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Duality results involving functions associated to nonempty subsets of locally convex spaces

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Abstract. In many papers on consumer theory and production analysis duality results between profit, revenue, cost, input, output and shortage functions are established. This functions are associated to certain subsets of $\mathbb{R}^n$. The aim of this paper is to study in a systematic way such duality results in locally convex spaces and to derive them under minimal hypotheses.

Resultados sobre dualidad mediante funciones asociadas a subconjuntos no vacios de espacios localmente convexos

Resumen. En muchos artículos sobre teoría del consumo y análisis de la producción, se establecen resultados de dualidad entre beneficios y costes, e inversiones y rendimientos, proponiéndose diversas funciones de insuficiencia asociadas a ciertos subconjuntos de $\mathbb{R}^n$. El objeto de este trabajo es el estudio sistemático de dichos resultados de dualidad en espacios localmente convexos, y su obtención bajo condiciones mínimas.

1 Introduction

In the sequel (X,τ) is a nontrivial separated real locally convex space with topological dual X^*; X^* is endowed with the weak-star topology $w^* := \sigma(X^*,\ X)$. So X^* becomes a separated locally convex space whose topological dual is (identified with) X. For $x \in X$ and $x^* \in X^*$ we set $\langle x, x^*\rangle := x^*(x)$. In the case X is just a real linear space we can see X as a separated locally convex space whose topology is generated by the family of all semi-norms defined on X; in this situation the topological dual of X coincides with the algebraic dual X' of X. We denote by $\mathbb{R}$ the set of real numbers, $\mathbb{R}_+ := [0,\infty[$, $\mathbb{R}_- :=]-\infty, 0]$ and $\mathbb{P} :=]0,\infty[$, where $\infty := +\infty$.

Consider $A, B \subset X$ and $\Gamma \subset \mathbb{R}$ (similar for X replaced by X^* or other locally convex space). We set

$$A + B := \{\, a + b \mid a \in A,\ b \in B \,\}, \qquad \Gamma A := \{\, sa \mid s \in \Gamma,\ a \in A \,\};$$

of course, $A + B = \emptyset$ if $A = \emptyset$ or $B = \emptyset$ and $\Gamma A = \emptyset$ when $\Gamma = \emptyset$ or $A = \emptyset$. Moreover, for $s \in \mathbb{R}$ we set $sA := \{s\}A$ and for $x \in X$ we set $x + A := \{x\} + A$. We denote by ${}^iA$ or $\operatorname{icr} A$, $\operatorname{int} A$, $\operatorname{cl} A$ or $\overline{A}$, $\operatorname{conv} A$ and $\operatorname{aff} A$ the intrinsic core, the interior, the closure, the convex hull and the affine hull of $A \subset X$, respectively; moreover, $\overline{\operatorname{conv}} A := \operatorname{cl}(\operatorname{conv} A)$ and $\overline{\operatorname{aff}} A := \operatorname{cl}(\operatorname{aff} A)$. To $A \subset X$ we associate the sets

$$A^0 := \{\, x^* \in X^* \mid \langle x, x^*\rangle \geq -1 \ \ \forall x \in A \,\},$$
$$A^+ := \{\, x^* \in X^* \mid \langle x, x^*\rangle \geq 0 \ \ \forall x \in A \,\}, \qquad A^- := -A^+,$$
$$A^\# := \{\, x^* \in X^* \mid \langle x, x^*\rangle > 0 \ \ \forall x \in A \setminus \{0\} \,\}.$$

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Note that A^0 is a (w^*-) closed convex set containing 0, A^+ is a (w^*-) closed convex cone and $A^\#$ is a convex cone (if nonempty). For $\emptyset \neq A \subset X$ the bipolar theorems give

$$A^{00} = \overline{\mathrm{conv}}(A \cup \{0\}) = \overline{\mathrm{conv}}([0,1]A), \qquad A^{++} = A^{--} = \overline{\mathrm{conv}}(\mathbb{R}_+ A).$$

These formulas give the possibility to recover A using the above polarity operations under certain conditions on A: $A = A^{00}$ if (and only if) A is a closed convex set containing 0 and $A = A^{++}$ if (and only if) A is a closed convex cone.

The asymptotic cone of the nonempty set $A \subset X$ is

$$A_\infty := \{\, u \in X \mid \exists (t_i)_{i\in I} \subset \mathbb{P},\ \exists (x_i)_{i\in I} \subset A \ :\ t_i \to 0,\ t_i x_i \to u \,\};$$

when X is a normed vector space we can take sequences instead of nets. When A is a closed convex set we have that $A_\infty = \bigcap_{t>0} t(A - a)$, where $a \in A$. We set $\emptyset_\infty := \{0\}$.

Recall that for $f\colon X \to \overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty, \infty\}$ the domain of f is the set $\operatorname{dom} f := \{\, x \in X \mid f(x) < \infty \,\}$ and the epigraph of f is the set $\operatorname{epi} f := \{\, (x,t) \in X \times \mathbb{R} \mid f(x) \le t \,\}$; f is proper if $\operatorname{dom} f \neq \emptyset$ and $f(x) > -\infty$ for every $x \in X$. The conjugate $f^*\colon X^* \to \overline{\mathbb{R}}$ of f is defined by $f^*(x^*) := \sup\{\, \langle x, x^* \rangle - f(x) \mid x \in X \,\}$; f^* is convex and weakly-star lower semicontinuous (lsc for short) and f^* is proper iff f is proper and minorized by a continuous affine functional (in which case $\operatorname{epi} f^{**} = \overline{\mathrm{conv}}(\operatorname{epi} f)$). We denote by $\overline{f}$ the function $\overline{f} : X \to \overline{\mathbb{R}}$ for which $\operatorname{epi} \overline{f} := \operatorname{cl}(\operatorname{epi} f)$; then $\overline{f}(x) = \liminf_{x' \to x} f(x')$. Furthermore, we use the notation $[f \le t] := \{\, x \in X \mid f(x) \le t \,\}$ for $t \in \mathbb{R}$, and similarly for $[f = t]$, $[f \ge t]$, $[f < t]$, $[f > t]$. If $g\colon X \to \overline{\mathbb{R}}$ is another function then the convolution of f and g is the function $f \Box g : X \to \overline{\mathbb{R}}$ defined by $(f \Box g)(x) := \inf\{\, f(u) +_e g(x-u) \mid u \in X \,\}$ (the sum "$+_e$" is defined in the next section).

2 Gauges and scalarization functions

Let $A \subset X$ be an arbitrary set. First we associate to A the following two functions

$$\sigma_A, \varsigma_A\colon X^* \to \overline{\mathbb{R}}, \qquad \sigma_A(x^*) := \sup_{x\in A} \langle x, x^* \rangle, \qquad \varsigma_A(x^*) := \inf_{x \in A} \langle x, x^* \rangle,$$

with the conventions $\sup \emptyset := -\infty$ and $\inf \emptyset := \infty$; hence $\sigma_\emptyset = -\infty$ and $\varsigma_\emptyset = +\infty$. The function σ_A is the support function of A. It is obvious that

$$\varsigma_A(x^*) = -\sigma_A(-x^*) \qquad \forall x^* \in X^*; \tag{1}$$

moreover $\sigma_A = \sigma_{\overline{\mathrm{conv}} A}$, $\varsigma_A = \varsigma_{\overline{\mathrm{conv}} A}$ and

$$\begin{aligned} \overline{\mathrm{conv}} A &= \{\, x \in X \mid \langle x, x^* \rangle \le \sigma_A(x^*) \quad \forall x^* \in X^* \,\} \\ &= \{\, x \in X \mid \langle x, x^* \rangle \ge \varsigma_A(x^*) \quad \forall x^* \in X^* \,\}. \end{aligned}$$

The above formulas show that we can recover A knowing σ_A or ς_A when A is a (nonempty) closed convex set.

Denoting by ι_A the indicator function of $A \subset X$, that is, $\iota_A\colon X \to \overline{\mathbb{R}}$ is defined by $\iota_A(x) := 0$ for $x \in A$ and $\iota_A(x) := \infty$ for $x \in X \setminus A$, it is clear that $\sigma_A := \iota_A^* := (\iota_A)^*$. When A is a nonempty set σ_A is a proper w^*-lsc sublinear functional.

To $A \subset X$ we associate also the gauges $\mu_A, \vartheta_A, \nu_A, \theta_A\colon X \to \overline{\mathbb{R}}$ defined by

$$\begin{aligned} \mu_A(x) &:= \inf\ \{\, \lambda > 0 \mid x \in \lambda A \,\}, & \vartheta_A(x) &:= \sup\{\, \lambda > 0 \mid \lambda x \in A \,\}, \\ \nu_A(x) &:= \sup\{\, \lambda > 0 \mid x \in \lambda A \,\}, & \theta_A(x) &:= \inf\ \{\, \lambda > 0 \mid \lambda x \in A \,\}; \end{aligned}$$

hence $\mu_\emptyset = \theta_\emptyset = +\infty$ and $\nu_\emptyset = \vartheta_\emptyset = -\infty$. Note that μ_A is the well known Minkowski functional associated to A; in Functional Analysis μ_A is considered mostly when A is an absorbing convex set in which case μ_A is a finite-valued sublinear function. Shephard's input and output functions are of type ν_A and μ_A with A subsets of $\mathbb{R}^n_+$. For some properties of μ_A and ν_A see f.i. [12]. In [5] one speaks about θ_A as the extended Farrell measure; moreover, in [5] one discusses arguments in favor of and against convexity axioms in DEA (Data Envelopment Analysis).

Note that the functions θ_A, μ_A are finite at $x \in X \setminus \{0\}$ if and only if $x \in \mathbb{P}A$. Moreover, $\mu_A = \mu_{]0,1]A} \geq 0$, $\vartheta_A = \vartheta_{]0,1]A}$, $\nu_A = \nu_{[1,\infty[A}$, $\theta_A = \theta_{[1,\infty[A} \geq 0$,

$$\begin{aligned} &\mu_A(tx) = t\mu_A(x), && \nu_A(tx) = t\nu_A(x) && \forall x \in X,\ \forall t \in \mathbb{P}, \\ &\theta_A(tx) = t^{-1}\theta_A(x), && \vartheta_A(tx) = t^{-1}\vartheta_A(x) && \forall x \in X,\ \forall t \in \mathbb{P} \end{aligned} \tag{2}$$

and

$$\mu_A(x) \leq \nu_A(x), \quad \theta_A(x) = 1/\nu_A(x), \quad \vartheta_A(x) = 1/\mu_A(x) \qquad \forall x \in \mathbb{P}A$$

with the conventions $1/\infty := 0$ and $1/0 := \infty$. In fact

$$\mu_A(x) = \frac{1}{0 \vee \vartheta_A(x)}, \quad 0 \vee \vartheta_A(x) = \frac{1}{\mu_A(x)}, \quad \theta_A(x) = \frac{1}{0 \vee \nu_A(x)}, \quad 0 \vee \nu_A(x) = \frac{1}{\theta_A(x)} \tag{3}$$

for every $x \in X$, where $s \vee t := \max\{s, t\}$ for $s, t \in \overline{\mathbb{R}}$. We use also the conventions

$$\begin{aligned} &0 \cdot_e \infty := \infty \cdot_e 0 := \infty, && 0 \cdot_e (-\infty) := (-\infty) \cdot_e 0 := 0, \\ &0 \cdot_h \infty := \infty \cdot_h 0 := 0, && 0 \cdot_h (-\infty) := (-\infty) \cdot_h 0 := -\infty \end{aligned}$$

(the indexes e and h are coming from epigraph and hypograph, respectively). We also use the conventions

$$(-\infty) +_e (+\infty) := (+\infty) +_e (-\infty) := +\infty, \qquad (-\infty) +_h (+\infty) := (+\infty) +_h (-\infty) := -\infty$$

($s + t$ being defined as usual in the other situations).

As for other operations on sets, one may ask when and how we can recover the set A knowing μ_A, ν_A, θ_A or ϑ_A. We have

$$A = \{ x \in X \mid \mu_A(x) \leq 1 \} = \{ x \in X \mid \vartheta_A(x) \geq 1 \} \qquad \text{if } A = \operatorname{cl} A = [0,1]A, \tag{4}$$

$$A = \{ x \in X \mid \nu_A(x) \geq 1 \} = \{ x \in X \mid \theta_A(x) \leq 1 \} \qquad \text{if } A = \operatorname{cl} A = [1,\infty[A. \tag{5}$$

In fact, instead of asking $A = \operatorname{cl} A$ we can assume that A is radially closed, or more precisely, that $A \cap \mathbb{R}x$ is closed for every $x \in X$; moreover, in such a case, the non zero finite values of these functions are attained.

Having now the set $A \subset X$ and the element $k \in X \setminus \{0\}$ we consider the function

$$\varphi_{A,k} \colon X \to \overline{\mathbb{R}}, \qquad \varphi_{A,k}(x) := \inf\{ t \in \mathbb{R} \mid x \in tk - A \},$$

and its counterpart

$$\psi_{A,k} \colon X \to \overline{\mathbb{R}}, \qquad \psi_{A,k}(x) := \sup\{ t \in \mathbb{R} \mid x \in tk + A \};$$

hence $\varphi_{\emptyset,k} = +\infty$ and $\psi_{\emptyset,k} = -\infty$. Note that $\varphi_{A,k}(x) > -\infty$ and $\psi_{A,k}(x) < \infty$ for every $x \in X$ if $k \notin -A_\infty$. Moreover,

$$\varphi_{A,sk} = s^{-1}\varphi_{A,k}, \qquad \psi_{A,sk} = s^{-1}\psi_{A,k} \quad \forall s \in \mathbb{P},$$

$$A \subset B \Rightarrow [\varphi_{B,k} \leq \varphi_{A,k}, \quad \psi_{A,k} \leq \psi_{B,k}], \tag{6}$$

$$\psi_{A,k} = \psi_{A+\mathbb{R}_+k,k}, \qquad \varphi_{A,k} = \varphi_{A+\mathbb{R}_+k,k}, \tag{7}$$

$$A + \mathbb{R}_+k \subset [\psi_{A,k} \geq 0], \qquad -A - \mathbb{R}_+k \subset [\varphi_{A,k} \leq 0] \tag{8}$$

and

$$\psi_{A,k}(x) = -\varphi_{-A,-k}(x) = -\varphi_{A,k}(-x) \quad \forall x \in X. \tag{9}$$

For this reason it is sufficient to study $\varphi_{A,k}$ or $\psi_{A,k}$. A detailed study of the function $\varphi_{A,k}$ in the case A closed and $A = A + \mathbb{R}_+ k$ is performed in [9, Section 2.3]; other properties of $\varphi_{A,k}$ are established in [13].

It is known and easy to prove that

$$\varphi_{A,k}(x + tk) = \varphi_{A,k}(x) + t, \qquad \psi_{A,k}(x + tk) = \psi_{A,k}(x) + t \qquad \forall x \in X, \quad \forall t \in \mathbb{R}.$$

Of course, if A is closed (or more generally, A is closed in the direction k, that is, $\{ t \in \mathbb{R} \mid x + tk \in A \}$ is closed in $\mathbb{R}$ for every $x \in X$) and $\varphi_{A,k}(x) \in \mathbb{R}$ then $x \in \varphi_{A,k}(x)k - A$, that is, the finite values of $\varphi_{A,k}$ are attained. Moreover, if $A + \mathbb{R}_+ k$ is closed then

$$A + \mathbb{R}_+ k = \{ x \in X \mid \varphi_{A,k}(-x) \le 0 \} = \{ x \in X \mid \psi_{A,k}(x) \ge 0 \}.$$

The function $\varphi_{A,k}$ was introduced by Gerstewitz (Tammer) and Iwanow in [8] and used by Chr. Tammer and her collaborators, as well as by D. T. Luc and others, mainly for scalarization of vector optimization problems; the framework was that of an ordered topological vector space. Luenberger [11, Def. 4.1] considered (practically) the same function, under the name of shortage function, in the context of production analysis (X being $\mathbb{R}^n$ and A a convex subset of $\mathbb{R}^n_+$) and Artzner et. al. [2] considered it in the context of mathematical finance (X being a space of Lebesgue integrable functions and A the corresponding positive cone). More historical facts about the use of the functions $\varphi_{A,k}$, $\psi_{A,k}$ in Functional Analysis and Mathematical Economics are given by A. H. Hamel in [10].

In production analysis the condition $A = A + \mathbb{R}_+ k$ is not granted. Because sometimes the results are not established in very precise terms in this context, the next three statements refer to the case when A might be different of $A + \mathbb{R}_+ k$. In the sequel we shall omit k if confusions cannot arrive (mainly in the proofs), that is, we shall write simply φ_A instead of $\varphi_{A,k}$ and ψ_A instead of $\psi_{A,k}$.

Proposition 1 *Assume that $A \subset X$ is closed, $K \subset X$ and $k \in X \setminus \{0\}$. If*

$$A = (A + \mathbb{R}_+ k) \cap K, \tag{10}$$

then

$$A = \{ x \in K \mid \psi_{A,k}(x) \ge 0 \} = \{ x \in K \mid \varphi_{A,k}(-x) \le 0 \}. \tag{11}$$

PROOF. If $A = \emptyset$ then clearly (11) holds. Assume that $A \ne \emptyset$. The inclusion $A \subset \{ x \in K \mid \psi_A(x) \ge 0 \}$ is clear. Take $x \in K$ with $\psi_A(x) \ge 0$. If $\psi_A(x) = 0$, since A is closed, we have that $x = x + \psi_A(x)k \in A$. Otherwise there exists $t > 0$ such that $x - tk \in A$. Then $x = x - tk + tk \in A + \mathbb{R}_+ k$, and so, by (10), $x \in A$. ■

When K is a cone and $k \in K$ condition (10) is implied by the condition $(A + K) \cap K = A$, while in the case $-k \in K$ condition (10) is implied by $(A - K) \cap K = A$. Note that in [1] one uses sets $L_i \subset \mathbb{R}^r$ and $P_i \subset \mathbb{R}^s$ with the properties $(L_i + \mathbb{R}^r_+) \cap \mathbb{R}^r_+ = L_i$ —called free disposability for inputs— and $(P_i - \mathbb{R}^s_+) \cap \mathbb{R}^s_+ = P_i$ —called free disposability for outputs.

As seen in (8), the set $[\psi_A \ge 0]$ includes always the set $A + \mathbb{R}_+ k$, and so it might be different of A even if A is closed and satisfies (10). Take $K := \mathbb{R}^2_+ \subset X := \mathbb{R}^2$, $k := (-1, -1)$, $A := [0, 1] \times [0, 1]$; then $[\psi_A \ge 0] = A + \mathbb{R}_+ k \ne A$. It is clear that the hypotheses of Proposition 1 are satisfied. Relation (2) in [7] could give the impression that $A = [\psi_A \ge 0]$ for A a closed convex set without asking $A = A + \mathbb{R}_+ k$. As observed in (6) we have that $A \subset B \subset X$ implies $\psi_A \le \psi_B$. Applying the previous proposition with $K = X$, if A and B are closed sets (in the direction k) such that $A = A + \mathbb{R}_+ k$ and $B = B + \mathbb{R}_+ k$, then $A \subset B$ if and only if $\psi_A \le \psi_B$.

Proposition 2 *Assume that $k \in X \setminus \{0\}$ and $A_i \subset X$ satisfies (10) for every $i \in I$ ($\ne \emptyset$). Then $A := \bigcap_{i \in I} A_i$ and $A' := \bigcup_{i \in I} A_i$ satisfy (10), too.*

PROOF. Since $A_i \subset K$ for every $i \in I$, we have that $A, A' \subset K$. Hence $A \subset (A + \mathbb{R}_+ k) \cap K$ and $A' \subset (A' + \mathbb{R}_+ k) \cap K$. Obviously, $(A + \mathbb{R}_+ k) \cap K \subset (A_i + \mathbb{R}_+ k) \cap K = A_i$ for every $i \in I$, and so $(A + \mathbb{R}_+ k) \cap K \subset A$. Let $x \in (A' + \mathbb{R}_+ k) \cap K$. Then $x = a' + tk \in K$ for some $a' \in A'$ and $t \geq 0$. Hence there exists $i \in I$ with $a' \in A_i$, and so $x \in K \cap (A_i + \mathbb{R}_+ k) = A_i \subset A'$. The conclusion follows. ■

Condition (10) will be used later on, too.

Proposition 3 *Let I be a nonempty set and $A_i \subset X$ for every $i \in I$. Then*

$$\psi_{\cup_{i\in I} A_i,k} = \sup_{i\in I} \psi_{A_i,k}, \quad \varphi_{\cup_{i\in I} A_i,k} = \inf_{i\in I} \varphi_{A_i,k}, \quad \psi_{\cap_{i\in I} A_i,k} \leq \inf_{i\in I} \psi_{A_i,k}, \quad \varphi_{\cap_{i\in I} A_i,k} \geq \sup_{i\in I} \varphi_{A_i,k}.$$

Moreover, if $K \subset X$ is closed and $(A_i + \mathbb{R}_+ k) \cap K = A_i$ (that is, A_i verifies (10)) for every $i \in I$, then

$$\psi_{\cap_{i\in I} A_i,k} = \inf_{i\in I} \psi_{A_i,k}, \qquad \varphi_{\cap_{i\in I} A_i,k} = \sup_{i\in I} \varphi_{A_i,k}. \tag{12}$$

In particular, if $A_i = A_i + \mathbb{R}_+ k$ for every $i \in I$ then (12) holds.

PROOF. Because $A := \bigcap_{i\in I} A_i \subset A_j \subset \bigcup_{i\in I} A_i =: A'$ for $j \in I$ we have that $\psi_A \leq \psi_{A_j} \leq \psi_{A'}$ for $j \in I$. This implies that $\psi_A \leq \inf_{i\in I} \psi_{A_i} \leq \sup_{i\in I} \psi_{A_i} \leq \psi_{A'}$. On the other hand it is clear that

$$\{\, t \in \mathbb{R} \mid x - tk \in \bigcup_{i\in I} A_i \,\} = \bigcup_{i\in I} \{\, t \in \mathbb{R} \mid x - tk \in A_i \,\},$$

whence $\psi_{A'} = \sup_{i\in I} \psi_{A_i}$. Similarly, $\varphi_{A'} = \inf_{i\in I} \varphi_{A_i}$.

Assume now that K is closed and $(A_i + \mathbb{R}_+ k) \cap K = A_i$ for every $i \in I$. Consider $x \in X$ and $s \in \mathbb{R}$ such that $s < \inf_{i\in I} \psi_{A_i}(x)$. Hence, for every $i \in I$ we have $s < \psi_{A_i}(x)$, and so there exists $t_i > s$ such that $x - t_i k \in A_i \subset K$. Set $\overline{s} := \inf\{\, t_i \mid i \in I \,\} \geq s$. Hence $x - \overline{s}k = x - t_i k + (t_i - \overline{s})k \in A_i + \mathbb{R}_+ k$ for every $i \in I$. Moreover, we have that $x - \overline{s}k \in \operatorname{cl} K = K$. Hence $x - \overline{s}k \in (A_i + \mathbb{R}_+ k) \cap K = A_i$ for every $i \in I$, and so $x - \overline{s}k \in A$. This shows that $\psi_A(x) \geq \overline{s} \geq s$. Hence $\psi_A = \inf_{i\in I} \psi_{A_i,k}$.

A similar argument yields $\varphi_A = \sup_{i\in I} \varphi_{A_i,k}$. ■

3 Duality relations involving the functions σ_A, ς_A, θ_A and ϑ_A

First we establish formulas for $\varsigma_A(x^*)$ with $x^* \in A^+$.

Proposition 4 *Let $A \subset X$ and set $\widetilde{A} := \overline{\operatorname{conv}}([1, \infty[A)$. Then for every $x^* \in A^+$ one has*

$$\begin{aligned} \varsigma_A(x^*) &= \inf\{\, \theta_A(x) \cdot \langle x, x^* \rangle \mid x \in \mathbb{P}A \,\} \\ &= \inf\{\, \theta_A(x) \cdot \langle x, x^* \rangle \mid x \in A \,\} \\ &= \inf\left\{\, \theta_A(x) \cdot_e \langle x, x^* \rangle \mid x \in A^{++} \,\right\} \\ &= \inf\left\{\, \theta_{\widetilde{A}}(x) \cdot_e \langle x, x^* \rangle \mid x \in A^{++} \,\right\}. \end{aligned} \tag{13}$$

PROOF. If $A = \emptyset$ (13) holds taking into account our conventions. Assume that $A \neq \emptyset$. Let $x^* \in A^+$. Then

$$\begin{aligned} \varsigma_A(x^*) &= \inf\{\, \langle x', x^* \rangle \mid x' \in A \,\} \\ &= \inf\{\, t \langle x, x^* \rangle \mid t > 0,\ x \in X,\ tx = x' \in A \,\} \\ &= \inf\{\, t \langle x, x^* \rangle \mid t > 0,\ x \in \mathbb{P}A,\ tx \in A \,\} \\ &= \inf\nolimits_{x\in\mathbb{P}A} [\langle x, x^* \rangle \cdot \inf\{\, t > 0 \mid tx \in A \,\}] \\ &= \inf\{\, \theta_A(x) \cdot \langle x, x^* \rangle \mid x \in \mathbb{P}A \,\}. \end{aligned}$$

For the second equality one uses (2), while the third is a rewriting of the first one because for $x \in A^{++} \setminus \mathbb{P}A$ we have that $\theta_A(x) = \infty$ and $\langle x, x^* \rangle \geq 0$. It is clear that $A^+ = \widetilde{A}^+$ and $\varsigma_A(x^*) = \varsigma_{\widetilde{A}}(x^*)$ for $x^* \in A^+$. Hence the last equality of (13) follows from the previous one replacing A by $\widetilde{A}$. ■

The next result establishes an estimate for $\varsigma_A(x^*)$ when $x^* \in X^*$.

Proposition 5 *Let $\emptyset \neq A \subset X$. Then*

$$\varsigma_A(x^*) \leq \inf \{ \theta_A(x) \mid x \in \mathbb{P}A, \ \langle x, x^* \rangle = 1 \} \tag{14}$$

for every $x^* \in X^*$, the inequality being strict for $x^* \in (X^* \setminus A^+) \cup (A^+ \cap A^-)$. Moreover, if $0 \notin A$ then

$$\varsigma_A(x^*) = \inf \{ \theta_A(x) \mid x \in \mathbb{P}A, \ \langle x, x^* \rangle = 1 \} = \inf \{ \theta_A(x) \mid x \in [x^* = 1] \} \tag{15}$$

for every $x^* \in A^\#$.

Proof. For $x^* \in A^+$ (14) follows immediately from (13). If $x^* \notin A^+$ then $\varsigma_A(x^*) < 0 \leq \lambda(x^*) := \inf \{ \theta_A(x) \mid x \in \mathbb{P}A, \ \langle x, x^* \rangle = 1 \}$; if $x^* \in A^+ \cap A^-$ then $\varsigma_A(x^*) = 0 < \infty = \lambda(x^*)$. Assume that $0 \notin A$ and fix $x^* \in A^\#$. Since $\langle x, x^* \rangle > 0$ for every $x \in \mathbb{P}A$, using (2) we may take $\langle x, x^* \rangle = 1$ in the second term of (13), getting so (15). ■

Assuming that $0 \notin A$, the natural question is if (15) is true for $x^* \in A^+ \setminus A^-$ ($\supset A^\#$). We give an affirmative answer under some additional conditions.

Proposition 6 *Let $A \subset X \setminus \{0\}$ be a convex set with ${}^iA \neq \emptyset$. Then*

$$\varsigma_A(x^*) = \inf \{ \theta_A(x) \mid x \in \mathbb{P}A, \ \langle x, x^* \rangle = 1 \} = \inf \{ \theta_A(x) \mid x \in [x^* = 1] \} \qquad \forall x^* \in A^+ \setminus A^-.$$

Proof. Set $A_0 := {}^iA$. Then $A_0^+ = A^+$, $\varsigma_{A_0} = \varsigma_A$ and $\theta_{A_0}(x) = \theta_A(x)$ for every $x \in \mathbb{P}A_0$. Moreover, $A^+ \setminus A^- \subset A_0^\#$. The first two assertions follow from the fact that $(1-t)x + ty \in {}^iA$ for $x \in {}^iA$, $y \in A$ and $t \in [0, 1[$. Since $A_0 \subset A$ it is clear that $\theta_{A_0} \geq \theta_A$. Take $x = s\overline{x}$ with $s \in \mathbb{P}$ and $\overline{x} \in A_0$, and $\theta_A(x) < \overline{t}$. Then there exists $t \in]0, \overline{t}[$ with $tx \in A$. Then $rtx + (1-r)\overline{x} = \big(rt + (1-r)s^{-1}\big)\, x \in A_0$ for all $r \in]0, 1[$. Hence $\theta_{A_0}(x) \leq rt + (1-r)s^{-1}$ for $r \in]0, 1[$. Letting $r \to 1$ we get $\theta_{A_0}(x) \leq t < \overline{t}$. Hence our claim is true. For the last claim take $x^* \in A^+ \setminus A^-$ and assume that $\langle x_0, x^* \rangle = 0$ for some $x_0 \in A_0$. Take $x \in A$. Then there exists $s > 0$ such that $x' := (1+s)x_0 - sx \in A$. It follows that $0 \leq \langle x', x^* \rangle = -s \langle x, x^* \rangle \leq 0$, whence $\langle x, x^* \rangle = 0$. Hence $-x^* \in A^+$, a contradiction.

Using Proposition 5 for A replaced by A_0 we obtain

$$\varsigma_A(x^*) \leq \inf \{ \theta_A(x) \mid x \in \mathbb{P}A, \ \langle x, x^* \rangle = 1 \} \leq \inf \{ \theta_A(x) \mid x \in \mathbb{P}A_0, \ \langle x, x^* \rangle = 1 \} = \varsigma_{A_0}(x^*)$$

for every $x^* \in A_0^\# \supset A^+ \setminus A^-$. The conclusion follows. ■

The next result is a kind of converse of Proposition 6.

Proposition 7 *Let $A \subset X \setminus \{0\}$ be a nonempty closed convex set such that $A = [1, \infty[A$, that is, A is semi-conic. Then $A_\infty = \operatorname{cl}(\mathbb{P}A)$. Moreover, for every $x \in A_\infty \setminus (-A_\infty)$ one has*

$$\theta_A(x) = \sup \left\{ \varsigma_A(x^*) \mid x^* \in A^+, \ \langle x, x^* \rangle = 1 \right\}. \tag{16}$$

Proof. We have that $x + sy = (1+s)\big(\frac{1}{1+s}x + \frac{s}{1+s}y\big) \in A$ for all $x, y \in A$ and $s \geq 0$ because A is convex and $[1, \infty[A \subset A$. Hence $A \subset A_\infty$, and so $\operatorname{cl}(\mathbb{P}A) \subset A_\infty$. On the other hand, for $u \in A_\infty$ we have that there exist the nets $(x_i)_{i \in I} \subset A$ and $(t_i)_{i \in I} \subset \mathbb{P}$ with $t_i \to 0$ and $t_i x_i \to u$. It follows that $u \in \operatorname{cl}(\mathbb{P}A)$. Hence $A_\infty = \operatorname{cl}(\mathbb{P}A)$.

First observe that for $sx \in A$ and $x^* \in X^*$ with $\langle x, x^* \rangle = 1$ we have that $s = \langle sx, x^* \rangle \geq \varsigma_A(x^*)$. This proves the inequality $\geq$ in (16) for every $x \in X$. Take $x \in A_\infty \setminus (-A_\infty)$ and $0 < s < \theta_A(x)$; this is possible because $\theta_A(x') > 0$ for every $x' \in X$ (A being closed and $0 \notin A$). Then $sx \notin A$. Using a separation theorem we get $x^* \in X^*$ and $r \in \mathbb{R}$ such that

$$\langle sx, x^* \rangle < r \leq \langle y, x^* \rangle \qquad \forall y \in A. \tag{17}$$

Since $A = A + A_\infty$ we obtain that $\langle u, x^* \rangle \geq 0$ for all $u \in A_\infty$, that is, $x^* \in (A_\infty)^+ = (\mathbb{P}A)^+ = A^+$. Because $x \in A_\infty$ we have that $\langle x, x^* \rangle \geq 0$, and so $r > 0$. If $\langle x, x^* \rangle > 0$ we may (and do) assume that $\langle x, x^* \rangle = 1$. From (17) we obtain that $s < r \leq \varsigma_A(x^*)$. Assume now that $\langle x, x^* \rangle = 0$. Because $x \in A_\infty \setminus (-A_\infty)$, there exists $\overline{x}^* \in A_\infty^+ = A^+$ such that $\langle x, \overline{x}^* \rangle = 1$. Then $x_t^* := tx^* + \overline{x}^* \in A^+$ and $\langle x, x_t^* \rangle = 1$ for $t \in \mathbb{R}_+$. Moreover, $\varsigma_A(x_t^*) \geq t\varsigma_A(x^*) + \varsigma_A(\overline{x}^*) \geq tr + \varsigma_A(\overline{x}^*) > s$ for $t > 0$ sufficiently large. Hence (16) holds. ■

Of course, in the conditions of Proposition 7, if $-x \in A_\infty$ then $\langle x, x^* \rangle \leq 0$ for every $x^* \in A^+$, and so (16) does not hold (the term in the right hand side of (16) is $-\infty$).

Note that the supremum in (16) could be not attained. To see this consider

$$A := \left\{ (x,y) \in \mathbb{R}^2 \mid x \in [0,1],\ y \geq 1 - \left(2x - x^2\right)^{1/2} \right\} \cup ([1,\infty[\times \mathbb{R}_+).$$

Then $A_\infty = \mathbb{R}_+^2 = A^+$, $\varsigma_A(a,1) = 1 + a - (1+a^2)^{1/2} < 1 = \theta(0,1)$ for every $a \geq 0$.

Proposition 8 *Let $\emptyset \neq A \subset X$. Then*

$$\begin{aligned}
\sigma_A(x^*) &= \sup \{ \vartheta_A(x) \cdot \langle x, x^* \rangle \mid x \in [x^* > 0] \cap \mathbb{P}A \} \\
&= \sup \{ \vartheta_A(x) \cdot_h \langle x, x^* \rangle \mid x \in [x^* \geq 0] \cap \mathbb{P}A \} \\
&= \sup \{ \vartheta_A(x) \cdot_h \langle x, x^* \rangle \mid x \in [x^* \geq 0] \cap A \} \\
&= \sup \left\{ \vartheta_A(x) \cdot_h \langle x, x^* \rangle \mid x \in [x^* \geq 0] \cap A^{++} \right\} \\
&= \sup \{ \vartheta_A(x) \mid x \in \mathbb{P}A,\ \langle x, x^* \rangle = 1 \} \\
&= \sup \{ \vartheta_A(x) \mid x \in X,\ \langle x, x^* \rangle = 1 \} > 0
\end{aligned} \tag{18}$$

for every $x^* \in X^* \setminus A^-$. Moreover, if $0 \in \overline{\operatorname{conv}} A$ then for every $x^* \in X^*$ one has

$$\sigma_A(x^*) = 0 \vee \sup \{ \vartheta_A(x) \mid x \in X,\ \langle x, x^* \rangle = 1 \}. \tag{19}$$

PROOF. Note first that $\sigma_A(x^*) > 0$ if and only if $x^* \in X^* \setminus A^-$. Let $x^* \in X^* \setminus A^-$; then there exists $x \in A$ such that $\langle x, x^* \rangle > 0$, and so $[x^* > 0] \cap \mathbb{P}A \neq \emptyset$. Then

$$\begin{aligned}
\sigma_A(x^*) &= \sup \{ \langle x', x^* \rangle \mid x' \in A \} \\
&= \sup \{ t \langle x, x^* \rangle \mid t > 0,\ x \in X,\ tx = x' \in A \} \\
&= \sup \{ t \langle x, x^* \rangle \mid t > 0,\ x \in [x^* > 0] \cap \mathbb{P}A,\ tx \in A \} \\
&= \sup\nolimits_{x \in [x^*>0] \cap \mathbb{P}A} \left[\langle x, x^* \rangle \cdot \sup \{ t > 0 \mid tx \in A \} \right] \\
&= \sup \{ \vartheta_A(x) \cdot \langle x, x^* \rangle \mid x \in [x^* > 0] \cap \mathbb{P}A \} > 0,
\end{aligned}$$

that is, (18) holds. Because for $x \in [x^* = 0] \cap \mathbb{P}A$ we have $\vartheta_A(x) \cdot_h \langle x, x^* \rangle = 0$, the second equality holds, too. For the third equality one uses (2), while the fourth is a rewriting of the second one because for $x \in [x^* \geq 0] \cap A^{++} \setminus \mathbb{P}A$ we have that $\vartheta_A(x) = -\infty$ and $\langle x, x^* \rangle \geq 0$. Taking into account (2), from (18) we get immediately the fifth equality, while for the last equality observe that $\vartheta_A(x) = -\infty$ for $x \in X \setminus \mathbb{P}A$.

Assume that $0 \in \overline{\operatorname{conv}} A$; therefore, $\sigma_A(x^*) = \sigma_{\overline{\operatorname{conv}} A}(x^*) \geq 0$ for every $x^* \in X^*$. Set $\xi(x^*) := \sup\{\vartheta_A(x) \mid x \in X,\ \langle x, x^* \rangle = 1\}$ for $x^* \in X^*$. It is clear that $\xi(0) = -\infty$ and so (19) holds for $x^* = 0$. If $x^* \in X^* \setminus A^-$ (19) clearly follows from the first part. Let $0 \neq x^* \in A^-$. Then $\sigma_A(x^*) = 0$ and $x \notin \mathbb{P}A$ for every $x \in [x^* = 1]$. Therefore, $\xi(x^*) = -\infty$ and once again we have that (19) holds. ■

Proposition 9 *Let $A \subset X$ be such that $0 \in \overline{\operatorname{conv}} A$. Then for every $x \in X$ we have*

$$0 \vee \vartheta_{\overline{\operatorname{conv}} A}(x) = \inf \{ \sigma_A(x^*) \mid x^* \in X^*, \ \langle x, x^* \rangle = 1 \}. \tag{20}$$

PROOF. Because $\sigma_A = \sigma_{\overline{\operatorname{conv}} A}$ we may (and do) assume that $A = \overline{\operatorname{conv}} A$. Set

$$\eta(x) := \inf \{ \sigma_A(x^*) \mid \langle x, x^* \rangle = 1 \}$$

for $x \in X$. Using (19) it is clear that $0 \vee \vartheta_A(x) \leq \eta(x)$ for every $x \in X$. For the converse inequality we consider several cases.

a) $x = 0$; then $\vartheta_A(x) = \infty$ and $\eta(x) = \infty$, and so (20) holds.

b) $x \in \mathbb{P}A \setminus \{0\}$ (hence $\vartheta_A(x) > 0$). Take $s > \vartheta_A(x)$ (if possible). Then $sx \notin A$, and so, by a separation theorem, there exists $x^* \in X^*$ such that $\langle sx, x^* \rangle > \sigma_A(x^*) \geq 0$. Hence $\langle x, x^* \rangle > 0$, and so we may (and do) assume that $\langle x, x^* \rangle = 1$. Therefore, $\eta(x) < s$; the conclusion follows.

c) $x \notin \mathbb{P}A$. Consider the function $\varphi\colon X \to \overline{\mathbb{R}}$ defined by $\varphi(u) = t$ if $u = tx$ and $\varphi(u) := \infty$ else. Then φ is a lsc convex function with $\varphi^*(x^*) = 0$ if $\langle x, x^* \rangle = 1$ and $\varphi^*(x^*) = \infty$ else. Moreover, we have that $(\iota_A \Box \varphi)(u) = \inf \{ t \in \mathbb{R} \mid u - tx \in A \}$. It is clear that $(\iota_A \Box \varphi)(0) = 0$. Let us prove that $\overline{\iota_A \Box \varphi}(0) = 0$. In the contrary case there exist $s > 0$ and a net $(u_i)_{i \in I}$ converging to 0 such that $(\iota_A \Box \varphi)(u_i) < -s$ for every $i \in I$. Therefore, for every $i \in I$ there exists $t_i > s$ with $x_i := u_i + t_i x \in A$. We may (and do) assume that $t_i \to t \in [s, \infty]$. If $t < \infty$ then $tx \in \operatorname{cl} A = A$, a contradiction. If $t = \infty$ then $x = \lim t_i^{-1} x_i$, and so $x \in A_\infty \subset A$, again a contradiction. Therefore, $\overline{\iota_A \Box \varphi}(0) = 0$; in particular $\overline{\iota_A \Box \varphi}$ is proper. Since $(\iota_A \Box \varphi)^* = \sigma_A + \varphi^*$, we obtain (see [14]) that

$$0 = \overline{\iota_A \Box \varphi}(0) = (\iota_A \Box \varphi)^{**}(0) = (\sigma_A + \varphi^*)^*(0) = -\inf_{x^* \in X^*} (\sigma_A + \varphi^*)(x^*) = -\eta(x).$$

Since $\vartheta_A(x) = -\infty$, we obtain that (20) holds in this case, too. ■

Observe that there exists $x \in X$ such that (20) is false if $0 \notin \overline{\operatorname{conv}} A$.

Note that using relations (1) and (3) one can establish duality results involving other combinations of the functions σ_A, ς_A, μ_A, ν_A, θ_A and ϑ_A.

4 Duality relations involving σ_A, ς_A and $\psi_{A,k}$

In [7, Rels. (18), (19)] the duality relations

$$\varsigma_A(x^*) = \inf_x \left(\langle x, x^* \rangle - \psi_A(x) \cdot \langle k, x^* \rangle \right) \tag{21}$$

and

$$\psi_A(x) = \inf_{x^*} \frac{\langle x, x^* \rangle - \varsigma_A(x^*)}{\langle k, x^* \rangle} \tag{22}$$

are given for $X = \mathbb{R}^n$ without mentioning from where x and x^* are taken. However in the context of [7] A $(= L(y))$ is a convex set (included in $\mathbb{R}^n_+$) and k $(= g_x)$ is an element of $\mathbb{R}^n_+ \setminus \{0\}$. In the sequel we shall try to find conditions which ensure the previous duality formulas.

In fact the first relation is true for x^* with $\langle k, x^* \rangle > 0$ taking $x \in X$, while in the second one must take $x^* \in X^*$ with $\langle k, x^* \rangle > 0$ (or, equivalently, $\langle k, x^* \rangle = 1$).

Proposition 10 *Let $x^* \in X^*$ with $\langle k, x^* \rangle \geq 0$ and $\emptyset \neq A \subset B \subset X$. Then*

$$\varsigma_A(x^*) = \inf_{x \in B} \left(\langle x, x^* \rangle - \psi_{A,k}(x) \cdot_h \langle k, x^* \rangle \right) \tag{23}$$

and

$$\sigma_A(x^*) = \sup_{x \in B} \left(\langle x, x^* \rangle + \psi_{A,-k}(x) \cdot_e \langle k, x^* \rangle \right). \tag{24}$$

If $\langle k, x^* \rangle > 0$ then (23) and (24) hold even if $A = \emptyset$.

PROOF. Let first $\langle k, x^* \rangle > 0$. If $A = \emptyset$ then (23) and (24) clearly hold. Assume that $A \neq \emptyset$. We have that

$$\begin{aligned}\inf_{x\in X}\left(\langle x, x^*\rangle - \psi_A(x)\cdot\langle k, x^*\rangle\right) &= \inf_{x\in X}\left(\langle x, x^*\rangle - \langle k, x^*\rangle\cdot\sup\{t \mid x - tk \in A\}\right)\\ &= \inf_{x\in X}\left(\inf\{\langle x, x^*\rangle - t\,\langle k, x^*\rangle \mid t\in\mathbb{R},\ x - tk \in A\}\right)\\ &= \inf\left\{\langle x - tk, x^*\rangle \mid x\in X,\ t\in\mathbb{R},\ x - tk\in A\right\}\\ &= \varsigma_A(x^*).\end{aligned}$$

Hence, if $A \subset B \subset X$ then

$$\begin{aligned}\inf_{x\in X}\left(\langle x, x^*\rangle - \psi_A(x)\cdot\langle k, x^*\rangle\right) &\le \inf_{x\in B}\left(\langle x, x^*\rangle - \psi_A(x)\cdot\langle k, x^*\rangle\right)\\ &\le \inf_{x\in A}\left(\langle x, x^*\rangle - \psi_A(x)\cdot\langle k, x^*\rangle\right)\\ &\le \inf_{x\in A}\langle x, x^*\rangle\end{aligned}$$

because $A \subset [\psi_A \ge 0]$. Therefore, (23) holds in this case.

Let now $\langle k, x^* \rangle = 0$. Then

$$\inf_{x\in X}\left(\langle x, x^*\rangle - \psi_A(x)\cdot_h\langle k, x^*\rangle\right) = \inf\{\langle x, x^*\rangle \mid x\in\mathbb{R}k + A\} = \inf\{\langle x, x^*\rangle \mid x\in A\} = \varsigma_A(x^*),$$

and so (23) holds in this case, too.

For obtaining (24) just use the formula $\sigma_A(x^*) = -\varsigma_A(-x^*)$ in (23). ■

If $\langle k, x^* \rangle < 0$ relation (21) does not hold.

Example 1 *Consider* $A := [1, \infty)$, $k := 1$ *and* $x^* := -1$, *one has* $\psi_A(x) = x - 1$ *for every* $x \in X = \mathbb{R}$, $\varsigma_A(x^*) = -\infty$ *and* $\inf_{x\in X}\{\langle x, x^*\rangle - \psi_A(x)\cdot\langle k, x^*\rangle\} = -1$.

Corollary 1 *Let* $x^* \in X^*$ *be such that* $\langle k, x^* \rangle \ge 0$ *and* $A \subset X$. *Then*

$$\varsigma_A(x^*) = \inf\{\langle x, x^*\rangle \mid x\in B\} \tag{25}$$

provided $A \subset B \subset [\psi_{A,k} \ge 0]$ *and*

$$\sigma_A(x^*) = \sup\{\langle x, x^*\rangle \mid x\in B\} \tag{26}$$

provided $A \subset B \subset [\psi_{A,-k} \ge 0]$.

PROOF. If $A = \emptyset$ then necessarily $B = \emptyset$, and so (25) and (26) hold. Assume that $A \neq \emptyset$. It is clear that $A \subset [\psi_A \ge 0]$; for $x \in [\psi_A \ge 0]$ clearly $\langle x, x^*\rangle - \psi_A(x)\cdot_h\langle k, x^*\rangle \le \langle x, x^*\rangle$. Using the preceding proposition (for $B = X$) we get

$$\begin{aligned}\varsigma_A(x^*) &= \inf_{x\in X}(\langle x, x^*\rangle - \psi_A(x)\cdot_h\langle k, x^*\rangle) \le \inf_{x\in[\psi_A\ge 0]}(\langle x, x^*\rangle - \psi_A(x)\cdot_h\langle k, x^*\rangle)\\ &\le \inf_{x\in[\psi_A\ge 0]}\langle x, x^*\rangle \le \inf_{x\in B}\langle x, x^*\rangle \le \inf_{x\in A}\langle x, x^*\rangle = \varsigma_A(x^*).\end{aligned}$$

Hence (25) holds. Replacing x^* by $-x^*$ and k by $-k$ in the preceding statement we get (26). ■

In what concerns relation (22) it is clear that one must have $\langle k, x^* \rangle \neq 0$. Also note that for $x \in X$ one has always

$$\psi_A(x) \le \inf\{\langle x, x^*\rangle - \varsigma_A(x^*) \mid x^*\in X^*,\ \langle k, x^*\rangle = 1\} =: \eta_A(x). \tag{27}$$

Indeed, let $x \in X$, $t \in \mathbb{R}$ and $x^* \in X^*$ be such that $x \in tk + A$ and $\langle k, x^* \rangle = 1$. Then $\varsigma_A(x^*) \le \langle x - tk, x^*\rangle = \langle x, x^*\rangle - t$, that is, $t \le \langle x, x^*\rangle - \varsigma_A(x^*)$. Therefore, the claim holds.

However, one cannot expect equality for every set A. One reason is that ψ_A is not necessarily concave, while η_A is concave (and $\eta_A = \eta_{\overline{\mathrm{conv}}A}$); moreover, as seen in (7), $\psi_A = \psi_{A+\mathbb{R}_+k}$ which is not the case with ς_A; however, $\varsigma_A(x^*) = \varsigma_{A+\mathbb{R}_+k}(x^*)$ if $\langle k, x^*\rangle > 0$. It is natural to consider only x^* with $\langle k, x^*\rangle > 0$ (simple examples can be given with $A = A + \mathbb{R}_+k$ convex for which strict inequality holds in (27)).

It is known that for A a nonempty closed convex set we have $(A_\infty)^+ = \mathrm{cl}\{\, x^* \mid \varsigma_A(x^*) > -\infty \,\}$ (use f.i. [14, Exercise 2.23] for $f = \iota_A$). Since $(A_\infty)^{++} = A_\infty$, we have

$$-k \in A_\infty \iff [\varsigma_A(x^*) > -\infty \Rightarrow \langle k, x^*\rangle \le 0]\,,$$

and so

$$-\,k \notin A_\infty \iff [\exists \overline{x}^* \in X^* : \varsigma_A(\overline{x}^*) > -\infty,\ \langle k, \overline{x}^*\rangle = 1]\,. \tag{28}$$

Note also that when A is a nonempty closed (not necessarily convex) set and $k \in X$ is such that $-k \notin A_\infty$ we have that $A + \mathbb{R}_+k$ is closed and $\psi_A(x) < \infty$ for every $x \in X$.

Proposition 11 *Assume that A is a closed convex set. Then*

$$\psi_{A,k}(x) = \inf\{\langle x, x^*\rangle - \varsigma_A(x^*) \mid x^* \in X^*,\ \langle k, x^*\rangle = 1\} \qquad \forall x \in X \tag{29}$$

provided $-k \notin A_\infty$, and

$$\psi_{A,-k}(x) = \inf\{\sigma_A(x^*) - \langle x, x^*\rangle \mid x^* \in X^*,\ \langle k, x^*\rangle = 1\} \qquad \forall x \in X \tag{30}$$

provided $k \notin A_\infty$.

PROOF. If $A = \emptyset$ then clearly (29) and (30) hold.

Assume that $A \ne \emptyset$. As observed above $A + \mathbb{R}_+k$ is closed and $\psi_A(x) < \infty$ for every $x \in X$. Fix $x \in X$. By (27) we have that $\psi_A(x) \le \eta_A(x)$. Take some $s \in \mathbb{R}$ such that $s > \psi_A(x) = \psi_{A+\mathbb{R}_+k}(x)$; then $x - sk \notin A + \mathbb{R}_+k$. Because $A + \mathbb{R}_+k$ is convex and closed there exist $x_0^* \in X^*$ and $s_0 \in \mathbb{R}$ such that $\langle x - sk, x_0^*\rangle < s_0 \le \langle a, x_0^*\rangle + t\,\langle k, x_0^*\rangle$ for all $a \in A$ and $t \in \mathbb{R}_+$. It follows that $\langle k, x_0^*\rangle \ge 0$ and $\langle x, x_0^*\rangle - s\,\langle k, x_0^*\rangle < s_0 \le \varsigma_A(x_0^*)$. If $\langle k, x_0^*\rangle \ne 0$, we may assume that $\langle k, x_0^*\rangle = 1$ (replacing x_0^* by $\langle k, x_0^*\rangle^{-1} x_0^*$ if necessary). Hence $\eta_A(x) \le \langle x, x_0^*\rangle - \varsigma_A(x_0^*) < s\,\langle k, x_0^*\rangle = s$. Assume that $\langle k, x_0^*\rangle = 0$. Because $-k \notin A_\infty$, by (28) there exists $\overline{x}^* \in X^*$ such that $\varsigma_A(\overline{x}^*) > -\infty$ and $\langle k, \overline{x}^*\rangle = 1$. Then $\langle k, \overline{x}^* + tx_0^*\rangle = 1$ and $\varsigma_A(\overline{x}^* + tx_0^*) \ge \varsigma_A(\overline{x}^*) + t\varsigma_A(x_0^*) \ge ts_0 + \varsigma_A(\overline{x}^*)$ for $t \ge 0$, and so

$$\begin{aligned}\eta_A(x) &\le \inf\{\, \langle x, \overline{x}^* + tx_0^*\rangle - \varsigma_A(\overline{x}^* + tx_0^*) \mid t \ge 0 \,\}\\ &\le \langle x, \overline{x}^*\rangle - \varsigma_A(\overline{x}^*) + \inf\{\, t \cdot [\langle x, x_0^*\rangle - s_0] \mid t \ge 0 \,\}\\ &= -\infty.\end{aligned}$$

Hence $\eta_A(x) < s$ in this case, too. It follows that $\eta_A(x) \le \psi_A(x)$. Therefore, (29) holds. Replacing k by $-k$ and x^* by $-x^*$ in (29) we get (30). ■

5 Other duality results

Inspired by [7], we consider the nonempty set $T \subset X \times Y$, where Y is another separated locally convex space with topological dual Y^*. As in [7] we associate the sets

$$P(x) := \{\, y \in Y \mid (x, y) \in T \,\}, \qquad L(y) := \{\, x \in X \mid (x, y) \in T \,\}$$

for $x \in X$ and $y \in Y$; of course, $P(x)$ and $L(y)$ are convex (resp. closed) if T is convex (resp. closed). Moreover, if T is closed and convex and $u \in \ker T_\infty := (T_\infty)^{-1}(0) := \{x \in X \mid (x, 0) \in T_\infty\}$, then $u \in (L(y))_\infty$ for any $y \in \mathrm{Pr}_Y(T)$; conversely, if $u \in (L(y))_\infty$ for some $y \in \mathrm{Pr}_Y(T)$ then $u \in \ker T_\infty$. A similar statement is valid with respect to the other variable.

Define now the sets

$$\overline{P}(x^*) := \{ y \mid \exists x \in X: \ y \in P(x), \ \langle x, x^* \rangle \leq 1 \} = \bigcup \{ P(x) \mid \langle x, x^* \rangle \leq 1 \} \subset \mathrm{Pr}_Y(T), \tag{31}$$

$$\overline{L}(y^*) := \{ x \mid \exists y \in Y: \ x \in L(y), \ \langle y, y^* \rangle \geq 1 \} = \bigcup \{ L(y) \mid \langle y, y^* \rangle \geq 1 \} \subset \mathrm{Pr}_X(T), \tag{32}$$

for $x^* \in X^*$ and $y^* \in Y^*$.

Lemma 1 *Let $T \subset X \times Y$ be nonempty.*

(a) *If T is convex then $\overline{P}(x^*)$ and $\overline{L}(y^*)$ are convex for all $x^* \in X^*$ and $y^* \in Y^*$.*

(b) *If T is closed and* $\dim X < \infty$ *then $\overline{P}(x^*)$ is closed for every $x^* \in (\ker T_\infty)^\#$.*

(c) *If T is closed and* $\dim Y < \infty$ *then $\overline{L}(y^*)$ is closed for every $y^* \in -(T_\infty(0))^\#$.*

PROOF. (a) Let $y_1, y_2 \in \overline{P}(x^*)$ and $s \in [0,1]$. There exist $x_1, x_2 \in X$ with $y_1 \in P(x_1)$ and $y_2 \in P(x_2)$, that is, $(x_1, y_1), (x_2, y_2) \in T$ and $\langle x_1, x^* \rangle \leq 1$, $\langle x_2, x^* \rangle \leq 1$. Hence $\langle s x_1 + (1-s) x_2, x^* \rangle \leq 1$ and, by the convexity of T, $(s x_1 + (1-s)x_2, s y_1 + (1-s) y_2) \in T$, that is, $s y_1 + (1-s) y_2 \in P(s x_1 + (1-s) x_2)$. Hence $s y_1 + (1-s) y_2 \in \overline{P}(x^*)$. The convexity of $\overline{L}(y^*)$ follows similarly.

(b) Because $\dim X < \infty$ we may (and do) assume X is a normed space. Fix some $x^* \in (\ker T_\infty)^\#$ and consider $y \in \mathrm{cl}(\overline{P}(x^*))$, that is, there exists $(y_i)_{i \in I} \subset \overline{P}(x^*)$ with $y_i \to y$. For every $i \in Y$ there exists $x_i \in X$ with $(x_i, y_i) \in T$ and $\langle x_i, x^* \rangle \leq 1$. Assume that $t_i := \|x_i\| \to \infty$ (on a subnet); hence $t_i \in \mathbb{P}$ for $i \geq i_0$. Passing to a subnet if necessary, $t_i^{-1} x_i \to u \neq 0$. Since $t_i^{-1} \to 0$ and $t_i^{-1}(x_i, y_i) \to (u, 0)$, we have that $u \in \ker T_\infty$. Moreover, clearly, $t_i^{-1} \geq \langle t_i^{-1} x_i, x^* \rangle \to \langle u, x^* \rangle$, and so $\langle u, x^* \rangle \leq 0$, contradicting the fact that $x^* \in (\ker T_\infty)^\#$. Therefore, there exists some i_0 such that $(x_i)_{i \geq i_0}$ is bounded, and so we may (and do) assume that $x_i \to x \in X$; hence $\langle x, x^* \rangle \leq 1$. Because T is closed we obtain that $(x, y) \in T$, and so $y \in P(x) \subset \overline{P}(x^*)$.

(c) The proof is similar to that of (b). ■

Throughout this section $k \in X \setminus \{0\}$ and $l \in Y \setminus \{0\}$ are fixed elements. Using Proposition 3 we obtain that

$$\psi_{\overline{L}(y^*),k} = \sup \{ \psi_{L(y),k} \mid \langle y, y^* \rangle \geq 1 \}, \qquad \psi_{\overline{P}(x^*),l} = \sup \{ \psi_{P(x),l} \mid \langle x, x^* \rangle \leq 1 \}. \tag{33}$$

Proposition 12 *Assume that T is a nonempty closed convex set such that $0 \in P(x)$ for every $x \in \mathrm{Pr}_X(T)$. Then*

$$L(y) = \bigcap \{ \overline{L}(y^*) \mid \langle y, y^* \rangle \geq 1 \} \qquad \forall y \in Y. \tag{34}$$

Moreover, if $k \in \ker T_\infty$ then

$$\psi_{L(y),k} = \inf \left\{ \psi_{\overline{L}(y^*),k} \mid \langle y, y^* \rangle \geq 1 \right\} \qquad \forall y \in Y. \tag{35}$$

PROOF. Let us set $\widetilde{L}(y) := \bigcap \{ \overline{L}(y^*) \mid \langle y, y^* \rangle \geq 1 \}$ for $y \in Y$. From (32) it is clear that $L(y) \subset \widetilde{L}(y)$ for every $y \in Y$. Fix $\overline{y} \in Y$ and let us show that $\widetilde{L}(\overline{y}) \subset L(\overline{y})$. Take some $\overline{x} \in X \setminus L(\overline{y})$. If $\overline{x} \notin \mathrm{Pr}_X(T)$, by (32) we have that $\overline{x} \notin \widetilde{L}(\overline{y})$. So let $\overline{x} \in \mathrm{Pr}_X(T)$; hence $(\overline{x}, 0) \in T$. Because $(\overline{x}, \overline{y}) \notin T$, by a separation theorem, there exist $(\overline{x}^*, \overline{y}^*) \in X^* \times Y^*$ and $s \in \mathbb{R}$ such that

$$\langle \overline{x}, \overline{x}^* \rangle + \langle \overline{y}, \overline{y}^* \rangle > s > \langle x, \overline{x}^* \rangle + \langle y, \overline{y}^* \rangle \qquad \forall (x, y) \in T.$$

Since $(\overline{x}, 0) \in T$ we obtain that $\langle \overline{y}, \overline{y}^* \rangle > s_0 := s - \langle \overline{x}, \overline{x}^* \rangle > 0$. We may (and do) assume that $s_0 = 1$. It follows that $\langle \overline{y}, \overline{y}^* \rangle > 1$ and $(\overline{x}, y) \in T \Rightarrow \langle y, \overline{y}^* \rangle < 1$. The last implication shows that $\overline{x} \notin L(y)$ for $\langle y, \overline{y}^* \rangle \geq 1$, that is, $\overline{x} \notin \overline{L}(\overline{y}^*)$. Since $\langle \overline{y}, \overline{y}^* \rangle \geq 1$, we have that $\overline{x} \notin \widetilde{L}(\overline{y})$.

Assume that $k \in \ker T_\infty$. Because $(k, 0) \in T_\infty$ we have that $L(y) + \mathbb{R}_+ k = L(y)$ for every $y \in Y$, and so $\overline{L}(y^*) + \mathbb{R}_+ k = \overline{L}(y^*)$ for every $y^* \in Y^*$. Applying Propositions 12 and 3 (with $K = X$) we get the conclusion. The proof is complete. ■

Proposition 13 *Assume that $T \subset X \times Y$ is a nonempty closed convex set. Then*

$$P(x) = \bigcap \left\{ \overline{P}(x^*) \mid \langle x, x^* \rangle \le 1 \right\} \qquad \forall x \in \ker T_\infty. \tag{36}$$

Moreover, assume that the closed set $F \subset Y$ and $l \in Y \setminus \{0\}$ are such that

$$P_Y(T) \subset F \qquad \text{and} \qquad [(x,y) \in T,\ t \ge 0,\ y + tl \in F] \Longrightarrow (x, y + tl) \in T. \tag{37}$$

Then

$$\psi_{P(x),l} = \inf \left\{ \psi_{\overline{P}(x^*),l} \mid \langle x, x^* \rangle \le 1 \right\} \qquad \forall x \in \ker T_\infty. \tag{38}$$

Proof. Let us set

$$\widetilde{P}(x) := \bigcap \left\{ \overline{P}(x^*) \mid \langle x, x^* \rangle \le 1 \right\}, \qquad x \in X. \tag{39}$$

From (31) it is clear that $P(x) \subset \widetilde{P}(x)$ for every $x \in X$. Fix $\overline{x} \in \ker T_\infty$ and let us show $\widetilde{P}(\overline{x}) \subset P(\overline{x})$. Take some $\overline{y} \in Y \setminus P(\overline{x})$. Because $(\overline{x}, \overline{y}) \notin T$, by a separation theorem, there exist $(\overline{x}^*, \overline{y}^*) \in X^* \times Y^*$ and $s \in \mathbb{R}$ such that

$$\langle \overline{x}, \overline{x}^* \rangle + \langle \overline{y}, \overline{y}^* \rangle < s < \langle x, \overline{x}^* \rangle + \langle y, \overline{y}^* \rangle \qquad \forall (x,y) \in T. \tag{40}$$

Fixing some $(x_0, y_0) \in T$ we have that $(x_0 + u, y_0) \in T$ for every $u \in \ker T_\infty$. From (40) we obtain that $\langle u, \overline{x}^* \rangle \ge 0$ for every $u \in \ker T_\infty$. In particular $\langle \overline{x}, \overline{x}^* \rangle \ge 0$. Using again (40) we obtain that $\langle \overline{y}, \overline{y}^* \rangle \le \langle \overline{x}, \overline{x}^* \rangle + \langle \overline{y}, \overline{y}^* \rangle < s$, whence $s_0 := s - \langle \overline{y}, \overline{y}^* \rangle > 0$. We may (and do) assume that $s_0 = 1$. It follows that $\langle \overline{x}, \overline{x}^* \rangle < 1$ and $(x, \overline{y}) \in T \Rightarrow \langle x, \overline{x}^* \rangle > 1$. The last implication shows that $\overline{y} \notin P(x)$ for $\langle x, \overline{x}^* \rangle \le 1$, that is, $\overline{y} \notin \overline{P}(\overline{x}^*)$. Since $\langle \overline{x}, \overline{x}^* \rangle \le 1$, we have that $\overline{y} \notin \widetilde{P}(\overline{x})$.

Assume that F and l satisfy (37). Condition (37) shows that $(P(x) + \mathbb{R}_+ l) \cap F = P(x)$ for every $x \in X$. By Proposition 2 we obtain that $(\overline{P}(x^*) + \mathbb{R}_+ l) \cap F = \overline{P}(x^*)$ for every $x^* \in X^*$. Using Proposition 3 we obtain that $\inf \left\{ \psi_{\overline{P}(x^*),l} \mid \langle x, x^* \rangle \le 1 \right\} = \psi_{\widetilde{P}(x),l}$. Since $P(x) = \widetilde{P}(x)$ we get (38). ■

The next example shows that for $x \in X \setminus \ker T_\infty$ it is possible that (36) is not verified under the hypotheses of Proposition 13.

Example 2 *Let $X = Y = \mathbb{R}$, $K := \mathbb{R}_+$, $a \in \mathbb{R}$ and $T_a := \left\{ (x,y) \in X \times Y \mid y \ge 0,\ x \ge a + y^2 \right\}$. Then $P_a(x) = [0, \sqrt{x - a}]$ for $x \ge a$ and $P_a(x) = \emptyset$ for $x < a$. Moreover, for $a \le 0$ one has*

$$\overline{P}_a(u) = \begin{cases} [0, \infty) & \text{if } u \le 0, \\ [0, \sqrt{1/u - a}] & \text{if } u > 0, \end{cases} \qquad \widetilde{P}_a(x) = \begin{cases} [0, \sqrt{-a}] & \text{if } x \le 0, \\ [0, \sqrt{x - a}] & \text{if } x > 0, \end{cases}$$

while for $a > 0$ one has

$$\overline{P}_a(u) = \begin{cases} [0, \infty) & \text{if } u \le 0, \\ [0, \sqrt{1/u - a}] & \text{if } 0 < u \le 1/a, \\ \emptyset & \text{if } u > 1/a, \end{cases} \qquad \widetilde{P}_a(x) = \begin{cases} \emptyset & \text{if } x < a, \\ [0, \sqrt{x - a}] & \text{if } x \ge a. \end{cases}$$

Hence $\widetilde{P}_a(x) = P_a(x)$ for every $x \in K$ (and every a), but $\widetilde{P}_a(x) \ne P_a(x)$ for $x \in X \setminus K$ when $a \le 0$ ($\widetilde{P}$ being defined in (39)).

Note that relations (33), (35) and (38), as well as (34) and (36) can be interpreted also as duality results.

6 Connections with duality results in economics literature

Taking $A := \{ x \in \mathcal{X} \mid u(x) \geq u \}$ and $B := \mathcal{X}$ Proposition 10 extends [11, Prop. 2.4] because $b(x,u) = \psi_{A,g}(x)$. It also extends [11, Prop. 4.1]. Indeed, using the notation from [11], we have $\pi(p) = \sigma_{\mathcal{Y}}(p)$ and $\sigma(g;y) = \varphi_{-\mathcal{Y},g}(y)$. So, from (24) and (9) we get for $g \cdot p > 0$, $A := -\mathcal{Y}$, $k := g$ and $x^* := p$,

$$\pi(p) = \sigma_{\mathcal{Y}}(p) = \sup_{x\in\mathbb{R}^m} \left(\langle x,p\rangle + \psi_{\mathcal{Y},-g}(x)\cdot\langle g,p\rangle\right) = \sup_{x\in\mathbb{R}^m} \left(x\cdot p - \sigma(g;y)g\cdot p\right),$$

that is, the conclusion of [11, Prop. 4.1].

In several papers on production analysis a technology is a nonempty set $T \subset \mathbb{R}^n_+ \times \mathbb{R}^m_+$ with $n, m \geq 1$ satisfying several axioms among the next ones (see [4, p. 353]):

(A1) T is closed.

(A2) Inputs and outputs are freely disposable; i.e., $(x,y) \in T$, $(x',y') \in \mathbb{R}^n \times \mathbb{R}^m$ and $x' \geq x$, $0 \leq y' \leq y$ imply $(x',y') \in T$ (here $x' \geq x$ means $x' - x \in \mathbb{R}^n_+$ and $y' \leq y$ means $y - y' \in \mathbb{R}^m_+$).

(A3) There is no free lunch; i.e., $(0,y) \in T$ implies $y = 0$.

(A4) Doing nothing is feasible; i.e., $(0,0) \in T$.

(A5) T is convex.

Sometimes instead of the (free) disposability axiom (A2) one uses the weak disposability axiom

(A2') $(x,y) \in T$, $s \in [1,\infty[$ and $t \in [0,1]$ imply $(sx,y) \in T$ and $(x,ty) \in T$.

Note that axiom (A2) is written in the form: "if $(x,y) \in T$ and $(x',-y') \geq (x,-y)$, then $(x',y') \in T$" in [4] (and other articles); this is equivalent to (A2) if one asks $(x',y') \in \mathbb{R}^n_+ \times \mathbb{R}^m_+$. Because the technology T is perfectly determined by the multifunction $P\colon \mathbb{R}^n_+ \rightrightarrows \mathbb{R}^m_+$ or $L\colon \mathbb{R}^m_+ \rightrightarrows \mathbb{R}^n_+$, sometimes one mentions the axioms in terms of P or L. We set $X := \mathbb{R}^n$ and $Y := \mathbb{R}^m$. Note that when (A1) and (A2') hold we have that $P(x)$ is a nonempty closed set with $[0,1]P(x) = P(x)$ for every $x \in \mathrm{Pr}_X(T)$ and $L(y)$ is a nonempty closed set with $[1,\infty[\,L(y) = L(y)$ for every $y \in \mathrm{Pr}_Y(T)$. Taking into account (4) and (5) the preceding remark proves that for $(x,y) \in X \times Y$ one has the equivalences

$$D_i(y,x) \geq 1 \iff x \in L(y) \iff (x,y) \in T \iff y \in P(x) \iff D_o(x,y) \leq 1$$

mentioned in [4, p. 353], where $D_i(y,x) := \nu_{L(y)}(x)$ and $D_o(x,y) := \mu_{P(x)}(y)$; D_i and D_o are the Shephard input and output distance functions. Besides D_i and D_o in production analysis other functions are considered, too (see [7] and the references therein):

$$\overrightarrow{D}_T(x,y;-g_x,g_y) := \psi_{T,(g_x,-g_y)}(x,y), \tag{41}$$

$$\overrightarrow{D}_o(x,y;g_y) := \psi_{P(x),-g_y}(y), \qquad \overrightarrow{D}_i(x,y;-g_x) := \psi_{L(y),g_x}(x), \tag{42}$$

$$I\overrightarrow{D}_o(w',y;g_y) := \psi_{\overline{P}(w'),-g_y}(y), \qquad I\overrightarrow{D}_i(x,p';-g_x) := \psi_{\overline{L}(p'),g_x}(x), \tag{43}$$

where $\overline{P}(w')$ and $\overline{L}(p')$ are defined as in (31), (32) for $w' \in X^* = \mathbb{R}^n$ and $p' \in Y^* = \mathbb{R}^m$ (corresponding to the sets $IP(w/C)$ and $IL(p/R)$); here $g_x \in \mathbb{R}^n_+$ and $g_y \in \mathbb{R}^m_+$ are nonnull in (42), (43) and $(g_x,g_y) \neq (0,0)$ in (41). Moreover,

$$\Pi(p,w) := \sigma_T(-w,p), \qquad R(x,p) := \sigma_{P(x)}(p), \qquad C(y,w) := \varsigma_{L(y)}(w),$$
$$IR(w',p) := \sigma_{\overline{P}(w')}(p), \qquad IC(p',w) := \varsigma_{\overline{L}(p')}(w).$$

Assuming that the axioms (A1), (A2'), (A3), (A4), (A5) hold and using also (3), from Propositions 8, 9, 6 and 7 we obtain that

$$\begin{aligned}
R(x,p) &= \sup_y \left\{ \frac{py}{D_o(x,y)} \;\middle|\; py > 0 \right\} && \forall x \in \mathrm{Pr}_X(T), \quad \forall p \in \mathbb{R}^m \setminus \{0\}, \\
\frac{1}{D_o(x,y)} &= \inf_p \left\{ \frac{R(x,p)}{py} \;\middle|\; py > 0 \right\} && \forall x \in \mathrm{Pr}_X(T), \quad \forall y \in \mathbb{R}^m \setminus \{0\}, \\
C(y,w) &= \inf_x \left\{ \frac{wx}{D_i(y,x)} \;\middle|\; wx > 0 \right\} && \forall y \in \mathrm{Pr}_Y(T), \quad \forall w \in \mathbb{R}^n_+ \setminus \{0\}, \\
\frac{1}{D_i(y,x)} &= \sup_w \left\{ \frac{C(y,w)}{wx} \;\middle|\; wx > 0 \right\} && \forall y \in \mathrm{Pr}_Y(T), \quad \forall x \in \mathbb{R}^n_+ \setminus \{0\},
\end{aligned}$$

respectively; in fact an attentive analysis shows that some hypotheses can be weakened. Such duality results are mentioned in [6, Rels. (10), (11)]; here one says "Shephard (Refs. 1, 9) proved that $C(y,w)$ is dual to $D_i(y,x)$ and that $R(x,p)$ is dual to $D_o(x,y)$. His duality theorems were stated as constrained optimization problems. Here, we follow Färe and Primont (Ref. 3) and state the dualities as unconstrained optimization problems".

In the sequel we assume that T satisfies the axioms (A1)–(A5). In this situation T_∞ is a subset of $\mathbb{R}^n_+ \times \mathbb{R}^m_+$ and satisfies the axioms (A1)–(A5), too; in particular $\mathbb{R}^n_+ \times \{0\} \subset T_\infty$, $\ker T_\infty = \mathbb{R}^n_+$ and $T_\infty(0) = \{0\}$. Moreover, $\mathrm{Pr}_X(T) = \mathbb{R}^n_+$ and $P(x)$ is a compact convex set containing 0 for every $x \in \mathrm{Pr}_X(T)$, and $L(y)$ is a nonempty closed convex set with $(L(y))_\infty = \mathbb{R}^m_+$ for every $y \in \mathrm{Pr}_Y(T)$. From axiom (A2) we obtain that

$$T = (T + \mathbb{R}_+ k) \cap (\mathbb{R}^n_+ \times \mathbb{R}^m_+) \qquad \forall k \in \mathbb{R}^n_+ \times (-\mathbb{R}^m_+);$$

hence (10) holds for $k \in \mathbb{R}^n_+ \times (-\mathbb{R}^m_+)$ (and $K := \mathbb{R}^n_+ \times \mathbb{R}^m_+$). Moreover

$$P(x) = (P(x) - \mathbb{R}^m_+) \cap \mathbb{R}^m_+ \quad \forall x \in X \qquad \text{and} \qquad L(y) = (L(y) + \mathbb{R}^n_+) \cap \mathbb{R}^n_+ \quad \forall y \in Y.$$

Because

$$-(g_x, -g_y) \notin T_\infty \,(\subset \mathbb{R}^n_+ \times \mathbb{R}^m_+), \qquad -g_x \notin (L(y))_\infty \,(\subset \mathbb{R}^n_+) \quad \text{and} \quad g_y \notin (P(x))_\infty \,(= \{0\}), \quad (44)$$

the functions $\overrightarrow{D}_T$, $\overrightarrow{D}_o$, $\overrightarrow{D}_i$ do not take the value $+\infty$; this is also true for $I\overrightarrow{D}_i$ (because $-g_x \notin (\overline{L}(p'))_\infty$ $(\subset \mathbb{R}^n_+)$) and for $I\overrightarrow{D}_o$ when $w' \in \mathbb{R}^n_{++} := \operatorname{int} \mathbb{R}^n_+$, because $\overline{P}(w')$ is compact in this case. Indeed, by Lemma 1 $\overline{P}(w')$ is closed. Assume that $\overline{P}(w')$ is not bounded. This means that there exists a sequence $((x_n, y_n))_{n \geq 1} \subset T$ with $\|y_n\| \to \infty$ and $\langle x_n, w' \rangle \leq 1$ for every $n \geq 1$. We may assume that $\|(x_n, y_n)\|^{-1}(x_n, y_n) \to (u,v) \in T_\infty \subset \mathbb{R}^n_+ \times \mathbb{R}^m_+$. Because $(0,0) \in T$, we get $(su, sv) \in T$ for $s \in \mathbb{R}_+$. If $u = 0$, then $(0,v) \in T$ and so $v = 0$ by (A3), contradicting $\|(u,v)\| = 1$. Hence $u \neq 0$. It follows that $\langle u, w' \rangle \leq 0$, contradicting the fact that $u \in \mathbb{R}^n_+ \setminus \{0\}$ and $w' \in \mathbb{R}^n_{++}$. Hence $\overline{P}(w')$ is bounded.

Note that for $w \notin \mathbb{R}^n_+$, taking into account (A2), we have that $\Pi(p,w) = \infty$, $C(y,w) = -\infty$ (if $L(y) \neq \emptyset$), $IC(p', w) = -\infty$ (if $\overline{L}(p') \neq \emptyset$); hence, in those results involving these quantities we may take $w \in \mathbb{R}^n_+$. Using Proposition 1 we obtain that

$$(x,y) \in T \iff \left[(x,y) \in \mathbb{R}^n_+ \times \mathbb{R}^m_+ \text{ and } \overrightarrow{D}_T(x,y;-g_x,g_y) \geq 0 \right]$$

and

$$y \in P(x) \iff \left[y \in \mathbb{R}^m_+ \text{ and } \overrightarrow{D}_o(x,y;g_y) \geq 0 \right];$$

taking into account (8) we see that [7, (2)] and [7, (5)]) do not hold.

Applying Proposition 10 we get [7, (16)] (and [4, (16)]; this is obtained under a differentiability assumption on solutions) for (those pairs (p,w) with) $pg_y + wg_x > 0$, [7, (18), (33)] for $wg_x > 0$, and [7, (20), (35)] for $pg_y > 0$. On the other hand, taking into account (44) and using Proposition 11 we get [7, (17), (19), (21)] in which the infimum is taken with respect to (w.r.t.) those p, w with $pg_y + wg_x > 0$ (or equivalently $pg_y + wg_x = 1$), w.r.t. w with $wg_x > 0$ and w.r.t. p with $pg_y > 0$, respectively. Moreover, by Lemma 1 we have that $\overline{P}(w')$ is a closed convex set containing 0 for every $w' \in \mathbb{R}^n_{++} = (\ker T_\infty)^\#$ and $\overline{L}(p')$ is a closed convex set for every $p' \in Y^* = \mathbb{R}^m = -(T_\infty(0))^\#$. Using again Proposition 11 we get [7, (34)] in which the supremum is taken w.r.t. w with $wg_x > 0$ for every $p' \in \mathbb{R}^m$ and [7, (36)] in which the infimum is taken w.r.t. p with $pg_y > 0$ for every $w' \in \mathbb{R}^n_{++}$.

We have that [7, (26)] and [7, (31)] follow immediately from (33) (if our interpretation, used throughout this section, that C and R are positive real numbers, is correct). Moreover, using Proposition 13 for $F = \mathbb{R}^m_+$ and $l = -g_y$ we get [7, (27)] for every $x \in \ker T_\infty = \mathbb{R}^n_+$, while using Proposition 12 for $k = g_x$ we get [7, (32)] for every $y \in Y$.

Recently one considered technologies in which the output space Y is a functions space. For example in [3] one considers the technology $\widetilde{T} \subset \mathbb{R}^n_+ \times L^2(\Omega, \mathbb{P}, \mathbb{R}^m_+)$ where $\mathbb{P}$ is a probability measure. Note that the duality results established in [3] can be deduced from the duality results from Section 4.

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References

[1] Agrell, P. J., Bogetoft, P., Brock, M. and Tind, J., (2005). Efficiency evaluation with convex pairs, *Adv. Model. Optim.*, **7**, 211–237.

[2] Artzner, P., Delbaen, F., Eber, J.-M. and Heath, D., (1999). Coherent measures of risk, *Math. Finance*, **9**, 203–228.

[3] Briec, W. and Cavaignac, L., (2009). An extension of the multi-output state-contingent production model, *Econom. Theory*, **39**, 43–64.

[4] Chambers, R. G., Chung, Y. and Färe, R., (1998). Profit, directional distance functions, and Nerlovian efficiency, *J. Optim. Theory Appl.*, **98**, 351–364.

[5] Cherchye, L., Kuosmanen, T. and Post, T., (2000). Why convexify? An assessment of convexity axioms in DEA, working paper, Helsinki School of Economics.

[6] Färe, R. and Grosskopf, S., (2000). On separability of the profit function, *J. Optim. Theory Appl.*, **105**, 609–620.

[7] Färe, R. and Primont, D., (2006). Directional duality theory, *Econom. Theory*, **29**, 239–247.

[8] Gerstewitz (Tammer), Chr. and Iwanow, E., (1985). Dualität für nichtkonvexe Vektoroptimierungsprobleme, *Wissenschaftliche Zeitschrift der Technischen Hochschule Ilmenau*, **31**, 2, 61–81.

[9] Göpfert, A., Tammer, C., Riahi, H. and Zălinescu, C., (2003). *Variational methods in partially ordered spaces*, CMS Books in Mathematics/Ouvrages de Mathématiques de la SMC 17, Springer, New York.

[10] Hamel, A. H., (2007). Translative sets and functions and its applications to risk measure theory and nonlinear separation, Preprint, University Halle–Wittenberg.

[11] Luenberger, D. G., (1992). New optimality principles for economic efficiency and equilibrium, *J. Optim. Theory Appl.*, **75**, 221–264.

[12] PENOT, J.-P. AND ZĂLINESCU, C., (2000). Harmonic sum and duality, *J. Convex Anal.*, **7** , 95–114.

[13] TAMMER, C. AND ZĂLINESCU, C., Lipschitz properties of the scalarization function and applications, *Optimization*, DOI: 10.1080/02331930801951033.

[14] ZĂLINESCU, C., (2002). *Convex Analysis in General Vector Spaces*, World Scientific, Singapore.

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