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A review of discrete-time risk models

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Abstract. In this paper, we present a review of results for discrete-time risk models, including the compound binomial risk model and some of its extensions. While most theoretical risk models use the concept of time continuity, the practical reality is discrete. For instance, recursive formulas for discrete-time models can be obtained without assuming a claim severity distribution and are readily programmable in practice. Hence the models, techniques used, and results reviewed here for discrete-time risk models are of independent scientific interest. Yet, results for discrete-time risk models can give, in addition, a simpler understanding of their continuous-time analogue. For example, these results can serve as approximations or bounds for the corresponding results in continuous-time models. This paper will serve as a detailed reference for the study of discrete-time risk models.

Una revista de modelos de riesgo en tiempo discreto

Resumen. En este artículo hacemos un repaso de los resultados para modelos de riesgo en tiempo discreto, incluyendo el modelo de riesgo binomial-compuesto, así como algunas de sus extensiones. Aunque gran parte de los modelos teóricos de riesgo se basen en el concepto de continuidad del tiempo, la realidad práctica es en sí discreta. Por ejemplo, en la práctica actuarial se programan fórmulas recursivas para modelos en tiempo discreto, sin necesidad de suponer una distribución de pérdidas conocida. Con lo cual estos modelos, las técnicas y los resultados que listamos para modelos de riesgo en tiempo discreto, generan un cierto interés científico propio. Pero más allá de sus aplicaciones directas, estos resultados para modelos en tiempo discreto también proporcionan un camino más simple hacia los modelos de riesgo análogos en tiempo continuo. Por ejemplo, los resultados en tiempo discreto pueden servir de aproximaciones o de cotas para sus resultados correspondientes en tiempo continuo. El propósito de este artículo es que pueda servir de referencia detallada para el estudio de modelos de riesgo en tiempo discreto.

1 Introduction

Problems associated with the calculation of ruin probabilities and ruin related quantities, for the continuous-time classical or Sparre Andersen risk model (Andersen [1, (1957)]), have received considerable attention in recent years, e.g., Dickson and Hipp [13, (2001)], Gerber and Shiu [15, 16, (1998, 2005)], Lin and Willmot [24, 25, (1999, 2000)], Willmot [33, (1999)], Li and Garrido [22, 23, (2004, 2005)], and references therein. These include studies of the distribution of the ruin time (finite-time ruin probabilities), the surplus before ruin, the deficit at ruin, the claim causing ruin, as well as moments of these variables. These

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quantities can be analyzed in a unified way through the expected discounted penalty function (Gerber-Shiu) function which was first introduced in Gerber and Shiu [15, (1998)] and is now one of the main research problems in ruin theory.

The compound binomial model, first proposed by Gerber [14, (1988)], is a discrete analog of the compound Poisson model in risk theory. It is a fully discrete-time model where premiums, claim amounts, and the initial surplus are assumed to be integer valued, but can be used as an approximation to the continuous-time compound Poisson model. Gerber [14, (1988)] and Shiu [30, (1989)] considered the ultimate ruin probability for the model, while Willmot [32, (1993)] studied finite-time ruin probabilities. Other references on the related topics, see, for example, Michel [26, (1989)], Dickson [11, (1994)], Cheng et al. [6, (2000)], Li and Garrido [21, (2002)], and Pavlova and Willmot [29, (2004)]. Several extensions to the compound binomial risk model can be found in Yuen and Guo [37, 38, (2001, 2006)], Cossette et al. [7, 8, (2003, 2004)], Li [19, 20, (2005)], Landriault [17, (2008)], Yang et al. [36, (2009)], and references therein.

Unlike continuous-time risk models, discrete-time risk models have not attracted much attention and the literature counts fewer contributions. Yet discrete-time risk models also have their special features and are closer to reality, results for discrete-time risk models can be simpler to understand than their analogue in the continuous-time setting. They are also of independent interest since formulas for discrete-time models are of a recursive nature and readily programmable in practice, while still reproducing the continuous analogue results as limiting cases. It is well known that explicit expressions for some ruin related quantities do not exist in continuous-time risk models with heavy tailed claims. Results for the discrete-time risk models can be used as approximations or bounds for the corresponding results in continuous time, see Dickson et al. [12, (1995)] and Cossette et al. [8, (2004)] for the approximating procedures.

The purpose of this paper is to review some of the results for discrete-time risk models in the actuarial literature. We focus on results on the expected discounted penalty functions and their special cases in the compound binomial model and its extensions, in particular the discrete-time Sparre Andersen model. The rest of the paper is structured as follows: in Section 2, we give the description of the compound binomial model and review results on ruin probabilities and ruin related quantities. Sections 3 and 4 review some results in the Sparre Andersen model with K_m inter-claim times and general inter-claim times, respectively. Finally, in Section 5, we review other extensions to the compound binomial model including time-correlated claims and general premium rates, the compound Markov binomial risk model, and the compound binomial model defined in a Markovian environment.

2 The compound binomial model

We start with the description of the compound binomial model. Assume that the premium income for each period is one and the number of claims up to time $t \in \mathbb{N}$ (or period t) is governed by a binomial process $\{\, N(t);\, t \in \mathbb{N}\,\}$ with

$$N(t) = I_1 + I_2 + \cdots + I_t, \qquad t \in \mathbb{N}^+, \tag{1}$$

with $N(0) = 0$, where I_1, I_2, ..., are i.i.d. Bernoulli random variables with mean $q \in (0,1)$. That is, in any time period there is at most one claim; the probability of having a claim is q and the probability of no claim is $1-q$. The occurrence of the claims in different time periods are assumed to be independent events. The claim amounts X_i are mutually independent, identically distributed positive integer-valued random variables with common probability function (p.f.) $p(x) = \mathbb{P}(X = x)$, for $x = 1, 2, \ldots$, with cumulative distribution function (c.d.f.) $P(x)$, probability generating function (p.g.f.) $\hat{p}(z) = \sum_{k=1}^{\infty} p(k) z^k$ and finite mean μ; they are also independent of $\{\, N(t);\, t \in \mathbb{N}\,\}$. Then the surplus of an insurance company at time t is described as

$$U(t) = u + t - \sum_{i=1}^{N(t)} X_i, \qquad t \in \mathbb{N}^+, \tag{2}$$

where $U(0) = u \in \mathbb{N}$ is the initial surplus. We further assume that $q\mu < 1$; providing a positive loading condition. This is the so-called discrete-time compound binomial risk model first proposed by Gerber [14, (1988)]. The compound binomial risk model defined in (2) can be rewritten as

$$U(t) = u + t - \sum_{i=1}^{t} Y_i, \qquad t \in \mathbb{N}^+, \tag{3}$$

where $Y_i = I_i X_i$ is the claim amount in period i with p.f. $b(0) = 1 - q$ and $b(x) = q\,p(x)$ for $x \in \mathbb{N}^+$.

We define the random variable $T = \min\{\, t \in \mathbb{N}^+;\, U(t) \le 0 \,\}$ to be the time (period) of ruin and

$$\psi(u) = \mathbb{P}\{\, T < \infty \,|\, U(0) = u \,\}, \qquad u \in \mathbb{N},$$

to be the ultimate ruin probability (also known as the eventual ruin probability, infinite-time ruin probability). By conditioning on what happened in the first period, it is easy to have that

$$\psi(0) = (1 - q)\,\psi(1) + q \tag{4}$$

and

$$\psi(u) = (1 - q)\,\psi(u + 1) + q \sum_{x=1}^{u} \psi(u + 1 - x)\,p(x) + q \sum_{x=u+1}^{\infty} p(x), \qquad u \in \mathbb{N}^+, \tag{5}$$

by the law of total probability. Note that if we know $\psi(0)$ the ruin probability with zero initial surplus, then equations (4) and (5) can be used to calculate the ruin probabilities $\psi(u)$, for $u \in \mathbb{N}^+$, recursively.

Based on the probabilistic argument that $\psi(u)$ is the ultimate probability of a visit at 0, Gerber [14, (1988)] derived the following results for the probability of ruin as well as the joint distribution of the surpluses immediately before and at ruin. The notation $a^{(k)} = k!\binom{a}{k}$ for the factorial powers of a is used below. Further, let $S_0 = 0$ and $S_k = X_1 + \cdots + X_k$ be the total claims in the first k periods. For $k \in \mathbb{N}$, denote by $g_x(k) = \mathbb{P}\{S_k = x\}$ its probability function and by $G_x(k) = \sum_{l=0}^{x} g_l(k)$ its cumulative distribution function.

Theorem 1 (Gerber [14, (1988)]) *The ultimate ruin probability for the compound binomial model in* (2) *can be expressed as*

$$\begin{aligned} \psi(0) &= q\mu, \\ \psi(u) &= (1 - q\mu) \sum_{k=1}^{\infty} \frac{1}{k!} \left(\frac{q}{1-q} \right)^{k} \mathbb{E}\left[(S_k - u)_+^{(k)} \, (1 - q)^{S_k - u} \right], \qquad u \in \mathbb{N}^+. \end{aligned}$$

Theorem 2 (Gerber [14, (1988)]) *For* $y = 0, 1, 2, \ldots$, *the joint probabilities of the surplus immediately before and at ruin for the compound binomial model in* (2) *is*

$$\mathbb{P}\left\{\, T < \infty,\, U(T-1) = x,\, U(T) = -y \,|\, U(0) = 0 \,\right\} = q\,p(x + 1 + y), \qquad x \in \mathbb{N}^+.$$

Furthermore, the probability function of the surplus at ruin is simply

$$\mathbb{P}\left\{\, T < \infty,\, U(T) = -y \,|\, U(0) = 0 \,\right\} = q\,[1 - P(y)], \qquad y \in \mathbb{N}.$$

The proofs of Theorems 1 and 2 essentially follow results developed by Gerber for sums of derivatives of probability generating functions and known martingale results.

Now let $\bar{\psi}(u) = 1 - \psi(u)$ be the corresponding non-ruin probability, with initial surplus $u \in \mathbb{N}$. Shiu [30, (1989)] derived following formulas for $\bar{\psi}(u)$, corresponding to those in Theorem 1, by alternative methods. Note that Shiu [30, (1989)] defines ruin as the event that the surplus $U(t)$ becomes strictly negative, whereas in Gerber [14, (1988)] the ruin is defined when the surplus $U(t)$ being negative or 0 (i.e. non-positive), for some $t \in \mathbb{N}^+$.

Theorem 3 (Shiu [30, (1989)]) *The non-ruin probabilities for the compound binomial model in (2) can be expressed as*

$$\bar{\psi}(0) = \frac{1-q\mu}{1-q},$$
$$\bar{\psi}(u) = \phi(0)\sum_{k=0}^{\infty}\left(\frac{-q}{1-q}\right)^{k}\mathbb{E}\left[\binom{u+k-S_k}{k}(1-q)^{S_k-u}1_{+}(u-S_k)\right], \qquad u \in \mathbb{N}^{+},$$

where $1_{+}(k) = 1$, for $k \in \mathbb{N}$, and 0 otherwise.

This particular result is obtained through the solution of a Volterra equation of the second kind.

Now we consider the finite time survival probability before time k with an integer-valued initial surplus u, defined by

$$\bar{\psi}(u;k) = \mathbb{P}\{\,U(j) \geq 0\,;\, j = 0,1,2,\ldots,k \,|\, U(0) = u\,\}, \qquad k,u \in \mathbb{N}. \tag{6}$$

The ultimate survival probability $\bar{\psi}(u)$ is the limiting case of (6), i.e.,

$$\bar{\psi}(u) = \lim_{k\to\infty}\bar{\psi}(u;k), \qquad u \in \mathbb{N}.$$

Explicit formulas are derived by Willmot [32, (1993)] for finite time survival probabilities using analytical techniques, such as Lagrange's expansions of generating functions. Note that for the theorem below the definition of ruin follows that of Shiu [30, (1989)] rather than Gerber [14, (1988)].

Theorem 4 (Willmot [32, (1993)]) *The finite time survival probabilities for the compound binomial model in (2) can be expressed as*

$$\bar{\psi}(0;k) = \frac{\sum_{l=0}^{k}(k-l+1)\,g_l(k+1)}{(1-q)(k+1)}, \qquad k \in \mathbb{N},$$
$$\bar{\psi}(u;k) = G_{u+k}(k) - (1-q)\sum_{l=0}^{k-1}\bar{\psi}(0,k-1-l)\,g_{u+l+1}(l), \qquad k,u \in \mathbb{N}^{+}.$$

Now for x, $y \in \mathbb{N}$ and $t \in \mathbb{N}^{+}$, define

$$f_3(x,y,t\,|\,u) = \mathbb{P}\,\{\,U(T-1) = x,\; U(T) = -y,\; T = t\,|\,U(0) = u\,\},$$

which is the joint probability distribution of the time of ruin, T, the surplus just before ruin, $U(T-1)$, and the surplus at ruin, $U(T)$. Let v $(0 < v < 1)$ be a discount factor and define

$$f_2(x,y\,|\,u) = \sum_{t=1}^{\infty} v^t f_3(x,y,t\,|\,u), \qquad x,\ y,\ u \in \mathbb{N}, \tag{7}$$

to be the "discounted" probability of ruin for an initial surplus u, such that the surplus before ruin is x and the deficit at ruin is y. Further, define the following two functions:

$$\begin{aligned} f_1(x\,|\,u) &= \sum_{y=0}^{\infty} f_2(x,y\,|\,u), \qquad x,u \in \mathbb{N}, \\ g(y\,|\,u) &= \sum_{x=0}^{\infty} f_2(x,y\,|\,u), \qquad y,u \in \mathbb{N}. \end{aligned} \tag{8}$$

Willmot points out that the results in Theorem 4 can also be obtained through probabilistic arguments, such as those of Gerber and Shiu for Theorems 1–3.

Cheng, Gerber and Shiu [6, (2000)] introduced Lundberg's fundamental equation for the compound binomial risk model in (2), that is,

$$q\,\mathbb{E}[r^{X-1}] + (1-q)\,r^{-1} = v^{-1}, \qquad r > 0. \tag{9}$$

It can be shown that equation (9) has a solution $r = \rho \in (0,1)$, and under some regularity conditions on the tail of the probability function, it has another solution $r = R > 1$.

An explicit formula for $f_2(x, y \,|\, 0)$ is obtained in Cheng, Gerber and Shiu [6, (2000)] where they show that $f_2(x, y \,|\, u)$ can be expressed in terms of $f_2(x, y \,|\, 0)$ and an auxiliary function $h(u)$. Note that in the results that follow, ruin is the event that $U(t) \le 0$, for some $t \ge 1$, which coincides with Gerber [14, (1988)].

Theorem 5 (Cheng, Gerber and Shiu [6, (2000)]) *For the compound binomial model in* (2),

$$f_2(x, y \,|\, 0) = q\,v\,\rho^x p(x+y+1), \qquad x, y \in \mathbb{N}, \tag{10}$$

$$f_2(x, y \,|\, u) = f_2(x, y \,|\, 0)\,[h(u) - \rho^{-x} h(u-x) I(u > x)], \qquad x \in \mathbb{N}^+, y \in \mathbb{N}, \tag{11}$$

where $h(u)$ *is defined as the solution of*

$$h(u) = \sum_{z=0}^{u-1} h(u-z)\, g(z \,|\, 0) + \rho^u, \qquad u \in \mathbb{N}^+,$$

and $g(z \,|\, 0) = \sum_{x=0}^{\infty} f_2(x, z \,|\, 0) = q\,v \sum_{x=0}^{\infty} \rho^x\, p(x+y+1)$*, for* $z \in \mathbb{N}$.

For the risk model described in (3), Li and Garrido [21, (2002)] further explored the following expected discounted penalty at ruin. Let $w(x, y)$, x, $y \in \mathbb{N}$ be the non-negative values of a penalty function payable at ruin. For $0 < v < 1$, define

$$\phi(u) = \mathbb{E}\left[v^T w\big(U(T-1), |U(T)|\big)\, I(T < \infty) \,\big|\, U(0) = u\right], \qquad u \in \mathbb{N}, \tag{12}$$

where the time of ruin T is the first time that the surplus in (3) becomes non-positive. The quantity $w\big(U(T-1), |U_1(T)|\big)$ can be interpreted as the penalty, payable at the time of ruin, for a surplus of $U(T-1)$ and a deficit of $|U(T)|$. Then ϕ is the expected discounted penalty (Gerber-Shiu) function, if v is viewed as a discount rate. The following results are proved using finite differences and probability generating functions.

Theorem 6 (Li and Garrido [21, (2002)]) *For the discrete-time risk model in* (3),

$$\phi(0) = v \sum_{x=0}^{\infty} \sum_{y=0}^{\infty} \rho^x\, w(x, y)\, b(x+y+1),$$

$$\phi(u) = v \sum_{x=0}^{u-1} \phi(u-x) \sum_{y=0}^{\infty} \rho^y\, b(x+y+1) + v\,\rho^{-u} \sum_{x=u}^{\infty} \rho^x \sum_{y=0}^{\infty} w(x, y)\, b(x+y+1),$$

where $u \in \mathbb{N}^+$ *and* $0 < \rho < 1$ *is the root of Lundberg's equation* $\hat{b}(s)/s = 1/v$.

Now define the compound geometric p.f. $k(u) = \sum_{n=0}^{\infty} \frac{\beta}{1+\beta}\big(\frac{1}{1+\beta}\big)^n l^{*n}(u)$, for $u \in \mathbb{N}$. Here β is defined as $1/(1+\beta) := \sum_{z=0}^{\infty} g(z|0) = (v-\rho)/(1-\rho)$, by (8) and (10), while $l(z) = (1+\beta)\, g(z|0)$, which is a proper p.f. on $\mathbb{N}$. Then the following explicit formula can be given for $\phi(u)$.

Theorem 7 (Li and Garrido [21, (2002)])

$$\phi(u) = \frac{1}{\beta} \sum_{z=0}^{u-1} M(u-z)\, k(z), \qquad u \in \mathbb{N}^+, \tag{13}$$

where $M(u) = (1+\beta)A(u)$ *and*

$$A(u) = \phi(0) - v\,\rho^{-u}\sum_{z=0}^{u-1}\rho^z\sum_{y=0}^{\infty} w(z,y)\,b(z+y+1), \qquad u\in\mathbb{N}^+,$$

with $A(0) = \phi(0)$.

For $x\in\mathbb{N}^+$ and $y\in\mathbb{N}$ fixed, set

$$w(t,s) = \begin{cases} 1, & \text{if } t=x,\ s=y,\\ 0, & \text{otherwise,}\end{cases}$$

then, in this particular case, $\phi(u)$ in (13) becomes (7). An application of Theorem 7 yields an alternative expression to (11), given in the following corollary.

Corollary 1 (Li and Garrido [21, (2002)])

$$f_2(x,y|u) = \gamma(u)\,f_2(x,y|0), \qquad x,\,u\in\mathbb{N}^+, \quad y\in\mathbb{N}, \tag{14}$$

where $f_2(x,y|0)$ *is given by* (10) *and*

$$\gamma(u) = \begin{cases} \frac{1}{\beta}\sum_{z=0}^{u-1}(1+\beta)\,\rho^{z-u}k(z), & \text{if } 1\le u\le x,\\ \frac{1}{\beta}\sum_{z=u-x}^{u-1}(1+\beta)\,\rho^{z-u}k(z) & \text{if } u>x.\end{cases}$$

In particular, if $v=1$, $\gamma(u)$ *simplifies to*

$$\gamma(u) = \begin{cases} \frac{1-\psi(u)}{1-\psi(0)}, & 1\le u\le x,\\ \frac{\psi(u-x)-\psi(u)}{1-\psi(u)}, & u>x.\end{cases}$$

Then (14) *simplifies to Dickson's formula in the compound binomial model. See Dickson* [10, (1992)] *for the continuous version of this formula.*

Denote by $\phi_T(u) = \mathbb{E}\big[v^T I(T<\infty)\,|\,U(0)=u\big]$, the p.g.f. of the ruin time T with initial reserve $u\in\mathbb{N}^+$. Clearly, $\phi_T(u)$ is a special case of (12) when $w(x,y)=1$. An application of Theorem 6 gives in the corollary below a recursive formula or discrete defective renewal equation for $\phi_T(u)$. Li and Garrido [21, (2002)] further shows in the theorem below that $\phi_T(u)$ can be expressed as a compound geometric tail.

Corollary 2 (Li and Garrido [21, (2002)])

$$\phi_T(u) = \sum_{z=0}^{u-1}\phi_T(u-z)\,g(z|0) + H(u), \qquad u\in\mathbb{N}^+, \tag{15}$$

where $g(z|0) = \sum_{x=0}^{\infty} v\,\rho^x\,p(x+z+1)$, *and* $H(u) = \phi_T(0) - \sum_{z=0}^{u-1} g(z|0)$ *with* $\phi_T(0) = H(0) = \frac{v-\rho}{1-\rho}$.

Theorem 8 (Li and Garrido [21, (2002)]) *The solution to equation* (15) *can be expressed as a compound geometric sum*

$$\phi_T(u) = \frac{\beta}{1+\beta}\sum_{n=1}^{\infty}\left(\frac{1}{1+\beta}\right)^n \bar{L}^{*n}(u-1), \qquad u\in\mathbb{N},$$

where $\bar{L}(u) = \sum_{z=u+1}^{\infty} l(z)$ *is the tail probability of* l, *while* $\bar{L}^{*n}$ *and* l^{*n} *are the* n*-th convolutions of* $\bar{L}$ *and* l, *respectively with* $\bar{L}(-1) = \bar{L}^{*n}(-1) = 1$.

Note that Theorem 8 gives a generalized Beekman's convolution formula for the discrete-time model. If $v = \rho = 1$, then it simplifies to Beekman's classical convolution formula for the ruin probability $\psi(u) = (1 - q\mu) \sum_{n=1}^{\infty} (q\mu)^n \bar{B}_1^{*n}(u - 1)$, where $\bar{B}_1$ is the tail probability of b_1, where b_1 is the equilibrium distribution of b, defined as

$$b_1(x) = \frac{\sum_{y=x+1}^{\infty} b(y)}{\sum_{x=0}^{\infty} \bar{B}(x)}, \qquad x \in \mathbb{N}.$$

See Li and Garrido [21, (2002)] for the definition of the higher order equilibrium distributions and their properties.

3 The discrete-time Sparre Andersen model with K_m inter-claim times

As an extension of the compound binomial risk model, the discrete-time Sparre Andersen model has been studied recently. Li [19, 20, (2005)] investigate the discounted penalty function in a Sparre Andersen risk model with discrete K_m inter-claim times. Pavlova and Willmot [29, (2004)] gives an expression of the expected discounted penalty function in the discrete-time stationary renewal risk model in terms of that in the corresponding ordinary renewal risk model, while Wu and Li [35, (2009)] considers the same function in a Sparre Andersen risk model with general inter-claim times. Wu and Li [34, (2008)] derives some results for the latter model with phase-type claims. We review here these results.

Consider a discrete-time compound renewal (Sparre Andersen) risk model in which the surplus can be described as

$$U(t) = u + t - \sum_{i=1}^{N(t)} X_i, \qquad t \in \mathbb{N}^+, \tag{16}$$

where $U(0) = u \in \mathbb{N}$ is the initial surplus. The counting process $\{ N(t) \,;\, t \in \mathbb{N} \}$ denotes the number of claims up to time t and is defined as $N(t) = \max\{ k : W_1 + W_2 + \cdots + W_k \leq t \}$, where the claim waiting times, W_i, are assumed to be i.i.d. positive integer-valued random variables with common p.f. $k(x) = \mathbb{P}\{W = x\}$, for $x = 1, 2, \ldots$. Denote by $\hat{k}(s) = \sum_{i=1}^{\infty} s^i\, k(i)$, $s \in \mathbb{C}$, its p.g.f..

All other assumptions are as in the compound binomial risk model, with the addition that $\{W_i \,;\, i \in \mathbb{N}^+\}$ and $\{X_i \,;\, i \in \mathbb{N}^+\}$ are independent, and $\mathbb{E}[W] = (1+\theta)\,\mathbb{E}[X] = (1+\theta)\mu$, $\theta > 0$, in order to have a positive loading factor. In the literature, the time of ruin T for the Sparre Andersen model is defined to be the first time when the surplus falls below zero, with $T = \infty$ if ruin does not occur. Note that when $k(x) = q\,(1-q)^{x-1}$, $x = 1, 2, \ldots$, the Sparre Andersen model in (16) reduces to the compound binomial model in (2).

We start with the results from Li [19, 20, (2005)] for a class of compound renewal risk process with inter-claim times having a discrete K_m distribution, i.e., the p.g.f. of the distribution function is a ratio of two polynomials of order $m \in \mathbb{N}^+$. More precisely, the p.g.f. of $k(x)$, $x \in \mathbb{N}^+$, can be expressed as

$$\hat{k}(s) = \frac{s\left[\prod_{i=1}^{m}(1-q_i) + \sum_{j=1}^{m-1} \beta_j\,(s-1)^j\right]}{\prod_{i=1}^{m}(1 - s\,q_i)}, \qquad \Re(s) < \min\left\{ \frac{1}{q_i} \,;\, 1 \leq i \leq m \right\},$$

where $0 < q_i < 1$, for $i = 1, 2, \ldots, m$, and the coefficients $\beta_1, \beta_2, \ldots, \beta_{m-1}$ are such that k is a p.f.. In particular, if $q_1, q_2, \ldots, q_m$ are distinct, by partial fractions, k can be expressed as a linear combination of m geometric distributions with parameters q_i:

$$k(x) = \sum_{i=1}^{m} \zeta_i\,(1-q_i)\,q_i^{x-1}, \qquad x = 1, 2, \ldots,$$

where $\zeta_1, \zeta_2, \ldots, \zeta_m$ are such that $\sum_{i=1}^m \zeta_i = 1$ and given explicitly by

$$\zeta_i = \frac{\sum_{k=1}^{m-1} \beta_k (1/q_i - 1)^k + \prod_{j=1}^m (1-q_j)}{(1-q_i)\left[\prod_{j=1,\, j\neq i}^m (1-q_j/q_i)\right]}, \qquad i = 1, 2, \ldots, m.$$

This class of distributions includes, as special cases, the shifted geometric, shifted or truncated negative binomial, discrete phase-type distributions, as well as linear combinations (including mixture) of these. Especially, the classical compound binomial risk model is a special case when $m = 1$.

By using the martingale arguments, Li [19, (2005)] obtains the generalized version of Lundberg's equation in (9), which is,

$$\hat{k}\left(\frac{v}{s}\right)\hat{p}(s) = 1, \qquad s \in \mathbb{C}, \tag{17}$$

and proves that for $0 < v < 1$ and $m \in \mathbb{N}^+$, equation (17) has exactly m roots, say $\rho_i(v)$, $i = 1, 2, \ldots, m$, with $0 < |\rho_i| < 1$. For simplicity, in what follows we assume that $\rho_1, \rho_2, \ldots, \rho_m$ are distinct.

Define T_r to be an operator on any real-valued function $f(x)$, $x \in \mathbb{N}^+$, by

$$T_r f(y) = \sum_{x=y}^{\infty} r^{x-y} f(x) = \sum_{x=0}^{\infty} r^x f(x+y), \qquad r \in \mathbb{C}, \quad y \in \mathbb{N}^+. \tag{18}$$

The continuous version of this operator can be found Dickson and Hipp [13, (2001)].

A recursive formula is derived for the expected discounted penalty (Gerber-Shiu) function defined as in (12), which can be used to analyze many quantities associated with the time of ruin, e.g., the surplus before ruin, the deficit at ruin, and the claim causing ruin. Furthermore, an explicit formula for the Gerber-Shiu function is given in terms of a compound geometric distribution function in Li [20, (2005)]. These results are given in the following theorems and corollaries.

Theorem 9 (Li [19, (2005)]) *For the discrete-time Sparre Andersen model* (16),

$$\phi(u) = \sum_{y=1}^{u} \phi(u-y)\, g(y|0) + H(u), \qquad u \in \mathbb{N}^+, \tag{19}$$

$$\phi(0) = \left[\prod_{i=1}^m \frac{\rho_i}{v q_i}\right] \sum_{j=1}^m c_j \sum_{x=0}^{\infty}\sum_{y=1}^{\infty} \rho_j^x\, w(x,y)\, p(x+y+1), \tag{20}$$

where $c_j = v\dfrac{\rho_j^{m-1}\prod_{i=1}^m (1-q_i) + \sum_{t=1}^{m-1}\beta_t \rho_j^{m-1-t}(v-\rho_j)^t}{\prod_{k=1,k\neq j}^m (\rho_j - \rho_k)}$, $g(y|0) = \sum_{x=0}^{\infty} f_2(x,y|0)$ *is given by*

$$g(y|0) = \left[\prod_{i=1}^m \frac{\rho_i}{v q_i}\right] \sum_{j=1}^m c_j T_{\rho_j} p(y+1), \qquad y \in \mathbb{N}^+, \tag{21}$$

in which T_r *is an operator defined by* (18)*, and*

$$H(u) = \left[\prod_{i=1}^m \frac{\rho_i}{v q_i}\right] \sum_{j=1}^m c_j T_{\rho_j} \omega(u+1), \qquad u \in \mathbb{N}^+,$$

with $\omega(u) = \sum_{x=1}^{\infty} w(u-1,x) p(u+t)$.

Expression (19) is a recursive formula for $\phi(u)$ with the starting point $\phi(0)$, given by (20). In particular, if $w(x,y) = 1$, then $\phi(u)$ simplifies to the p.g.f. of the time of ruin T, with respect to the discount factor v, defined by $\phi_T(u)$. Similar to Corollary 2 for the discrete-time risk model, Li [19, (2005)] derives the following result for the discrete-time Sparre Andersen model.

Corollary 3 (Li [19, (2005)]) *For the discrete-time Sparre Andersen model* (16),

$$\phi_T(u) = \sum_{y=1}^{u} \phi_T(u-y)\, g(y|0) + \sum_{y=u+1}^{\infty} g(y|0), \qquad u \in \mathbb{N}^+,$$

where $g(y|0)$ *is given by* (21)*. The starting value of the above recursion is*

$$\phi_T(0) = 1 - \left[1 - \hat{k}(v)\right] \prod_{i=1}^{m} \left[\frac{\rho_i(1 - v\, q_i)}{v\, q_i(1-\rho_i)}\right].$$

Furthermore, $\phi_T(u)$ *can be expressed as the following compound geometric tail*

$$\phi_T(u) = \frac{\beta}{1+\beta} \sum_{n=1}^{\infty} \left(\frac{1}{1+\beta}\right)^n \bar{L}^{*n}(u), \qquad u \in \mathbb{N},$$

where β *is such as that* $1/(1+\beta) = \phi_T(0)$, $\bar{L}^{*n}$ *is the* n*-th convolutions of* $\bar{L}$*, while* $\bar{L}(u) = \sum_{z=u+1}^{\infty} l(z)$ *is the tail probability of* l *with definition* $l(y) = (1+\beta) g(y|0)$.

Note that Theorem 7 is still true for the surplus $U(t)$ in the discrete-time Sparre Andersen model case. In particular, the discounted joint p.f. $f_2(x, y|u)$ can be obtained as a special case as follows.

Theorem 10 (Li [20, (2005)]) *For* $x \in \mathbb{N}$*, and* $y \in \mathbb{N}^+$*,*

$$f_2(x,\, y\,|\, u) = \begin{cases} \left(\frac{1+\beta}{\beta}\right)\left(\frac{\prod_{i=1}^m \rho_i}{\prod_{i=1}^m v q_i}\right) p(x+y+1) \sum_{j=1}^m c_j\, \rho_j^{x-u} \sum_{n=0}^{u} \rho_j^n\, a(n), & 0 \le u \le x, \\ \left(\frac{1+\beta}{\beta}\right)\left(\frac{\prod_{i=1}^m \rho_i}{\prod_{i=1}^m v q_i}\right) p(x+y+1) \sum_{j=1}^m c_j\, \rho_j^{x-u} \sum_{n=u-x}^{u} \rho_j^n\, a(n), & u > x, \end{cases}$$

where $a(u)$ *is a compound geometric p.f. with*

$$a(u) = \frac{\beta}{1+\beta} \sum_{n=1}^{\infty} \left(\frac{1}{1+\beta}\right)^n l^{*n}(u), \qquad u \in \mathbb{N}.$$

Li [20, (2005)] shows that the distributions of the surplus before ruin, the deficit at ruin, and the claim causing ruin, can also be obtained through the compound geometric p.f. $a(u)$. Furthermore, $a(u)$ is in closed form when the claim amounts distribution belongs to the K_m family or the claim amounts distribution has a finite support.

Pavlova and Willmot [29, (2004)] considers the discrete stationary or discrete equilibrium renewal model, which is a generalization of the discrete renewal (or Sparre Andersen) risk model. For the discrete (ordinary) renewal process, we have the same assumptions as described at the beginning of this section. In the equilibrium renewal model case, we assume that W_1, independent of $\{W_2, W_3, \ldots\}$ and $\{X_1, X_2, \ldots\}$, has c.d.f. $K_1(x) = 1 - \bar{K}_1(x)$ and p.f. $k_1(x) = \bar{K}(x-1)/\mathbb{E}[W_2]$ for $x \in \mathbb{N}^+$. The insurer's surplus at time t is described by (16), and the relative security loading $\theta > 0$ is assumed to satisfy $(1+\theta)\mathbb{E}[W_2]/\mathbb{E}[X_2] = 1$.

Let $T = \inf\{\, t \in \mathbb{N}^+ : U(t) < 0\,\}$ be the ruin time in the ordinary discrete renewal model, and its discounted penalty (Gerber-Shiu) function is defined by (12). Furthermore, denote by T_e the time of ruin and by $\phi_e(u)$ the discounted penalty function in the discrete equilibrium renewal risk model, defined as follows:

$$\phi_e(u) = \mathbb{E}\left[v^{T_e} w\big(U(T_e - 1), |U(T_e)|\big)\, I(T_e < \infty) \,\big|\, U(0) = u\right], \qquad u \in \mathbb{N}.$$

The main result in Pavlova and Willmot [29, (2004)] is given by the following theorem, which describes the relationship between the Gerber-Shiu discounted penalty functions in the ordinary and the stationary discrete renewal risk models.

Theorem 11 (Pavlova and Willmot [29, (2004)]) *The Gerber-Shiu discounted penalty functions in the ordinary and the stationary (equilibrium) discrete renewal risk models are related by*

$$\phi_e(u) = \frac{1}{1+\theta} \sum_{y=0}^{u} \phi(u-y)\, p_1(y+1) + \nu_v(u), \qquad u \in \mathbb{N}^+,$$

where

$$\nu_v(u) = \sum_{j=u+1}^{\infty} v^{j-u}\, \tau(j) - \frac{1-v}{1+\theta} \sum_{j=u+1}^{\infty} v^{j-u-1} \sum_{y=1}^{j} \phi(j-y)\, p_1(y), \qquad u \in \mathbb{N}^+,$$

with $\tau(j) = \left[\sum_{y=j+1}^{\infty} w(j, y-j)\, p(y)\right] / \mathbb{E}[W_2]$, $j \in \mathbb{N}$ *and* $p_1(x) = \bar{P}(x-1)/\mu$.

Pavlova and Willmot [29, (2004)] further considers a special case when $k(x) = q(1-q)^{x-1}$, $x \in \mathbb{N}^+$. Then $\bar{K}(x) = (1-q)^x$ and $\mathbb{E}[W_2] = 1/q$, yielding that $k_1(x) = k(x)$, $x \in \mathbb{N}^+$ and $\phi_e(u) = \phi(u)$. The defective renewal equation for $\phi(u)$, stated in Theorem 6, can also be obtained through Theorem 11.

4 Discrete-time Sparre Andersen model with general inter–claim times

In this section, we shall review some recent results in the discrete-time Sparre Andersen model with general inter-claim times. We assume that the inter-claim times have a common p.f. $k(x)$ for $x = 1, 2, \ldots$ with p.g.f. $\hat{k}(z) = \sum_{x=1}^{\infty} z^x k(x)$.

Wu and Li [35, (2009)] shows that the recursive formula (defective renewal equation) for the Gerber-Shiu function given in Theorem 9 can be extended for the Sparre Andersen model with K_m inter-claim times to general inter-claim times as follows.

Theorem 12 (Wu and Li [35, (2009)])

$$\phi(u) = \sum_{y=1}^{u} \phi(u-y)\, g(y|0) + H(u), \qquad u \in \mathbb{N}^+,$$

where $g(y|0) = \sum_{x=0}^{\infty} p_x(y)\, f_1(x|0)$ *with* $p_x(y) = p(x+y+1)/\bar{P}(x+1)$, *and*

$$H(u) = \sum_{y=u+1}^{\infty} \sum_{x=0}^{\infty} w(x+u, y-u) p_x(y) f_1(x|0).$$

In particular, if $w(x,y) = 1$, $\phi_T(u)$ *satisfies the following recursive formula*

$$\phi_T(u) = \sum_{y=1}^{u} \phi_T(u-y) g(y|0) + \sum_{y=u+1}^{\infty} g(y|0)$$

and has the following explicit expression

$$\phi_T(u) = (1-\xi_v) \sum_{n=1}^{\infty} \xi_v^n \bar{L}_1^{*n}(u), \qquad u \in \mathbb{N},$$

where $\xi_v = \phi_T(0) = \sum_{x=0}^{\infty} f_1(x|0)$ *is to be determined,* $L_1^{*n}(u) = 1 - \bar{L}_1^{*n}(u)$ *is the d.f. of the* n*-fold convolution of* $L_1(u) = \sum_{y=0}^{u} g(y|0)/\xi_v$.

Although $\phi(u)$ and $\phi_T(u)$ satisfy a defective renewal equation and in particular $\phi_T(u)$ can be expressed as a compound geometric tail for the Sparre Andersen model with general inter-claim times, however, unlike those results for the Sparre Andersen model with K_m inter-claim times, ξ_v, $g(y|0)$ and $H(u)$ are unknown for general claim severity distributions and unspecified penalty functions. Wu and Li [35, (2009)] derives explicit results for the Gerber-Shiu function with some specially chosen penalty functions when claim amounts follow a zero-truncated geometric distribution, a degenerate distribution at a constant size of 2, and the mixture of n geometric distributions. Wu and Li [34, (2008)] derives compact matrix expressions for the ruin probability and the distribution of the deficit at ruin.

4.1 Constant claims

In this subsection, we assume that claim amounts are constant and of size 2, i.e., $p(x) = I(x = 2)$ for $x = 1, 2, \ldots$. It follows from Wu and Li [35, (2009)] that the p.g.f. of the time of ruin T has the following expression

$$\phi_T(u) = \xi_v^{u+1}, \qquad u \in \mathbb{N},$$

where $0 < \xi_v < 1$ is the unique solution of the equation $\xi_v^2 = \hat{k}(v\xi_v)$. In particular, the ruin probability

$$\psi(u) = \xi_1^{u+1}, \qquad u \in \mathbb{N},$$

where $\xi_1 = \lim_{v\to 1} \xi_v$ is the solution of $\xi_1^2 = \hat{k}(\xi_1)$. In general, the Gerber-Shiu function $\phi(u)$ has the following expression

$$\phi(u) = \phi(0)\xi_v^u = w(0,1)\xi_v^{u+1}, \qquad u \in \mathbb{N}.$$

4.2 Zero-truncated geometric claims

Now assume that claim amounts have a zero-truncated geometric distribution with $p(x) = (1-\pi)\pi^{x-1}$, for $x \in \mathbb{N}^+$, and $0 < \pi < 1$. Then Wu and Li [35, (2009)] shows that

$$\phi_T(u) = \xi_v\big[\pi + \xi_v\,(1-\pi)\big]^u, \qquad u \in \mathbb{N}, \tag{22}$$

where $0 < \xi_v < 1$ is the unique solution of the following equation

$$\xi_v = \hat{k}\big\{v\big[\pi + \xi_v(1-\pi)\big]\big\}.$$

In particular, the ruin probability

$$\psi(u) = \xi_1\left[\pi + \xi_1(1-\pi)\right]^u, \qquad u \in \mathbb{N}, \tag{23}$$

where $0 < \xi_1 < 1$ is the unique solution of the following equation $\xi_1 = \hat{k}\big\{\pi + \xi_1(1-\pi)\big\}$.

The moment of the time of ruin given that ruin has occurred may be calculated by differentiating (22) and setting $v = 1$ as follows

$$\mathbb{E}\left[T \,|\, T < \infty, U(0) = u\right] = \frac{\hat{k}'\big\{\pi + \xi_1(1-\pi)\big\}\big[\pi/\xi_1 + (1-\pi)(1+u)\big]}{1 - (1-\pi)\,\hat{k}'\left\{\pi + \xi_1\,(1-\pi)\right\}}.$$

For the following specially chosen penalty function

$$w(x,y) = s^x w_1(y), \qquad 0 < s \le 1,$$

with $w_1(y)$ being a univariate function, the Gerber-Shiu function with the following new notation

$$\phi_{v,s}(u) = \mathbb{E}\left[v^T s^{U(T-1)} w_1\big(|U(T)|\big)\, I(T < \infty) \,|\, U(0) = u\right], \qquad u \in \mathbb{N}, \tag{24}$$

has the following expression

$$\phi_{v,s}(u) = \eta_v(s)\left\{\xi_v \alpha(\pi s)^u + (1-\alpha)\phi_T(u)\right\}, \qquad u \in \mathbb{N},$$

where

$$\alpha = \frac{\pi(1-s)}{\pi(1-s)+\xi_v(1-\pi)},$$

$$\eta_v(s) = \frac{\mathbb{E}\left[w_1(X_1)\right]\hat{k}(v\pi s)s^{-1}}{\alpha\xi_v + (1-\alpha)\hat{k}(v\pi s)}.$$

In particular, if $v = 1$ and $w_1(y) = 1$, then

$$\begin{aligned}\phi_{1,s}(u) &= \mathbb{E}\left[s^{U(T-1)} I(T<\infty) \,|\, U(0)=u\right]\\ &= \frac{\hat{k}(\pi s)s^{-1}}{\alpha\,\psi(0)+(1-\alpha)\hat{k}(\pi s)}\left[\psi(0)\alpha\,(\pi s)^u + (1-\alpha)\,\psi(u)\right], \qquad u \in \mathbb{N}.\end{aligned}$$

The first moment of the surplus before ruin can be calculated by

$$\mathbb{E}\left[U(T-1)\,I(T<\infty)\,|\,U(0)=u\right] = \lim_{s\to 1^-}\frac{\partial}{\partial s}\phi_{1,s}(u) = \frac{\pi}{1-\pi}\left[\frac{\psi(u)}{\hat{k}(\pi)} - \pi^u\right] - \psi(u), \qquad u \in \mathbb{N}.$$

In this case, the literature does not yet give results for the Gerber-Shiu function with more general penalty functions $w(x, y)$.

4.3 Mixed geometric claim amounts

In this section let the claim amounts have a mixed geometric distribution with coefficients $0 < \chi_j < 1$, such that $\sum_{j=1}^n \chi_j = 1$, i.e.,

$$p(x) = \sum_{j=1}^n \chi_j \varpi_j(x), \qquad x \in \mathbb{N}^+,$$

where $\varpi_j(x) = (1-\pi_j)\pi_j^{x-1}$, $j = 1, 2, \ldots, n$, is a geometric p.f. with parameter $0 < \pi_j < 1$. The p.g.f. of $p(x)$ is given by

$$\hat{p}(z) = \sum_{j=1}^n \chi_j \frac{(1-\pi_j)z}{1-\pi_j z}.$$

Lemma 1 of Wu and Li [35, (2009)] shows that the following generalized Lundberg's equation

$$\hat{k}(v/z)\,\hat{p}(z) = 1$$

has n roots, say, $R_1, R_2, \ldots, R_n$, with $|R_i| > 1$. For simplicity, we assume that $R_1, R_2, \ldots, R_n$ are distinct.

Wu and Li [35, (2009)] shows that the generating function of the Gerber-Shiu function defined in (24) has the following form:

$$\hat{\phi}_{v,s}(z) = \sum_{u=0}^{\infty} s^u \phi_{v,s}(u) = \frac{\hat{\beta}_{v,s}(z)\prod_{l=1}^n R_l}{\prod_{j=1}^n (R_j - z)\prod_{i=1}^n (1-\pi_i s z)}, \tag{25}$$

where

$$\hat{\beta}_{v,s}(z) = \left[\prod_{l=1}^n (1-\pi_l z)\right]\sum_{j=1}^n \hat{\alpha}_{v,s}(R_j)\prod_{i=1, i\neq j}^n \frac{R_i - z}{(R_i - R_j)}$$

is a polynomial of degree $2n-1$, with

$$\hat{\alpha}_{v,s}(z) = \sum_{j=1}^{n} \delta_j(s) \left[\prod_{\substack{i=1 \\ i \neq j}}^{n} (1 - \pi_i s z) \right] \mathbb{E}\big[w_1(Z_j)\big],$$

$$\delta_j(s) = \sum_{x=0}^{\infty} s^x \frac{\chi_j \pi_j^{x+1}}{\bar{P}(x+1)} f_1(x|0),$$

and Z_j being a discrete r.v. that has p.f. $\varpi_j(x)$ for $j = 1, 2, \ldots, n$.

Using partial fractions, (25) is rewritten as

$$\hat{\phi}_{v,s}(z) = \sum_{j=1}^{n} \gamma_{v,s}(j) \frac{R_j}{(R_j - z)} + \sum_{j=1}^{n} \kappa_{v,s}(j) \frac{1}{(1 - \pi_j s z)}, \tag{26}$$

where

$$\gamma_{v,s}(j) = \frac{\hat{\beta}_{v,s}(R_j) \prod_{l=1, l \neq j}^{n} R_l}{\prod_{i=1, i \neq j}^{n} (R_i - R_j) \prod_{k=1}^{n} (1 - \pi_k s R_j)},$$

$$\kappa_{v,s}(j) = \frac{\hat{\beta}_{v,s}\big((\pi_j s)^{-1}\big) \prod_{l=1}^{n} R_l}{\prod_{i=1}^{n} \left[R_i - (\pi_j s)^{-1}\right] \prod_{i=1, i \neq j}^{n} (1 - \pi_i / \pi_j)}.$$

Finally, the inversion of (26) yields

$$\phi_{v,s}(u) = \sum_{j=1}^{n} \gamma_{v,s}(j) R_j^{-u} + \sum_{j=1}^{n} \kappa_{v,s}(j) (\pi_j s)^u, \qquad u \in \mathbb{N}.$$

Note that $\gamma_{v,s}(j)$ and $\kappa_{v,s}(j)$ are unknown coefficients as $\hat{\beta}_{v,s}(z)$ depends on $f_1(x|0)$ which has not been determined. Wu and Li [35, (2009)] shows that

$$\kappa_{v,s}(j) = \frac{\chi_j \, \hat{k}(v \pi_j s) \, \mathbb{E}\left[w_1(Z_j)\right] / s}{1 - \sum_{i=1}^{n} \chi_i (1 - \pi_i) / (\pi_j s - \pi_i) \, \hat{k}(v \pi_j s)}, \qquad j = 1, \ldots, n.$$

Let $\mathbf{A} = (a_{i,j})_{n \times n}$ and $\mathbf{B}(s) = (b_{i,j}(s))_{n \times n}$ be two matrices, where $a_{i,j} = \hat{\varpi}_i(R_j)$ and $b_{i,j}(s) = (1 - \pi_i)/(\pi_i - \pi_j s)$. Denote $\vec{\boldsymbol{\gamma}}_{v,s} = (\gamma_{v,s}(1), \ldots, \gamma_{v,s}(n))^\top$ and $\vec{\boldsymbol{\kappa}}_{v,s} = (\kappa_{v,s}(1), \ldots, \kappa_{v,s}(n))^\top$ as two n-dimensional column vectors. Then

$$\vec{\boldsymbol{\gamma}}_{v,s} = \mathbf{A}^{-1} \mathbf{B}(s) \vec{\boldsymbol{\kappa}}_{v,s}.$$

4.4 Phase-type claims

In this subsection, we assume that the distribution of claim amounts is a discrete phase-type (PH) distribution with representation $(\vec{\boldsymbol{\alpha}}, \mathbf{T})$, where $\vec{\boldsymbol{\alpha}} = (\alpha_1, \ldots, \alpha_n)$, see Asmussen [4, (2000)] for an introduction to PH distributions. Here the claim amounts take values on $\mathbb{N}^+$, which implies that $\sum_{i=1}^{n} \alpha_i = 1$. Then the probability function of X, is then given by

$$p(x) = \vec{\boldsymbol{\alpha}} \, \mathbf{T}^{x-1} \, \vec{\mathbf{t}}^\top, \qquad x \in \mathbb{N}^+,$$

where $\mathbf{T} = (t_{i,j})_{n \times n}$ is a sub-stochastic matrix and $\vec{\mathbf{t}}^\top = (t_{1,0}, t_{2,0}, \ldots, t_{n,0})^\top = (\mathbf{I} - \mathbf{T}) \, \vec{\mathbf{1}}^\top$ with $\mathbf{I}$ being the $n \times n$ identity matrix and $\vec{\mathbf{1}} = (1, 1, \ldots, 1)$ being an $n \times 1$ row vector. The corresponding distribution function of X is given by $P(x) = 1 - \vec{\boldsymbol{\alpha}} \, \mathbf{T}^x \, \vec{\mathbf{1}}^\top$ and the k-th factorial moment of X is then

$$\mathbb{E}[X^{(k)}] = k! \, \vec{\boldsymbol{\alpha}} \, (\mathbf{I} - \mathbf{T})^{-k} \, \mathbf{T}^{k-1} \, \vec{\mathbf{1}}^\top,$$

where $w^{(k)} = w(w-1)(w-2)\cdots(w-k+1)$ is the k-th factorial power of w. The p.g.f. of X is

$$\hat{p}(z) = z\vec{\boldsymbol{\alpha}}\,(\mathbf{I} - z\mathbf{T})^{-1}\,\vec{\mathbf{t}}^{\top} = \frac{a_1 z + a_2 z^2 + \cdots + a_m z^m}{1 + b_1 z + b_2 z^2 + \cdots + b_m z^m}. \tag{27}$$

Note that the discrete phase-type distributions belong to the K_m family as the p.g.f. in (27) is the ratio of two polynomials of order m.

Formal definitions of discrete phase-type distributions date back to the mid 1970's, see Neuts [27, (1975)]. By in large, however, research has focused more on the study of continuous phase-type distributions. Detailed discussions of continuous phase-type distributions can be found in Neuts [28, (1981)] and Latouche and Ramaswami [18, (1999)]. Brief overviews of either discrete or continuous phase-type distributions and their properties can be found in Asmussen [3, 4, (1992, 2000)], Stanford and Stroiński [31, (1994)], Bobbio et al. [5, (2003)].

As surveyed in the following theorems, Wu and Li [34, (2008)] gives a matrix expression for the ruin probability and the distribution of the deficit at ruin for the Sparre Andersen model with general inter-claim times and phase-type claims in the following theorems.

Theorem 13 (Wu and Li [34, (2008)]) *For a discrete-time Sparre Andersen risk model, as defined in* (16)*, if the claim severity distribution is a discrete phase-type with representation* $(\vec{\boldsymbol{\alpha}}, \mathbf{T})$*, then*

$$\psi(u) = \vec{\boldsymbol{\alpha}}_{+}(\mathbf{T} + \vec{\mathbf{t}}^{\top}\,\vec{\boldsymbol{\alpha}}_{+})^{u}\,\vec{\mathbf{1}}^{\top}, \qquad u \in \mathbb{N}^{+}, \tag{28}$$

where the row vector $\vec{\boldsymbol{\alpha}}_{+}$ *satisfies equation* $\vec{\boldsymbol{\alpha}}_{+} = \varphi(\vec{\boldsymbol{\alpha}}_{+})$*, with*

$$\varphi(\vec{\boldsymbol{\alpha}}_{+}) = \vec{\boldsymbol{\alpha}}\,\hat{k}\left(\mathbf{T} + \vec{\mathbf{t}}^{\top}\,\vec{\boldsymbol{\alpha}}_{+}\right).$$

Here $\vec{\boldsymbol{\alpha}}_{+}$ *can be computed as* $\vec{\boldsymbol{\alpha}}_{+} = \lim_{n\to\infty}\vec{\boldsymbol{\alpha}}_{+}^{(n)}$*, where*

$$\vec{\boldsymbol{\alpha}}_{+}^{(0)} = \vec{\mathbf{0}} \quad \textit{and} \quad \vec{\boldsymbol{\alpha}}_{+}^{(n)} = \varphi(\vec{\boldsymbol{\alpha}}_{+}^{(n-1)}), \qquad n \in \mathbb{N}^{+}.$$

Note that when $n = 1$, i.e., the claim amounts have a truncated geometric distribution, then the ruin probability in (28) simplifies to the expression in (23).

Define $F(u,y) = \mathbb{P}(T < \infty, |U(T)| \le y \,|\, U(0) = u)$ and $\psi(u,y) = \mathbb{P}(T < \infty, |U(T)| > y | U(0) = u)$. Then it is obvious that $F(u,y) = \psi(u) - \psi(u,y)$, for $u, y \in \mathbb{N}$.

Theorem 14 (Wu and Li [34, (2008)])

$$\psi(u,y) = \vec{\boldsymbol{\alpha}}_{+}\left(\mathbf{T} + \vec{\mathbf{t}}^{\top}\,\vec{\boldsymbol{\alpha}}_{+}\right)^{u}\mathbf{T}^{y}\,\vec{\mathbf{1}}^{\top}, \qquad u, y \in \mathbb{N}.$$

Open problems for the discrete-time Sparre Andersen model with general inter-claim times and phase-type claims, include finding compact matrix expressions for the Gerber-Shiu functions with some special choices of the penalty functions such as $w(x,y) = 1$ and $w(x,y) = s^{x} w_1(y)$.

For general inter-claim times, Cossette et al. [9, (2006)] gives an upper bound and an asymptotic expressions for the ruin probability, and study how to use the discrete-time Sparre Andersen model to approximate the continuous-time one.

5 Other discrete-time risk models

In this section, we give a brief review of the literature of other discrete-time risk models that extend the compound binomial risk model.

Yuen and Guo [37, (2001)] studies the ruin probability in the compound binomial risk model with time correlated claims. In each unit time period, there is a main claim with a probability $0 < q < 1$. This main claim can produce a by-claim, that may be settled in the same period with probability $0 < \theta < 1$, or may be delayed to the next period with probability $1 - \theta$.

Unlike the classical compound Poisson risk model in which a unit premium rate can be assumed without loss of generality (by appropriately rescaling the time unit as well as the claim sizes), it is clear that such reasoning does not hold for the compound binomial model. Landriault [17, (2008)] studies the evaluation of the generalized expected discounted penalty function in the compound binomial risk model in which the premium rate per unit time is $c \in \mathbb{N}^+$, rather than 1 as in the classical setting.

Cossette et al. [7, (2003)] presents a compound Markov binomial model, as another extension of the compound binomial model. Here, the binomial process $\{N(t)\,;\, t \in \mathbb{N}\}$ defined in (1) is extended to a Markov binomial process for which $\{\, I_t\,;\, t \in \mathbb{N}\,\}$ is a stationary homogeneous Markov chain with state space $\{0, 1\}$ and transition probability matrix

$$\mathbf{P} = \begin{pmatrix} p_{00} & p_{01} \\ p_{10} & p_{11} \end{pmatrix},$$

where $p_{ij} = \mathbb{P}(I_{t+1} = j \,|\, I_t = i)$, for $t = 0, 1, 2, \ldots$ and $i, j \in \{0, 1\}$. The positive loading condition is $\mu p_{01} < p_{01} + p_{10}$. When $p_{01} = p_{11} = q$, the Markov binomial process simplifies to the binomial process.

For the compound Markov binomial risk model, Cossette et al. [7, (2003)] provides recursive formulas for the computation of ruin probabilities over finite and infinite time horizons. A Lundberg exponential bound for the ruin probability is derived. Yuen and Guo [38, (2006)] studies the expected discounted penalty functions and the joint distribution of the surplus before ruin and the deficit at ruin conditional on the initial state of the Markov chain.

Cossette et al. [8, (2004)] proposes a compound binomial model defined in a Markovian environment as an extension to the compound binomial model. This model is a discrete analogue of the Markov-modulated compound Poisson risk model which was proposed by Asmussen [2, (1989)]. In the compound binomial model defined in a Markovian environment, the claim occurrences and the claim amounts are both regulated by an underlying Markov environment process, denoted by $\{J_t\,;\, t \in \mathbb{N}\}$, which is a homogeneous and irreducible discrete-time Markov process with state space $\{1, 2, \ldots, n\}$. The one step transition probability matrix is given by $\mathbf{\Gamma} = (\gamma_{i,j})_{i,j=1}^n$, where $\gamma_{i,j} = \mathbb{P}(J_{t+1} = j \,|\, J_t = i)$. The sequence of claim occurrences $\{I_t\,;\, t \in \mathbb{N}\}$ and the sequence of claim amounts $\{X_i\,;\, i \in \mathbb{N}^+\}$ in the compound binomial model in (2) are governed by $\{\, J_t\,;\, t \in \mathbb{N}\,\}$ such that, given $[J_t = i]$, I_t is Bernoulli distributed with mean $q_i \in (0, 1)$, the claim amount X_t has a p.f. $p_i(x)$ for $x \in \mathbb{N}^+$ and $i = 1, 2, \ldots, n$. Furthermore, given $[J_t = i]$, I_t and X_t are independent.

For the above compound binomial model defined in a Markovian environment, Cossette et al. [8, (2004)] presents an algorithm for the computation of the ruin probability and the distribution of aggregate claims for a fixed time period. Moreover, Cossette et al. [8, (2004)] shows that the compound binomial model defined in a Markovian environment can be used to approximate the continuous-time Markov-modulated risk model. For the same model, Yang et al. [36, (2009)] studies the discounted joint distribution of the surplus before ruin and the deficit at ruin.

6 Concluding Remarks

There are several motivations to this brief review of results for discrete-time risk models, including the compound binomial risk model and some of its extensions. First, because most theoretical risk models use the concept of time continuity, and yet practical reality is discrete. Recursive formulas for discrete-time models are obtained without assuming a claim severity distribution and are readily programmable in practice. This review wants to serve as a reminder that these models can be useful.

Also, although the techniques used and the results obtained for discrete-time risk models are of independent scientific interest, in addition they provide a simpler understanding of continuous-time risk models. For instance, these results can serve as approximations or bounds for the corresponding results in analogue continuous-time models.

Finally, the review highlights the problems that remain open in the theory, and provides an exhaustive bibliography of recent results, that should facilitate and encourage future research work in this area.

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