly positive value of R which solves the equation

$$G(R,Z) = 1, \tag{4}$$

for that particular Z, when such a root exists. The map $Z \mapsto R_Z$ is a well defined functional in the set

$$\mathcal{Z}^+ = \{Z \in \mathcal{Z} :\ (4) \text{ admits a positive solution}\}.$$

Denoting by u, $u > 0$, the initial reserve and if a reinsurance policy $Z \in \mathcal{Z}$ is in force year after year, then the probability of ultimate ruin is

$$\psi_Z(u) = \Pr\left\{u + \sum_{k=1}^{n} L_{Z_k}(w) < 0, \text{for some } n = 1, 2, \ldots\right\}$$

and it is well known that the probability of ruin satisfies the Lundberg inequality:

$$\psi_Z(u) \leq \exp(-uR_Z).$$

The main problem that the authors solve is:

Problem 1 *Find* $\left(\hat{R}, \hat{Z}\right) \in \,]0,+\infty[\, \times \mathcal{Z}^+$ *such that* $\hat{R} = R_{\hat{Z}} = \max\{R_Z : Z \in \mathcal{Z}^+\}$.

A policy $\hat{Z} \in \mathcal{Z}$ is said to be **optimal for the adjustment coefficient criterion** if $\left(R_{\hat{Z}}, \hat{Z}\right)$ solves this problem.

Considering the exponential utility function with coefficient of risk aversion $R > 0$, $U_R(w) = -\mathrm{e}^{-Rw}$, the expected utility of wealth obtained by the insurance company in a given unit of time is $E[U_R(L_Z)] = -G(R,Z)$.

A policy $Z \in \mathcal{Z}$ is said to be **optimal for the expected utility criterion** with coefficient of risk aversion R if it solves, for that particular R (a fixed constant), the following problem:

Problem 2 *Find* $\hat{Z} \in \mathcal{Z}$*, such that* $E\left[U_R\left(L_{\hat{Z}}\right)\right] = \max\{E[U_R(L_Z)] : Z \in \mathcal{Z}\}$.

It follows immediately that a policy is optimal for the expected utility criterion if and only if it is a minimizer of the functional $Z \mapsto G(R,Z)$, with the same (fixed) value of R being considered.

The authors prove that the adjustment coefficient problem can be solved in two steps:

1. For each $R \in \,]0,+\infty[$ find Z_R, the respective optimal policy for the expected utility criterion. Equivalently, find $Z_R = \arg\min\{G(R,Z) : Z \in \mathcal{Z}\}$;

2. Solve the equation with one single real variable

$$G\left(R, Z_R\right)=1.$$

This is to say that the maximal adjustment coefficient equals the coefficient of risk aversion for which the maximal expected utility that can be attained is -1. The optimal policy for the adjustment coefficient criterion coincides with the optimal policy for the expected utility criterion for this particular value of the coefficient of risk aversion. The importance of this relation is that one can concentrate on the expected utility of wealth problem, which is from the mathematical point of view a much easier problem.

The authors prove that there is always an optimal policy for the expected utility criterion and that all the optimal policies are equivalent from the economic point of view, in the sense that the net result between premiums and claims, and hence the profit, is the same with probability 1. An equivalent result is then true for the adjustment coefficient problem.

In order to deduce optimal necessary conditions the authors use needle-like perturbations. Fix a reinsurance policy, $Z \in \mathcal{Z}$. For each $v>0,\ \varepsilon>0, \alpha \in[0,1]$, consider the perturbed reinsurance policy

$$Z_{v, \alpha, \varepsilon}(y)= \begin{cases}Z(y), & \text{if } y \notin[v, v+\varepsilon]\,; \\ \alpha y, & \text{if } y \in[v, v+\varepsilon]\end{cases}$$

and assume that $\Delta \pi_Z(y)=\lim_{\alpha \rightarrow Z(y) / y}\ \lim_{\varepsilon \rightarrow 0^{+}} \frac{\pi\left(Z_{y, \alpha, \varepsilon}\right)-\pi(Z)}{\varepsilon(\alpha y-Z(y))}$ defines a function $y \mapsto \Delta \pi_Z(y)$ in a domain having probability equal to one. One important class of functionals for which $\Delta \pi_Z$ is defined with probability one for each $Z \in \mathcal{Z}$ is the class of functionals of the type

$$\pi(Z)=\gamma\left(\int_0^{+\infty} Q(y, Z(y)) f(y)\, \mathrm{d} y\right), \qquad Z \in \mathcal{Z},$$

where $Q\colon \mathbb{R}^2 \mapsto \mathbb{R}^n$, $\gamma\colon \mathbb{R}^n \mapsto \mathbb{R}$ are smooth functions. Indeed, for functionals of this class, we have:

$$\Delta \pi_Z(v)=D \gamma \cdot \frac{\partial Q}{\partial z}(v, Z(v))\, f(v), \qquad \text{a.e. } v>0,$$

for any $Z \in \mathcal{Z}$. Here, $D\gamma$ denotes the differential of $\gamma(x)$, evaluated at $x=\int_0^{+\infty} Q(y, Z(y)) f(y)\, \mathrm{d} y$. For this particular class of functionals they prove the following theorem:

Theorem 8 *Suppose that $Z \in \mathcal{Z}$ is optimal for the expected utility criterion with the particular coefficient of risk aversion $R>0$. Then, Z satisfies the following conditions.*

$$\begin{cases}\mathrm{e}^{-R L_Z(y)} \geq G(R, Z)\, D \gamma \cdot \frac{\partial Q}{\partial z}(y, Z(y))\,, & \text{if } Z(y)=y; \\ \mathrm{e}^{-R L_Z(y)}=G(R, Z)\, D \gamma \cdot \frac{\partial Q}{\partial z}(y, Z(y))\,, & \text{if } 0<Z(y)<y; \\ \mathrm{e}^{-R L_Z(y)} \leq G(R, Z)\, D \gamma \cdot \frac{\partial Q}{\partial z}(y, Z(y))\,, & \text{if } Z(y)=0,\end{cases}$$

with probability equal to one.

This theorem is used to calculate the optimal policies first when the premium calculation principle used is the expected value principle, making way for Theorem 9.

Theorem 9 *Assume the reinsurance premium is computed by the expected value principle. For each positive value of the coefficient of risk aversion, there is an optimal policy for the expected utility criterion. There is an optimal policy for the adjustment coefficient criterion. The optimal policy for any of the above criteria is unique and it is a stop-loss contract.*

Note that in the results of this article the amount to be spent with reinsurance is not limited, as it happened in the previous ones.

When considering that the reinsurance premium principle is a convex variance related premium principle, i.e. that it is a convex premium principle of the form

$$\pi(Z) = E[Z] + g(\mathrm{Var}(Z)),$$

where $g\colon [0,+\infty[\mapsto [0,+\infty[$ is a function smooth in $]0,+\infty[$ such that $g(0) = 0$ and $g'(x) > 0,\ \forall x \in]0,+\infty[$ (which happens for the standard deviation principle and the variance principle with $g(x) = \beta\sqrt{x}$ and $g(x) = \beta x$, respectively) the authors prove that:

Theorem 10 *If it is used a convex variance related principle to calculate the reinsurance premium, then for each positive value of the coefficient of risk aversion, there is an optimal policy for the expected utility criterion. There is an optimal policy for the adjustment coefficient criterion. The optimal policy for any of the above criteria must be economically equivalent to one of the following policies:*

(a) $Z \equiv 0$, *(no risk is reinsured);*

(b) *a contract satisfying*

$$y = Z(y) + \frac{1}{R}\ln\frac{Z(y)+\alpha}{\alpha}, \qquad \text{a.e. } y \geq 0,$$

where $\alpha > 0$ is a constant such that

$$\alpha = \frac{1}{2g'\left(\mathrm{Var}(Z)\right)} - E[Z],$$

and R is the risk-aversion coefficient or the maximal adjustment coefficient, according to which optimality criterion is being considered.

If g' is bounded in a neighbourhood of zero, then $Z \equiv 0$ cannot be optimal for any of the two criteria.

The authors also provide an example where this optimal arrangement is compared with the best stop loss treaty.

2.2.5 Optimal retention for a stop-loss reinsurance under the VaR and CTE risk measures (Jun Cai, Ken Seng Tan, 2007 [6])

Cai and Tan deal with the problem of determining the optimal retention M in a stop loss reinsurance, minimizing the value-at-risk (VaR) and the conditional tail expectation (CTE). Y, nonnegative, with cumulative distribution function $F(y) = \Pr[Y \leq y]$ and survival function $S(y) = \Pr[Y > y]$, is the aggregate loss for an insurance portfolio or an insurer. $I^M = Y \wedge M$ is the retained claim amount and $Z^M = (Y - M)_+$ is the ceded total claim. $F(y)$ is assumed to be a one-to-one continuous function on $(0,\infty)$ with a possible jump at 0 and $S^{-1}(y)$ exists for $0 < y < S(0)$. Furthermore, the authors consider $S^{-1}(0) = \infty$ and $S^{-1}(y) = 0$, $S(0) \leq y \leq 1$. The stop-loss reinsurance premium, calculated with the expected value principle, is $\pi(M) = (1+\rho)\delta(M)$, where $\delta(M) = \mathrm{E}[Z^M] = \int_M^\infty S(y)\,\mathrm{d}y$ is the net premium and $\rho > 0$ is the relative safety loading. $T = I^M + \pi(M)$ is the insurer's total cost.

The VaR measure, as it is well known, has the advantage of simplicity: being the $100(1-\alpha)$th percentile of I^M, the probability of the risk exceeding such a value is no greater then α, $0 < \alpha < S(0) \leq 1$, often selected to be a small value. Formally, the VaR of the insurer's retained loss at a confidence level $1-\alpha$ is $\mathrm{VaR}_{I^M}(M,\alpha) = \inf\left\{\, y : \Pr[I^M > y] \leq \alpha \,\right\}$ and the VaR of the insurer's total cost is $\mathrm{VaR}_T(M,\alpha) = \inf\{y : \Pr[T > t] \leq \alpha\}$. If I^M (T) has a one-to-one continuous distribution function on $[0,\infty)$, then the VaR is unique.

VaR is not a coherent risk measure and provides no information on the severity of the shortfall for the risk beyond the threshold. The CTE is intuitively appealing, since it captures the expected magnitude of the loss, given that risk exceeds or is equal to its VaR. When the risk is continuous, it is a coherent risk measure. Formally, $\mathrm{CTE}_{I^M}(M,\alpha) = \mathrm{E}\left[I^M | I^M \geq \mathrm{VaR}_{I^M}(M,\alpha)\right]$ and $\mathrm{CTE}_T(M,\alpha) = \mathrm{E}\left[T | T \geq \mathrm{VaR}_T(M,\alpha)\right]$ The explicit inclusion of M emphasizes that the two risk measures are functions of the retention limit.

The VaR optimization consists in determining the optimal retention M^* such that

$$\mathrm{VaR}_T(M^*,\alpha) = \min_{M>0} \{\mathrm{VaR}_T(M,\alpha)\}.$$

Noting that

$$S_{I^M}(y) = \begin{cases} S(y), & 0 \leq y < M \\ 0, & y \geq M \end{cases}$$

and

$$\mathrm{VaR}_{I^M}(M,\alpha) = \begin{cases} M, & 0 < M \leq S^{-1}(\alpha) \\ S^{-1}(\alpha), & M > S^{-1}(\alpha), \end{cases}$$

it follows that

$$\mathrm{VaR}_T(M,\alpha) = \mathrm{VaR}_{I^M}(M,\alpha) + \pi(M)$$

and Theorem 11 is established.

Theorem 11 *The optimal retention M^* exists and is given by $M^* = S^{-1}(\rho^*)$, $\rho^* = 1/(1+\rho)$, if and only if $\alpha < \rho^* < S(0)$ and $S^{-1}(\alpha) \geq S^{-1}(\rho^*) + \pi(S^{-1}(\rho^*))$. The minimum* VaR *of T is given by* $\mathrm{VaR}_T(M^*,\alpha) = M^* + \pi(M^*)$.

Remark 5 *It is of interest to note that the optimal retention depends only on the assumed loss distribution and the reinsurer's loading factor.*

Remark 6 *The following corollary gives the sufficient condition for the existence of the optimal retention M^*, and it is very easy to apply.*

Corollary 1 *If $\alpha < \rho^* < S(0)$ and $S^{-1}(\alpha) > (1+\rho)\mathrm{E}Y$, then $M^* = S^{-1}(\rho^*)$ and the minimum* VaR *of T is* $\mathrm{VaR}_T(M^*,\alpha) = M^* + \pi(M^*)$.

Moving on to the CTE optimization, it consists in determining the optimal retention $\widetilde{M}$ such that

$$\mathrm{CTE}_T(\widetilde{M},\alpha) = \min_{M>0} \{\mathrm{CTE}_T(M,\alpha)\}.$$

Simple calculations show that $\mathrm{CTE}_T(M,\alpha) = \mathrm{E}[I^M + \pi(M) | I^M + \pi(M) \geq \mathrm{VaR}_T(M,\alpha)]$ can be decomposed as $\mathrm{CTE}_T(M,\alpha) = \mathrm{CTE}_{I^M}(M,\alpha) + \pi(M)$. Performing a few more computations, it follows that

$$\mathrm{CTE}_T(M,\alpha) = \begin{cases} M + \pi(M), & 0 < M \leq S^{-1}(\alpha) \\ S^{-1}(\alpha) + \pi(M) + \dfrac{1}{\alpha} \displaystyle\int_{S^{-1}(\alpha)}^{M} S(y)\,\mathrm{d}y, & M > S^{-1}(\alpha) \end{cases}$$

and Theorem 12 can be stated and proved.

Theorem 12 *The optimal retention $\widetilde{M} > 0$ exists and is given by $\widetilde{M} = S^{-1}(\rho^*)$, if and only if $0 < \alpha < \rho^* < S(0)$. Moreover, $\widetilde{M} \geq S^{-1}(\rho^*)$ if and only if $0 < \alpha = \rho^* < S(0)$.*

Remark 7 *Comparing to the* VaR *optimization, it is of interest to note that both criteria yield the same optimal retentions, but the optimality condition for the optimization based on CTE is less restrictive, providing an added advantage of adopting this criterion.*

Remark 8 *The* VaR *based optimization has an alternative justification from the point of view of a minimum capital requirement. By assuming risk* Y*, the insurer charges an insurance premium* c *and at the same time sets aside a minimum capital* K*, so that its probability of insolvency is at most* α*. In other words, given* α *and* c*, the minimum capital* K *is the minimum solution of the inequality* $\Pr[T > K + c] \leq \alpha$*. In practice, insurers prefer to set aside as little capital as possible while satisfying the insolvency constraint. So, from the definition of* $\mathrm{VaR}_T(M, \alpha)$*, we immediately conclude that* $K = \mathrm{VaR}_T(M, \alpha) - c$*. Since* $\mathrm{VaR}_T(M^*, \alpha) = \min_{M>0}\{\mathrm{VaR}_T(M, \alpha)\}$ *than the capital requirement is also minimized at the insolvency constraint.*

2.2.6 Optimal reinsurance under VaR and CTE risk measures (Jun Cai, Ken Seng Tan, Chengguo Weng, Yi Zhang, 2008 [7])

Let again Y be a nonnegative random variable representing the aggregate claims initially assumed by an insurer. The cumulative distribution function of Y is $F(y)$, a continuous strictly increasing function on $(0, \infty)$, with a possible jump at 0, which allows Y to be a random sum $\sum_{i=1}^{N} X_i$, an important special case in actuarial loss models. The survival function of Y is $S(y)$. Consider a reinsurance arrangement such that the insurer cedes part of its loss, say $Z(Y)$, $0 \leq Z(Y) \leq Y$, to a reinsurer and retains $I(Y) = Y - Z(Y)$. $Z(y)$ is therefore the ceded loss function and $I(y)$ is the retained loss function. Let $\pi(Z)$ denote the reinsurance premium and let $T(Z) = I(Y) + \pi(Z)$ denote the total risk exposure of the insurer in the presence of reinsurance.

As usual, the insurer is now concerned with $T(Z)$ instead of Y and the objective is to find an appropriate choice of the ceded loss function, in order to provide an effective way of reducing its risk exposure. Since $T(Z(y))$ captures the overall cost of insuring a loss for a ceded loss function Z, a prudent risk management is to ensure that the risk measures associated with $T(Z(y))$ are as small as possible.

Motivated by Cai and Tan [6, (2007)], the authors strive to determine the optimal ceded loss functions that, respectively, minimize VaR and CTE of the total cost $T(Z)$. This search is done in the class $\mathcal{Z}$ of ceded loss functions $Z(y)$, defined on $[0, \infty)$ and satisfying $0 \leq Z(y) \leq y$, that are non-decreasing convex functions —and excluding $Z(y) \equiv 0$. They assume that the reinsurance premium is calculated according to the expected value principle, i.e., $\pi(Z) = (1 + \rho)\mathrm{E}[Z(Y)]$, with $\rho > 0$.

The VaR optimization consists now in determining Z^* such that

$$\mathrm{VaR}_{T(Z^*)}(\alpha) = \min_{Z \in \mathcal{Z}} \left\{ \mathrm{VaR}_{T(Z)}(\alpha) \right\},$$

$$\mathrm{VaR}_{T(Z)}(\alpha) = \inf \left\{ t : \Pr[T(Z) > t] \leq \alpha \right\}, \qquad 0 < \alpha < S(0).$$

Defining $\mathrm{VaR}_{I(Y)}(\alpha)$ as the VaR of the retained loss random variable $I(Y)$, the translation invariance property of VaR allows us to write $\mathrm{VaR}_{T(Z)}(\alpha) = \mathrm{VaR}_{I(Y)}(\alpha) + \pi(Z)$.

In a preliminary stage, the authors define a subclass $\mathcal{H}$ of $\mathcal{Z}$, which consists of all non-negative functions $h(y)$ with the form $h(y) = \sum_{j=1}^{n} c_{n,j}(y - M_{n,j})_+$, $y \geq 0$; $n = 1, 2, \ldots$, defined on $[0, \infty)$, where $c_{n,j} > 0$ and $0 \leq M_{n,1} \leq M_{n,2} \leq \cdots \leq M_{n,n}$, $n = 1, 2, \ldots$, and they formally show that any function in $\mathcal{Z}$ is the limit of a sequence of functions in $\mathcal{H}$. Consequently, by using some convergence results on VaR (and CTE), they prove that the optimal functions in $\mathcal{H}$ which minimize the VaR (and the CTE) of the total cost $T(h)$ for $h \in \mathcal{H}$, also optimally minimize the VaR (and CTE) of the total cost $T(Z)$ for $Z \in \mathcal{Z}$.

Under the assumption that the reinsurance premium is determined using the expectation premium principle, it follows that the reinsurance premium on the ceded loss $h(y) \in \mathcal{H}$ is $\pi(h) = (1 + \rho)\mathrm{E}[h(Y)] = (1 + \rho)\left\{\sum_{j=1}^{n} c_{n,j} \int_{M_{n,j}}^{\infty} S(y)\,\mathrm{d}y\right\}$.

Moreover, by defining $A_{n,i} = 1 - \sum_{j=1}^{i} c_{n,j}$ and $B_{n,i} = 1 - \sum_{j=1}^{i} c_{n,j} M_{n,j}$, $i = 1, \ldots, n$, it is easy to show that the retained loss is

$$\begin{aligned} I_h(Y) &= Y - h(Y) \\ &= Y - \sum_{j=1}^{n} c_{n,j}(Y - M_{n,j})_+ = \begin{cases} Y, & Y \le M_{n,1} \\ A_{n,i}Y + B_{n,i}, & M_{n,i} \le Y \le M_{n,i+1},\ i = 1, \ldots, n-1 \\ A_{n,n}Y + B_{n,n}, & Y \ge M_{n,n}. \end{cases} \end{aligned}$$

After some trivial calculations, the authors derive the expression for the VaR of $T(h)$, at a confidence level $1 - \alpha$:

$$\mathrm{VaR}_{T(h)}(\alpha) = \begin{cases} S^{-1}(\alpha) + \pi(h), & S^{-1}(\alpha) \le M_{n,1} \\ A_{n,i}S^{-1}(\alpha) + B_{n,i} + \pi(h), & M_{n,i} \le S^{-1}(\alpha) \le M_{n,i+1},\ i = 1, \ldots, n-1 \\ A_{n,n}S^{-1}(\alpha) + B_{n,n} + \pi(h), & S^{-1}(\alpha) \ge M_{n,n}. \end{cases}$$

Before stating Theorem 13 below, giving the solution, four lemmas are proved and the following notations are introduced:

$$\rho^* = \frac{1}{1+\rho}; \qquad M^* = S^{-1}(\rho^*); \qquad v(y) = y + \frac{1}{\rho^*}\int_y^\infty S_Y(t)\,\mathrm{d}t, \quad y \ge 0;$$
$$u(y) = S^{-1}(y) + \frac{1}{\rho^*}\int_{S^{-1}(y)}^\infty S_Y(t)\,\mathrm{d}t, \quad y \ge 0.$$

Theorem 13 *For a given confidence level* $1 - \alpha$, $0 < \alpha < S(0)$:

(a) *If* $\rho^* < S(0)$ *and* $S^{-1}(\alpha) > u(\rho^*)$, *then* $\min_{Z \in \mathcal{Z}} \left\{\mathrm{VaR}_{T(Z)}(\alpha)\right\} = u(\rho^*)$ *and the minimum* VaR *is attained at* $Z^*(y) = (y - M^*)_+$.

(b) *If* $\rho^* < S(0)$ *and* $S^{-1}(\alpha) = u(\rho^*)$, *then* $\min_{Z \in \mathcal{Z}} \left\{\mathrm{VaR}_{T(Z)}(\alpha)\right\} = S^{-1}(\alpha)$ *and the minimum* VaR *is attained at* $Z^*(y) = r(y - M^*)_+$, *for any constant* r *such that* $0 < r \le 1$.

(c) *If* $\rho^* \ge S(0)$ *and* $S^{-1}(\alpha) > v(0)$, *then* $\min_{Z \in \mathcal{Z}} \left\{\mathrm{VaR}_{T(Z)}(\alpha)\right\} = v(0)$ *and the minimum* VaR *is attained at* $Z^*(y) = y$.

(d) *If* $\rho^* \ge S(0)$ *and* $S^{-1}(\alpha) = v(0)$, *then* $\min_{Z \in \mathcal{Z}} \left\{\mathrm{VaR}_{T(Z)}(\alpha)\right\} = S^{-1}(\alpha)$ *and the minimum* VaR *is attained at* $Z^*(y) = rx$, *for any constant* r *such that* $0 < r \le 1$.

Remark 9 *Theorem 13 establishes that for the proposed optimal reinsurance model, the optimal reinsurance is a stop-loss reinsurance in case (a), a change-loss reinsurance in case (b), and a quota-share reinsurance in cases (c) and (d), depending on the risk measures's level of confidence and the safety loading for the reinsurance premium.*

To identify optimal reinsurance under CTE risk measure the deduction process is analogous to the situation of VaR criterion (but J. Cai et al. emphasize that is considerably more complicated to discuss the optimal ceded loss functions under the CTE criterion than the VaR criterion).

Recalling that the optimal functions in $\mathcal{H}$ which minimize the CTE of the total cost $T(h)$ for $h \in \mathcal{H}$, also optimally minimize the CTE of the total cost $T(Z)$ for $Z \in \mathcal{Z}$, the problem to be solved is now to find $h^* \in \mathcal{H}$ satisfying the condition

$$\mathrm{CTE}_{T(h^*)}(\alpha) = \min_{h \in \mathcal{H}} \left\{\mathrm{CTE}_{T(h)}(\alpha)\right\}.$$

After again proving four lemmas, Theorem 14 with the solution is stated:

Theorem 14 *For a given confidence level $1-\alpha$, $0<\alpha<S(0)$:*

(a) *If $\alpha<\rho^*<S(0)$, then $\min_{Z\in\mathcal{Z}}\{\mathrm{CTE}_{T(Z)}(\alpha)\}=u(\rho^*)$ and the minimum CTE is attained at $Z^*(y)=(y-M^*)_+$.*

(b) *If $\alpha=\rho^*<S(0)$, then $\min_{Z\in\mathcal{Z}}\{\mathrm{CTE}_{T(Z)}(\alpha)\}=u(\rho^*)$ and the minimum CTE is attained at any $Z^*(y)=\sum_{j=1}^{n}c_{n,j}(y-M_{n,j})_+\in\mathcal{H}$ such that $M^*\leq M_{n,1}\leq M_{n,2}\leq\cdots\leq M_{n,n}$ and $n=1,2,\ldots$.*

(c) *If $\alpha<S(0)\leq\rho^*$, then $\min_{Z\in\mathcal{Z}}\{\mathrm{CTE}_{T(Z)}(\alpha)\}=u(\rho^*)$ and the minimum CTE is attained at $Z^*(y)=y$.*

As a suggestion, we believe that perhaps it would be of interest to compare the optimal solutions contained in Theorems 13 and 14 with the contract excluded by hypothesis: $Z(y)\equiv 0$.

2.2.7 Optimal reinsurance with general risk measures (Alejandro Balbás, Beatriz Balbás, Antonio Heras, 2009 [1])

Consider an insurance company that in a given period of time receives a premium c and has to pay a non-negative random amount $Y\in L^p_+$, where L^p is the Banach space of $\mathbb{R}$-valued r.v. Y on Ω such that $\mathrm{E}|y|^p<\infty$, $p\in[1,\infty)$, $(\Omega,\mathcal{F},\mathrm{Pr})$ a probability space. Let $\eta\colon L^p\to\mathbb{R}$ be the general risk function that the insurer uses in order to control the risk of its final (at the end of the period) wealth.

Suppose that a reinsurance contract is signed in such a way that the company will cede $Z\in L^p$ and will retain $I=Y-Z$; the reinsurance premium principle is given by a continuous convex function $\pi\colon L^p\to\mathbb{R}$ and $P>0$ is the highest amount that the insurer will pay for the contract. The purpose is to choose retention I^* (which is equivalent to choose $Z^*\in\mathcal{Z}$) so as to solve problem P1:

$$\begin{cases}\text{Minimize}_{I\in\mathcal{I}} & \eta(c-I-\pi(Z))\\ & \pi(Z)\leq P,\end{cases}\tag{P1}$$

where

$$\mathcal{I}=\{I\colon\ [0,+\infty[\,\mapsto\mathbb{R}|\ I\text{ is measurable and }0\leq I(y)\leq y,\ \forall y\geq 0\}\,.$$

Note that $\mathcal{I}\equiv\mathcal{Z}$ and that the risk measure η is calculated at L_Z, with L_Z defined by (3). Note also that the authors work with this general risk function till the last section of the paper, where they propose three particular risk functions: $\sigma_1(I)=\mathrm{E}|I-\mathrm{E}(I)|$; $\sigma_2(I)=(\mathrm{E}(|I-\mathrm{E}(I)|^2))^{1/2}$; and $\mathrm{CVaR}_\alpha(I)=\max\{-\mathrm{E}(IW);W\in L^\infty,0\leq W\leq 1/\alpha,0<\alpha<1\}$, $\forall I\in L^1$. If it was not for the constraint $\pi(Z)\leq P$, we could regard this problem as a generalization of the expected utility of wealth problem.

The risk function η is in general non-differentiable and so is problem P1 above. Still, if we define the convex set $\Delta_\eta=\{W\in L^q;-\mathrm{E}(IW)\leq\eta(I),\forall I\in L^p\}$, $q\in(1,\infty]$, $1/p+1/q=1$, and assume that Δ_η is $\sigma(L^q,L^p)$-compact and also that $\eta(I)=\max\{-\mathrm{E}(IW):W\in\Delta_\eta\}$ holds for every $I\in L^p$, then it is possible to see that P1 is equivalent to problem P2:

$$\begin{cases}\text{Minimize}_{I\in\mathcal{I}} & \theta\\ & \theta+\mathrm{E}((c-I-\pi(Z))W)\geq 0,\ \forall W\in\Delta_\eta\\ & \pi(Z)\leq P\\ & \theta\in\mathbb{R}\end{cases}\tag{P2}$$

in the sense that I solves problem P1 if and only if there exists $\theta\in\mathbb{R}$ such that (θ,I) solves the equivalent problem P2, in which case $\theta=\eta(c-I-\pi(Z))$ holds. Further assuming that $\mathrm{E}(\cdot)$ remains constant on Δ_η, $\mathrm{E}(W)=\widetilde{E}\geq 0$ for every $W\in\Delta_\eta$, and that $\eta(I)\geq-\mathrm{E}(I)\widetilde{E}$ holds for every $I\in L^p$, the two preceding problems are equivalent to problem P3:

$$\begin{cases}\text{Minimize}_{I\in\mathcal{I}} & \theta\\ & \theta+(c-\pi(Z))\widetilde{E}-\mathrm{E}(IW)\geq 0,\quad\forall W\in\Delta_\eta\\ & \pi(Z)\leq P\\ & \theta\in\mathbb{R}\end{cases}\tag{P3}$$

which is easier to solve, supposing that it is a convex problem. Observe that Δ_η is composed of those linear functions that are lower than the risk measure η; every $W \in \Delta_\eta$ can be understood as a particular scenario, $-\mathrm{E}(IW)$ being a distorted expectation of I under the scenario given by W. After some calculations and intermediate results, the following theorem is proved.

Theorem 15 (Variational Principle) *Suppose that $I^* \in \mathcal{I}$ in L^p and $Z^* = Y - I^*$. I^* is a solution to problem P1 if and only if there exist $\tau^* \in \mathbb{R}^+$ and $W^* \in \Delta_\eta$ such that*

$$\begin{cases} \mathrm{E}(I^*W^*) \geq \mathrm{E}(I^*W), & \forall W \in \Delta_\eta \\ \mathrm{E}(I^*W^*) + \big(\widetilde{E} + \tau^*\big)\pi(Z^*) \leq \mathrm{E}(IW^*) + \big(\widetilde{E} + \tau^*\big)\pi(Z), & \forall I \in \mathcal{I} \\ \pi(Z^*) \leq P \\ \tau^*(\pi(Z^*) - P) = 0. \end{cases}$$

Notice that τ^* is the Lagrange multiplier associated to the budget constraint.

The authors stress out that these are necessary and sufficient conditions and therefore are a quite useful tool. Despite the generality of the analysis carried out, the solutions of the conditions in the theorem will obviously depend on the specific assumptions about the premium principle that the reinsurer applies. Balbás et al. proceed then by using the expected value premium principle, i.e. $\pi(Z) = (1+\rho)\mathrm{E}[Z(Y)]$, $\forall Z \in L^p$, with $\rho > 0$, in which case $\theta^* = \mathrm{E}(I^*W^*) - (c - (1+\rho)\mathrm{E}(Z^*)\widetilde{E}$.

Focusing afterwards on verifying whether the most usual reinsurance contracts, quota share and stop loss, solve these conditions, the conclusions that follow are:

- quota share contracts are never optimal in practice: for expectation bounded risk measures it would be necessary that $\rho = 0$, which does not hold; for deviation measures, Y should be zero-variance.
- as to stop loss contracts, they are optimal with retention M, that is to say, $I^M = \min\{Y, M\}$ is the optimal contract in the conditions of Theorem 16 —that are very easy to verify in practice, according to Balbás et al.

Theorem 16 *Suppose that $\Pr(Y > M) > 0$ and $(1+\rho)\mathrm{E}(Y - I^M) = P$. Then: I^M solves problem P1 if and only if there exists $W^* \in \Delta_\rho$ such that:*

(a) $W^* \leq \widetilde{E} + \tau^*$,

(b) $W^*(\omega) = (1+\rho)\tilde{E} + \tau^*$, $\omega \in \Omega_M = \{\omega \in \Omega; Y(\omega) > M\}$, *and*

(c) $\mathrm{E}(I^M W^*) \geq \mathrm{E}(I^M W)$, $\forall W \in \Delta_\eta$.

In such a case $\theta^* = \mathrm{E}(I^M W^*) - (c - (1+\rho)\mathrm{E}(Y - I^M)\widetilde{E}$.

In the last section of the paper, the authors propose particular risk functions and summarize their results in three more theorems. In the first two they provide results considering that η is σ_2 and σ_1, respectively, the general p-deviation being defined $\sigma_p(I) = (\mathrm{E}(|I - \mathrm{E}(I)|^p))^{1/p} = \|I - \mathrm{E}(I)\|_p$. In the third it is assumed that $\eta = \mathrm{CVaR}_\alpha(I)$, $\mathrm{CVaR}_\alpha(I) = (1/\alpha)\int_0^\alpha \mathrm{VaR}_t(I)\,\mathrm{d}t = \max\{-\mathrm{E}(IW); W \in L^\infty, 0 \leq W \leq 1/\alpha, 0 < \alpha < 1\}$, $\forall I \in L^1$, a definition that guarantees that the CVaR is always coherent and expectation bounded. In the three cases, when it is possible to identify a solution, this is of the stop loss type. As it would be reasonable to expect, when the assumptions are equivalent to those given by Cai et al. in the previous section, the solutions in the third theorem are equivalent to their solutions as well.

2.2.8 Final comments

The articles presented here, with the exception of [16], deal with optimal reinsurance when the reinsurance program is arranged on the aggregate claims. In most cases the results can be generalized to individual reinsurance, the most common way of placing reinsurance (for instance, the work presented in [9] generalizes [15], if the number of claims is Poisson, Binomial or Negative Binomial).

Most of the displayed results lead to stop loss reinsurance, or a variant of it (change loss), with the exception of [15]. This difference is not just related to the objective function and the premium principle, but also to the fact that in [15] there is no constraint on the amount of money to spend with the reinsurance premium. The constraint $\pi(Z) \leq P$ is active (holds as an equality) whenever functions like the variance of the retained risk are chosen. In our opinion it would be more interesting to replace this constraint with one on the reinsurance loading (of the type $\pi(Z) - E(Z)) \leq C$), which is the real reinsurance cost.

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