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On the ultradistributions of Beurling type

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Abstract. Let Ω be a nonempty open set of the k-dimensional euclidean space $\mathbb{R}^k$. In this paper, we show that if S is an ultradistribution in Ω, belonging to a class of Beurling type stable under differential operators, then S can be represented in the form $\sum_{\alpha\in\mathbb{N}_0^k} D^\alpha f_\alpha$, where f_α is a complex function defined in Ω which is Lebesgue measurable and essentially bounded in each compact subset of Ω. Other structure results on certain ultradistributions are obtained, too.

Ultradistribuciones de tipo Beurling

Resumen. Sea Ω un conjunto abierto no vacío del espacio euclídeo . En este artículo se demuestra que si S es una ultradistribución en Ω, perteneciente a una clase de tipo Beurling que sea estable frente a operadores diferenciales, entonces S se puede representar en la forma $\sum_{\alpha\in\mathbb{N}_0^k} D^\alpha f_\alpha$, donde f_α es una función compleja definida en Ω que es Lebesgue medible y esencialmente acotada en cada subconjunto compacto de Ω. También se obtienen otros resultados de estructura de ciertas ultradistribuciones.

1 Introduction and notation

Throughout this paper all linear spaces are assumed to be defined over the field $\mathbb{C}$ of complex numbers. We write $\mathbb{N}$ for the set of positive integers and by $\mathbb{N}_0$ we mean the set of nonnegative integers. If E is a locally convex space, E' will be its topological dual and $\langle\cdot\,,\cdot\rangle$ will denote the standard duality between E and E'. Given a Banach space X, $B(X)$ denotes its closed unit ball and X^* is the Banach space conjugate of X. Given a positive integer k, if $\alpha := (\alpha_1, \alpha_2, \dots, \alpha_k)$ is a multiindex of order k, i.e., an element of $\mathbb{N}_0^k$, we put $|\alpha|$ for its length, that is, $|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_k$, and $\alpha! := \alpha_1!\alpha_2!\cdots\alpha_k!$.

Given a complex function f defined in the points $x = (x_1, x_2, \dots, x_k)$ of an open subset O of the k-dimensional euclidean space $\mathbb{R}^k$, and being infinitely differentiable, we write

$$D^\alpha f(x) := \frac{\partial^{|\alpha|} f(x)}{\partial x_1^{\alpha_1}\partial x_2^{\alpha_2}\dots\partial x_k^{\alpha_k}}, \qquad x\in O, \quad \alpha\in\mathbb{N}_0^k.$$

We consider a sequence M_0, M_1, ..., M_n, ... of positive numbers satisfying the following conditions:

1. $M_0 = 1$.

2. Logarithmic convexity:
$$M_n^2 \le M_{n-1}M_{n+1}, \qquad n\in\mathbb{N}.$$

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3. Non-quasi-analyticity:

$$\sum_{n=1}^{\infty} \frac{M_{n-1}}{M_n} < \infty.$$

Let us take a nonempty open set Ω in $\mathbb{R}^k$. A complex function f, defined and infinitely differentiable in Ω, is said to be *ultradifferentiable of class* (M_n) whenever, given $h > 0$ and a compact subset K of Ω, there is $C > 0$ such that

$$|D^\alpha f(x)| \leq C h^{|\alpha|} M_{|\alpha|}, \qquad x \in K, \quad \alpha \in \mathbb{N}_0^k.$$

We put $\mathcal{E}^{(M_n)}(\Omega)$ to denote the linear space over $\mathbb{C}$ formed by all the ultradifferentiable functions of class (M_n) defined in Ω, with the ordinary topology, [2]. By $\mathcal{D}^{(M_n)}(\Omega)$ we denote the linear subspace of $\mathcal{E}^{(M_n)}(\Omega)$ formed by those functions which have compact support.

We now choose a fundamental sequence of compact subsets of Ω:

$$K_1 \subset K_2 \subset \cdots \subset K_m \cdots$$

If K is an arbitrary compact subset of Ω, we use $\mathcal{D}^{(M_n)}(K)$ to denote the subspace of $\mathcal{E}^{(M_n)}(\Omega)$ formed by those functions which have their support in K. We then have that

$$\mathcal{D}^{(M_n)}(\Omega) = \bigcup_{m=1}^{\infty} \mathcal{D}^{(M_n)}(K_m).$$

We consider $\mathcal{D}^{(M_n)}(\Omega)$ as the inductive limit of the sequence $(\mathcal{D}^{(M_n)}(K_m))$ of Fréchet spaces. The elements of the topological dual $\mathcal{D}^{(M_n)'}(\Omega)$ of $\mathcal{D}^{(M_n)}(\Omega)$ are called *ultradistributions of Beurling type* in Ω. We assume that $\mathcal{D}^{(M_n)'}(\Omega)$ has its strong topology.

By $\mathcal{K}(\Omega)$ we mean the linear space over $\mathbb{C}$ of the complex functions defined in Ω which are continuous and have compact support. If K is any compact subset of Ω, $\mathcal{K}(K)$ is the subspace of $\mathcal{K}(\Omega)$ formed by the functions with support contained in K. If f is in $\mathcal{K}(K)$, we put

$$|f|_\infty := \sup_{x \in \Omega} |f(x)|,$$

and assume that $\mathcal{K}(K)$ is endowed with the norm $|\cdot|_\infty$.

We consider $\mathcal{K}(\Omega)$ as the inductive limit of the sequence $(\mathcal{K}(K_m))$ of Banach spaces. A Radon measure in Ω is an element of the topological dual $\mathcal{K}'(\Omega)$ of $\mathcal{K}(\Omega)$. Given a Radon measure u in Ω and a compact subset K of Ω, we put $\|u\|(K)$ for the norm of the restriction of u to the Banach space $\mathcal{K}(K)$.

In [2, p. 76], a structure theorem for ultradistributions of Beurling type in Ω is given. It can be stated as follows:

Result a) *If S is an element of $\mathcal{D}^{(M_n)'}(\Omega)$ and G is an open subset of Ω which is relatively compact, for each $\alpha \in \mathbb{N}_0^k$, we may find an element v_α in the conjugate of the Banach space $\mathcal{K}(\overline{G})$, whose norm we represent by $\|v_\alpha\|$, such that, for some $h > 0$,*

$$\sup_{\alpha \in \mathbb{N}_0^k} h^{|\alpha|} M_{|\alpha|} \|v_\alpha\| < \infty$$

and

$$S_{|G} = \sum_{\alpha \in \mathbb{N}_0^k} D^\alpha v_\alpha.$$

The above result is of local character, for the elements v_α, $\alpha \in \mathbb{N}_0^k$, depend on G. In [4], we give a structure theorem of global character for the ultradistributions of Beurling type in Ω. This theorem contains result a) as a particular case and can be stated as follows:

Result b) *If S is an element of $\mathcal{D}^{(M_n)'}(\Omega)$, then there is a family $(\, u_\alpha : \alpha \in \mathbb{N}_0^k \,)$ of Radon measures in Ω such that*

$$\langle \varphi, S \rangle = \sum_{\alpha \in \mathbb{N}_0^k} \langle D^\alpha \varphi, u_\alpha \rangle, \qquad \varphi \in \mathcal{D}^{(M_n)}(\Omega),$$

where the series converges absolutely and uniformly on every bounded subset of $\mathcal{D}^{(M_n)}(\Omega)$. Also, given a compact subset K of Ω, there is $h > 0$ such that

$$\sup_{\alpha \in \mathbb{N}_0^k} h^{|\alpha|} M_{|\alpha|} \|u_\alpha\|(K) < \infty.$$

We now put $\mathcal{L}^\infty_{\mathrm{loc}}(\Omega)$ for the linear space over $\mathbb{C}$ formed by the complex functions defined in Ω, which are Lebesgue-measurable and essentially bounded in every compact subset of Ω. The elements of this space are considered as Radon measures on Ω in the usual way. If f is in $\mathcal{L}^\infty_{\mathrm{loc}}(\Omega)$ and K is a compact subset of Ω, we write $|f|_{K,\infty}$ for the essential supremum of $|f|$ in K.

We say that the sequence $M_0, M_1, \ldots, M_n, \ldots$ satisfies the stability condition for differential operators provided there are $A > 0$ and $h > 0$ such that

$$M_{n+1} \leq A h^n M_n, \qquad n \in \mathbb{N}_0. \tag{1}$$

In this paper, we give a structure theorem for ultradistributions of Beurling type in Ω, which contains the following result as a particular case:

Result c) *If $M_0, M_1, \ldots, M_n, \ldots$ satisfies condition (1) and S is an element of $\mathcal{D}^{(M_n)'}(\Omega)$, then there is a family $(\, f_\alpha : \alpha \in \mathbb{N}_0^k \,)$ of elements of $\mathcal{L}^\infty_{\mathrm{loc}}(\Omega)$ such that, given an arbitrary compact subset K of Ω, there is $h > 0$ with*

$$\sup_{\alpha \in \mathbb{N}_0^k} h^{|\alpha|} M_{|\alpha|} |f_\alpha|_{K,\infty} < \infty$$

and

$$S = \sum_{\alpha \in \mathbb{N}_0^k} D^\alpha f_\alpha.$$

2 Basic constructions

Let X be a Banach space. We put $\|\cdot\|$ for the norm of X and also for the norm of X^*. Given $r \in \mathbb{N}$ and $\alpha \in \mathbb{N}_0^k$, we put, for each $x \in X$,

$$|x|_{r,\alpha} := \frac{r^{|\alpha|} \|x\|}{M_{|\alpha|}}.$$

We denote by $X_{r,\alpha}$ the linear space X provided with the norm $|\cdot|_{r,\alpha}$. By $X^*_{r,\alpha}$ we mean the Banach space conjugate of $X_{r,\alpha}$ with $|\cdot|_{r,\alpha}$ as its norm. Clearly, if u is in X^*, then

$$|u|_{r,\alpha} = \frac{M_{|\alpha|}}{r^{|\alpha|}} \|u\|.$$

We put Z_r for the linear space over $\mathbb{C}$ of the families $(\, x_\alpha : \alpha \in \mathbb{N}_0^k \,)$ of elements of X, which we shall just denote by (x_α), such that

$$\|(x_\alpha)\|_r := \sup_{\alpha \in \mathbb{N}_0^k} \frac{r^{|\alpha|} \|x_\alpha\|}{M_{|\alpha|}} < \infty.$$

We assume that Z_r is provided with the norm $\|\cdot\|_r$ It then follows that $Z_r \supset Z_{r+1}$ and that the canonical injection from Z_{r+1} into Z_r is continuous.

We write Z to denote the Fréchet space given by the projective limit of the sequence (Z_r) of Banach spaces. We assume Z' endowed with the strong topology.

Given β in $\mathbb{N}_0^k$, we put Z^β for the subspace of Z whose elements (x_α) satisfy that $x_\alpha = 0$ when α is distinct from β. We then have that Z^β is topologically isomorphic to X and, considering Z^β as a subspace of Z_r, then it is isometric to $X_{r,\beta}$.

If u is an arbitrary element of Z' and $r \in \mathbb{N}$, we put

$$\|u\|_{(r)} := \sup \left\{ \left|\langle (x_\alpha), u\rangle\right| : (x_\alpha) \in B(Z_r) \cap Z \right\}.$$

For each $u \in Z'$ and each $\beta \in \mathbb{N}_0^k$, we identify, in the usual manner, the restriction of u to Z^β with an element u_β of X^*.

If (x_α) is an element of Z and β is in $\mathbb{N}_0^k$, we write

$$x_\alpha^\beta := \begin{cases} x_\beta, & \text{if } \alpha = \beta, \\ 0, & \text{if } \alpha \neq \beta. \end{cases}$$

Clearly, (x_α^β) belongs to Z and, for each $r \in \mathbb{N}$,

$$\|(x_\alpha^\beta)\|_r \leq \|(x_\alpha)\|_r.$$

The next proposition unifies Proposition 1 and the Note in [4].

Proposition 1. *If M is a bounded subset of Z', then there is r in $\mathbb{N}$ such that*

$$\sup_{\substack{\alpha \in \mathbb{N}_0^k \\ u \in M}} r^{-|\alpha|} M_{|\alpha|} \|u_\alpha\| \leq 1$$

and

$$\langle (x_\alpha), u\rangle = \sum_{\alpha \in \mathbb{N}_0^k} \langle x_\alpha, u_\alpha\rangle, \qquad u \in M, \quad (x_\alpha) \in Z,$$

where the series converges absolutely and uniformly when u varies in M and (x_α) varies in any given bounded subset of Z.

PROOF. If M° is the polar set of M in Z, we find $r \in \mathbb{N}$ such that $B(Z_r) \cap Z$ is contained in M°. Then, for each $u \in M$, we have, if we fix $\beta \in \mathbb{N}_0^k$,

$$\begin{aligned} 1 \geq \|u\|_{(r)} &= \sup \left\{ \left|\langle (x_\alpha), u\rangle\right| : (x_\alpha) \in B(Z_r) \cap Z \right\} \\ &\geq \sup \left\{ \left|\langle (x_\alpha^\beta), u\rangle\right| : (x_\alpha) \in B(Z_r) \cap Z \right\} \\ &= \sup \left\{ \left|\langle (x_\beta), u_\beta\rangle\right| : |x_\beta|_{r,\beta} \leq 1 \right\} = |u_\beta|_{r,\beta} \\ &= \frac{M_{|\beta|}}{r^{|\beta|}} \|u_\beta\| \end{aligned}$$

from where we deduce

$$\sup_{\substack{\alpha \in \mathbb{N}_0^k \\ u \in M}} r^{-|\alpha|} M_{|\alpha|} \|u_\alpha\| \leq 1.$$

We take (x_α) in Z and we see that $\left((x_\alpha^\beta) : \beta \in \mathbb{N}_0^k\right)$ is summable in Z to (x_α). Let s, q be in $\mathbb{N}$. We then have

$$\begin{aligned} \Big\|(x_\alpha) - \sum_{|\beta| \leq q} (x_\alpha^\beta)\Big\|_s &= \sup_{|\alpha| > q} \frac{s^{|\alpha|} \|x_\alpha\|}{M_{|\alpha|}} \\ &= \sup_{|\alpha| > q} \frac{(2s)^{|\alpha|} \|x_\alpha\|}{2^{|\alpha|} M_{|\alpha|}} \\ &\leq \frac{1}{2^q} \sup_{\alpha \in \mathbb{N}_0^k} \frac{(2s)^{|\alpha|} \|x_\alpha\|}{M_{|\alpha|}} \\ &= \frac{1}{2^q} \|(x_\alpha)\|_{2s} \end{aligned}$$

and the conclusion follows. From

$$(x_\alpha) = \sum_{\beta\in\mathbb{N}_0^k} (x_\alpha^\beta)$$

in Z, we obtain

$$\big\langle (x_\alpha), u\big\rangle = \sum_{\beta\in\mathbb{N}_0^k} \big\langle (x_\alpha^\beta), u\big\rangle = \sum_{\beta\in\mathbb{N}_0^k} \langle x_\beta, u_\beta\rangle, \qquad u \in Z'.$$

We consider now a bounded subset B of Z. We find $b > 0$ such that $B \subset bB(Z_{2kr})$. We choose arbitrary elements $(x_\alpha) \in B$ and $u \in M$. We fix $\beta \in \mathbb{N}_0^k$. Then

$$\begin{aligned}
\big|\langle x_\beta, u_\beta\rangle\big| &\le \|x_\beta\| \cdot \|u_\beta\| \\
&= \frac{(2kr)^{|\beta|}\|x_\beta\|}{M_{|\beta|}} \cdot \frac{M_{|\beta|}\|u_\beta\|}{(2kr)^{|\beta|}} \\
&\le \frac{1}{(2k)^{|\beta|}} \|(x_\alpha)\|_{2kr} \sup_{\substack{\alpha\in\mathbb{N}_0^k \\ u\in M}} r^{-|\alpha|} M_{|\alpha|}\|u_\alpha\| \\
&\le \frac{1}{(2k)^{|\beta|}} b
\end{aligned}$$

and, since

$$\sum_{\beta\in\mathbb{N}_0^k} \frac{1}{(2k)^{|\beta|}} = 2,$$

the conclusion follows. ■

The following proposition may be found in [4].

Proposition 2. *Let* $\{\, v_\alpha : \alpha \in \mathbb{N}_0^k \,\}$ *a family of elements of* X^* *such that there is* $h > 0$ *with*

$$\sup_{\alpha\in\mathbb{N}_0^k} h^{|\alpha|} M_{|\alpha|}\|v_\alpha\| < \infty.$$

Then, there is a unique element $u \in Z'$ *such that* $u_\alpha = v_\alpha$, $\alpha \in \mathbb{N}_0^k$.

3 The space $\mathcal{D}_{(L^p)}^{(M_n)}(\Omega)$

We put $L^p(\Omega$ and $\mathcal{L}^p(\Omega)$, $1 \le p \le \infty$, for the classical Lebesgue spaces. If $f \in \tilde{f} \in \mathcal{L}^p(\Omega)$, $1 \le p < \infty$, we write

$$\|f\|_p = \|\tilde{f}\|_p = \left(\int_\Omega |f|^p \,\mathrm{d}x\right)^{1/p},$$

and, if $f \in \tilde{f} \in L^\infty(\Omega)$, then

$$\|f\|_\infty = \|\tilde{f}\|_\infty \ = \mathrm{supess}\{\, |f(x)| : x \in \Omega \,\}.$$

$\mathcal{D}_{L^p}(\mathbb{R}^k)$, $1 \le p < \infty$, is the classical L. Schwartz's space, [3, p. 199]. We put $\mathcal{B}_{L^p}(\Omega)$ for the linear space over $\mathbb{C}$ of the complex functions f defined in Ω which are infinitely differentiable and such that $D^\alpha f$ is in $\mathcal{L}^p(\Omega)$, $\alpha \in \mathbb{N}_0^k$. We assume that $\mathcal{B}_{L^p}(\Omega)$ is endowed with the metrizable locally convex topology such that a sequence (f_n) in $\mathcal{B}_{L^p}(\Omega)$ converges to the origin if and only if $(\|D^\alpha f_n\|_p)$ converges to zero for each $\alpha \in \mathbb{N}_0^k$. We then have that $\mathcal{B}_{L^p}(\Omega)$ is a Fréchet space. Clearly, $\mathcal{B}_{L^p}(\mathbb{R}^k)$ coincides with $\mathcal{D}_{L^p}(\mathbb{R}^k)$.

Given $r \in \mathbb{N}$ and $1 \leq p < \infty$, we put $\mathcal{B}_{L^p}^{(M_n),1/r}(\Omega)$ for the linear space over $\mathbb{C}$ of the functions $f \in \mathcal{B}_{L^p}(\Omega)$ which satisfy:

$$\|f\|_{p,1/r} := \sup_{\alpha \in \mathbb{N}_0^k} \frac{r^{|\alpha|}\|D^\alpha f\|_p}{M_{|\alpha|}} < \infty.$$

We assume $\mathcal{B}_{L^p}^{(M_n),1/r}(\Omega)$ provided with the norm $\|\cdot\|_{p,1/r}$. Given a Cauchy sequence (f_m) in $\mathcal{B}_{L^p}^{(M_n),1/r}(\Omega)$, it is immediate that (f_m) is a Cauchy sequence in $\mathcal{B}_{L^p}(\Omega)$ and hence it converges in this space to a function f. Given $\varepsilon > 0$, there is a positive integer m_0 such that

$$\|f_m - f_s\|_{p,1/r} < \varepsilon, \qquad m, s \geq m_0.$$

Then, for those values of m and s, and for each $\alpha \in \mathbb{N}_0^k$, we have that

$$\frac{r^{|\alpha|}\|D^\alpha f_m - D^\alpha f_s\|_p}{M_{|\alpha|}} < \varepsilon$$

and therefore, for $m \geq m_0$,

$$\frac{r^{|\alpha|}\|D^\alpha f_m - D^\alpha f\|_p}{M_{|\alpha|}} \leq \varepsilon,$$

from where we deduce that f belongs to $\mathcal{B}_{L^p}^{(M_n),1/r}(\Omega)$ and that $\|f_m - f\|_{p,1/r} \leq \varepsilon, m \geq m_0$. Consequently, $\mathcal{B}_{L^p}^{(M_n),1/r}(\Omega)$ is a Banach space.

It is plain that $\mathcal{B}_{L^p}^{(M_n),\frac{1}{r+1}}(\Omega)$ is contained in $\mathcal{B}_{L^p}^{(M_n),1/r}(\Omega)$ and also that the canonical injection from $\mathcal{B}_{L^p}^{(M_n),\frac{1}{r+1}}(\Omega)$ into $\mathcal{B}_{L^p}^{(M_n),1/r}(\Omega)$ is continuous. We denote by $\mathcal{B}_{L^p}^{(M_n)}(\Omega)$ the projective limit of the sequence $(\mathcal{B}_{L^p}^{(M_n),1/r}(\Omega))$ of Banach spaces. We assume that the topological dual $\mathcal{B}_{L^p}^{(M_n)\prime}(\Omega)$ of $\mathcal{B}_{L^p}^{(M_n)}(\Omega)$ is endowed with the strong topology.

In this section we substitute the Banach space X of the previous section by $L^p(\Omega)$. Then, every element of Z_r is a family $(\,\tilde{f}_\alpha : \alpha \in \mathbb{N}_0^k\,)$ of elements of $L^p(\Omega)$ such that

$$\|(\tilde{f}_\alpha)\|_r = \sup_{\alpha \in \mathbb{N}_0^k} \frac{r^{|\alpha|}\|\tilde{f}_\alpha\|_p}{M_{|\alpha|}} < \infty.$$

If f belongs to $\mathcal{B}_{L^p}^{(M_n)}(\Omega)$, we put $\tilde{D}^\alpha f$ for the element of $L^p(\Omega)$ to which $D^\alpha f$ belongs, $\alpha \in \mathbb{N}_0^k$. By V_r we represent the linear subspace of Z_r formed by those families $(\,\tilde{D}^\alpha f : \alpha \in \mathbb{N}_0^k\,)$ such that $f \in \mathcal{B}_{L^p}^{(M_n),1/r}(\Omega)$. Let

$$\Phi_r \colon \mathcal{B}_{L^p}^{(M_n),p,1/r}(\Omega) \longrightarrow V_r$$

be such that

$$\Phi_r(f) = (\tilde{D}^\alpha f), \qquad f \in \mathcal{B}_{L^p}^{(M_n),1/r}(\Omega).$$

Then, Φ_r is a linear onto isometry. We put $V := \cap\{\,V_r : r \in \mathbb{N}\,\}$ considered as a subspace of Z. Let

$$\Phi \colon \mathcal{B}_{L^p}^{(M_n)}(\Omega) \longrightarrow V$$

be such that

$$\Phi(f) = (\tilde{D}^\alpha f), \qquad f \in \mathcal{B}_{L^p}^{(M_n)}(\Omega).$$

Clearly, Φ is a topological isomorphism from $\mathcal{B}_{L^p}^{(M_n)}(\Omega)$ onto V.

In the following, we fix $1 \leq p < \infty$. By q we denote the conjugate of p, i.e., $q = \infty$ when $p = 1$, and, if $p > 1$ then $\frac{1}{p} + \frac{1}{q} = 1$.

Proposition 3. *For each j in a set J, let $(\, f_{\alpha,j} : \alpha \in \mathbb{N}_0^k\,)$ be a family of elements of $\mathcal{L}^q(\Omega)$ such that there is $h > 0$ with*

$$\sup_{\substack{\alpha\in\mathbb{N}_0^k\\ j\in J}} h^{|\alpha|} M_{|\alpha|} \|f_{\alpha,j}\|_q < \infty$$

Then, there is a bounded subset $\{\, S_j : j \in J\,\}$ of $\mathcal{B}_{L^p}^{(M_n)'}(\Omega)$ such that

$$\langle \varphi, S_j\rangle = \sum_{\alpha\in\mathbb{N}_0^k} \int_\Omega D^\alpha \varphi \cdot f_{\alpha,j}\,\mathrm{d}x, \qquad j\in J, \quad \varphi \in \mathcal{B}_{L^p}^{(M_n)}(\Omega),$$

where the series converges absolutely and uniformly when j varies in J and φ varies in any given bounded subset of $\mathcal{B}_{L^p}^{(M_n)}(\Omega)$.

PROOF. We identify in the usual fashion $f_{\alpha,j}$ with a continuous linear functional on $L^p(\Omega)$ whose norm is $\|f_{\alpha,j}\|_q$. We apply Proposition 2 to obtain, for each j in J, a unique element u_j in Z' whose restriction to Z^α coincides with $f_{\alpha,j}$, $\alpha \in \mathbb{N}_0^k$. If we fix j in J, we apply Proposition 1 for $M = \{u_j\}$ and so obtain that

$$\big\langle (\tilde{g}_\alpha), u_j \big\rangle = \sum_{\alpha\in\mathbb{N}_0^k} \int_\Omega g_\alpha \cdot f_{\alpha,j}\,\mathrm{d}x, \qquad g_\alpha \in \tilde{g}_\alpha, \quad (\tilde{g}_\alpha) \in Z. \tag{2}$$

We find $r \in \mathbb{N}$ such that $1/r < h$. We fix $(\tilde{g}_\alpha)$ in Z. We then have

$$\begin{aligned}
\big|\big\langle (\tilde{g}_\alpha), u_j \big\rangle\big| &\le \sum_{\alpha\in\mathbb{N}_0^k} \int_\Omega |g_\alpha| \cdot |f_{\alpha,j}|\,\mathrm{d}x \\
&\le \sum_{\alpha\in\mathbb{N}_0^k} \|g_\alpha\|_p \cdot \|f_{\alpha,j}\|_q \\
&\le \sum_{\alpha\in\mathbb{N}_0^k} \frac{(2kr)^{|\alpha|}\|g_\alpha\|_p}{M_{|\alpha|}} \frac{1}{(2k)^{|\alpha|}} r^{-|\alpha|} M_{|\alpha|} \|f_{\alpha,j}\|_q \\
&\le \sum_{\alpha\in\mathbb{N}_0^k} \|(\tilde{g}_\alpha)\|_{2kr} \cdot \frac{1}{(2k)^{|\alpha|}} \sup_{\substack{\gamma\in\mathbb{N}_0^k\\ j\in J}} h^{|\gamma|} M_{|\gamma|} \|f_{\gamma,j}\|_q \\
&= 2\|(\tilde{g}_\alpha)\|_{2kr} \cdot \sup_{\substack{\gamma\in\mathbb{N}_0^k\\ j\in J}} h^{|\gamma|} M_{|\gamma|} \|f_{\gamma,j}\|_q
\end{aligned}$$

and thus

$$\sup_{j\in J} |\langle (\tilde{g}_\alpha), u_j\rangle < \infty.$$

Applying now the Theorem of Banach-Steinhaus, we obtain that $\{\, u_j : j \in J\,\}$ is a bounded subset of Z'. Proposition 1 yields that, for $M = \{\, u_j : j \in J\,\}$, the series in (2) converges absolutely and uniformly when j varies in J and $(\tilde{g}_\alpha)$ varies in any given bounded subset of Z.

We put w for the mapping Φ considered from $\mathcal{B}_{L^p}^{(M_n)}(\Omega)$ into Z. Let ${}^t w$ be the transpose of w. We write

$$S_j :=^t w(u_j), \qquad j \in J.$$

Then $\{\, S_j : j \in J\,\}$ is a bounded subset of $\mathcal{B}_{L^p}^{(M_n)'}(\Omega)$. On the other hand, for each $\varphi \in \mathcal{B}_{L^p}^{(M_n)}(\Omega)$, we have

$$\big\langle (\tilde{D}^\alpha \varphi), u_j \big\rangle = \langle w(\varphi), u_j\rangle = \langle \varphi, ^t w(u_j)\rangle = \langle \varphi, S_j\rangle.$$

Consequently, for each φ of $\mathcal{B}_{L^p}^{(M_n)}(\Omega)$ and each $j \in J$, it follows

$$\langle \varphi, S_j \rangle = \big\langle (\tilde{D}^\alpha \varphi), u_j \big\rangle = \sum_{\alpha \in \mathbb{N}_0^k} D^\alpha \varphi \cdot f_{\alpha,j} \, \mathrm{d}x.$$

Finally, when φ varies in a bounded subset of $\mathcal{B}_{L^p}^{(M_n)}(\Omega)$, $(\tilde{D}^\alpha \varphi)$ varies in a bounded subset of Z. The conclusion is now obvious. ■

Proposition 4. *If $\{ S_j : j \in J \}$ is a bounded subset of $\mathcal{B}_{L^p}^{(M_n)'}(\Omega)$, there are $h > 0$ and, for each $j \in J$, a family $(f_{\alpha,j} : \alpha \in \mathbb{N}_0^k)$ of elements of $\mathcal{L}^q(\Omega)$ such that*

$$\sup_{\substack{\alpha \in \mathbb{N}_0^k \\ j \in J}} h^{|\alpha|} M_{|\alpha|} \| f_{\alpha,j} \|_q < \infty$$

and

$$\langle \varphi, S_j \rangle = \sum_{\alpha \in \mathbb{N}_0^k} \int_\Omega D^\alpha \varphi \cdot f_{\alpha,j} \, \mathrm{d}x, \qquad j \in J, \quad \varphi \in \mathcal{B}_{L^p}^{(M_n)}(\Omega),$$

where the series converges absolutely and uniformly when j varies in J and φ varies in any given bounded subset of $\mathcal{B}_{L^p}^{(M_n)}(\Omega)$.

PROOF. We have

$$^t w \colon Z' \longrightarrow \mathcal{B}_{L^p}^{(M_n)'}(\Omega)$$

is onto. It is easy to verify that there is a bounded subset $\{ u_j : j \in J \}$ in Z' such that

$$^t w(u_j) = S_j, \qquad j \in J.$$

We put $f_{\alpha,j}$ for the element of $\mathcal{L}^q(\Omega)$ given by the restriction of u_j to X^α. We apply Proposition 1 for $M = \{ u_j : j \in J \}$ and so obtain $r \in \mathbb{N}$ such that

$$\sup_{\substack{\alpha \in \mathbb{N}_0^k \\ j \in J}} r^{-|\alpha|} M_{|\alpha|} \| f_{\alpha,j} \|_q < \infty$$

and

$$\big\langle (\tilde{D}^\alpha \varphi), u_j \big\rangle = \sum_{\alpha \in \mathbb{N}_0^k} \langle \tilde{D}^\alpha \varphi, \tilde{f}_{\alpha,j} \rangle = \sum_{\alpha \in \mathbb{N}_0^k} \int_\Omega D^\alpha \varphi \cdot f_{\alpha,j} \, \mathrm{d}x, \qquad j \in J, \quad \varphi \in \mathcal{B}_{L^p}^{(M_n)}(\Omega),$$

where the series converges absolutely and uniformly when j varies in J and φ varies in any given bounded subset of $\mathcal{B}_{L^p}^{(M_n)}(\Omega)$. Finally, for each $j \in J$, we have

$$\big\langle (\tilde{D}^\alpha \varphi), u_j \big\rangle = \big\langle w(\varphi), u_j \big\rangle = \big\langle \varphi, ^t w(u_j) \big\rangle = \langle \varphi, S_j \rangle$$

and the conclusion follows. ■

Given a compact subset K of Ω and $r \in \mathbb{N}$, we put $\mathcal{D}_{(L^p)}^{(M_n),1/r}(K)$ to denote the subspace of $\mathcal{B}_{L^p}^{(M_n),1/r}(\Omega)$ whose elements have their support in K. If (f_m) is a sequence in $\mathcal{D}_{(L)^p}^{(M_n),1/r}(K)$ which converges to f in $\mathcal{B}_{L^p}^{(M_n),1/r}(\Omega)$, there is a subsequence (f_{m_i}) of (f_m) which converges to f almost everywhere. Since $f_{m_i}(x) = 0$, $x \in \Omega \setminus K$, we have that f belongs to $\mathcal{D}_{(L^p)}^{(M_n),1/r}(K)$, from where we get that this space is a Banach space. We put $\mathcal{D}_{(L^p)}^{(M_n)}(K)$ for the projective limit of the sequence $(\mathcal{D}_{(L^p)}^{(M_n),1/r}(K))$ of Banach spaces. It is immediate that $\mathcal{D}^{(M_n)}_{(L^p)}(K)$ coincides with the subspace of $\mathcal{B}^{(M_n)}_{L^p}(\Omega)$ formed by the functions with support in K. We now write

$$\mathcal{D}^{(M_n)}_{(L^p)}(\Omega) := \bigcup_{r=1}^{\infty} \mathcal{D}^{(M_n)}_{(L^p)}(K_r)$$

and assume that $\mathcal{D}^{(M_n)}_{(L^p)}(\Omega)$ is the inductive limit of the sequence $\left(\mathcal{D}^{(M_n)}_{(L^p)}(K_r)\right)$ of Fréchet spaces. We also assume that the topological dual $\mathcal{D}^{(M_n)'}_{(L^p)}(\Omega)$ of $\mathcal{D}^{(M_n)}_{(L^p)}(\Omega)$ is endowed with the strong topology.

If $g \in \mathcal{L}^{p_1}(\mathbb{R}^k)$ and $l \in \mathcal{L}^{p_2}(\mathbb{R}^k)$, with $1 \le p_1,\, p_2 \le \infty$ and $1/p_1 + 1/p_2 \ge 1$, then the convolution of g and l exists almost everywhere. We extend this convolution to the whole of $\mathbb{R}^k$ by assigning the zero value for the points where it is not defined. Thus $g * l$ belongs to $\mathcal{L}^s(\mathbb{R}^k)$, where $\frac{1}{s} = \frac{1}{p_1} + \frac{1}{p_2} - 1$ and we then have

$$\|g * l\|_s \le \|g\|_{p_1} \cdot \|l\|_{p_2}. \tag{3}$$

This property will be used in the proof of the next result.

Proposition 5. *The linear space $\mathcal{D}^{(M_n)}(\Omega)$ is dense in $\mathcal{D}^{(M_n)}_{(L^p)}(\Omega)$.*

PROOF. We may assume that $\overset{\circ}{K}_1 \neq \emptyset$ and that $K_m \subset \overset{\circ}{K}_{m+1}$, $m = 1, 2, \ldots$. Given $\rho > 0$, we write $B(\rho)$ for the closed ball in $\mathbb{R}^k$ with center in the origin and radius ρ. We take f in $\mathcal{D}^{(M_n)}_{(L^p)}(\Omega)$. We find a positive integer m such that $f \in \mathcal{D}^{(M_n)}_{(L^p)}(K_m)$. We choose a sequence (ψ_i) in $\mathcal{D}^{(M_n)}(\mathbb{R}^k)$ satisfying:

(i) $\psi_i(x) \ge 0$, $x \in \mathbb{R}^k$.

(ii) $\int_{\mathbb{R}^k} \psi_i(x)\,\mathrm{d}x = 1$.

(iii) $\operatorname{supp}\psi_i \subset B(\rho_i)$, $\rho_1 > \rho_2 > \cdots > \rho_i > \cdots$,

$$\lim_i \rho_i = 0$$

and $K_m + B(\rho_1) \subset K_{m+1}$.

We extend f to $\mathbb{R}^k$ by putting $f(x) = 0$, $x \in \mathbb{R}^k \setminus \Omega$. We set $f_i := f * \psi_i$, $i \in \mathbb{N}$. We see next that (f_i) is a sequence in $\mathcal{D}^{(M_n)}(K_{m+1})$ which converges to f in $\mathcal{D}^{(M_n)}_{(L^p)}(K_{m+1})$. For each $\alpha \in \mathbb{N}_0^k$, we have

$$D^\alpha f_i(x) = \int_{\mathbb{R}^k} f(y)(D^\alpha \psi_i)(x-y)\,\mathrm{d}y, \qquad x \in \mathbb{R}^k,$$

and hence f_i is in $\mathcal{D}^{(M_n)}(K_{m+1})$.

Let us take $\varepsilon > 0$ and $r \in \mathbb{N}$. We find a positive integer s_0 such that

$$\left(\frac{r}{r+1}\right)^{s_0} \|f\|_{p,1/r+1} < \frac{\varepsilon}{4}.$$

Given $\alpha \in \mathbb{N}_0^k$, we have, for $x \in \mathbb{R}^k$,

$$\begin{aligned}|D^\alpha f_i(x) - D^\alpha f(x)| &\le \int_{\mathbb{R}^k} |(D^\alpha f)(x-y) - D^\alpha(x)| \cdot \psi_i(y)\,\mathrm{d}y \\ &\le \sup\left\{|(D^\alpha f)(x-y) - D^\alpha f(x)| \colon y \in B(\delta_i)\right\}\end{aligned}$$

and so, if μ is the Lebesgue measure in $\mathbb{R}^k$, we may find $i_0 \in \mathbb{N}$ such that

$$|D^\alpha f_i(x) - D^\alpha f(x)| < \frac{\varepsilon}{2r^{s_0}\mu(K_{m+1})}, \qquad i \ge i_0, \quad x \in \mathbb{R}^k, \quad |\alpha| \le s_0.$$

Then

$$\|D^\alpha f_i - D^\alpha f\|_p \le \frac{\varepsilon}{2r^{s_0}}, \qquad i \ge i_0, \quad |\alpha| \le s_0.$$

Applying (3) for $p_1 = p$, $p_2 = 1$, $g = D^\alpha f$ and $l = \psi_i$, we obtain

$$\|D^\alpha f_i\|_p = \|(D^\alpha f) * \psi_i\|_p \le \|D^\alpha f\|_p \cdot \|\psi_i\|_1 = \|D^\alpha f\|_p.$$

Therefore, for $i \ge i_0$, we have that

$$\begin{aligned}
\|f - f_i\|_{p,1/r} &= \sup_{\alpha\in\mathbb{N}_0^k} \frac{r^{|\alpha|}\|D^\alpha(f-f_i)\|_p}{M_{|\alpha|}} \\
&\le \sup_{|\alpha|\le s_0} \frac{r^{|\alpha|}\|D^\alpha(f-f_i)\|_p}{M_{|\alpha|}} + \sup_{|\alpha|> s_0} \frac{r^{|\alpha|}\|D^\alpha(f-f_i)\|_p}{M_{|\alpha|}} \\
&\le \frac{\varepsilon}{2} + \sup_{|\alpha|>s_0} \left(\frac{r}{r+1}\right)^{|\alpha|} \frac{(r+1)^{|\alpha|}(\|D^\alpha f\|_p + \|D^\alpha f_i\|_p)}{M_{|\alpha|}} \\
&\le \frac{\varepsilon}{2} + \left(\frac{r}{r+1}\right)^{s_0} \sup_{\alpha\in\mathbb{N}_0^k} \frac{2(r+1)^{|\alpha|}|D^\alpha f\|_p}{M_{|\alpha|}} \\
&= \frac{\varepsilon}{2} + \left(\frac{r}{r+1}\right)^{s_0} \cdot 2\|f\|_{p,1/r+1} \\
&< \varepsilon,
\end{aligned}$$

from where the conclusion follows. ■

The previous proposition tells us that the elements of $\mathcal{D}_{(L^p)}^{(M_n)'}(\Omega)$ may be considered as ultradistributions.

Proposition 6. *If $\varphi \in \mathcal{B}_{(L^p)}^{(M_n)}(\Omega)$ and $g \in \mathcal{D}^{(M_n)}(\Omega)$, then $g\varphi$ is in $\mathcal{D}_{(L^p)}^{(M_n)}(\Omega)$.*

PROOF. The support of $g\varphi$ is a compact subset of Ω. We take a positive integer r. We find a constant C_r such that

$$|D^\alpha g(x)| \le C_r (2r)^{-|\alpha|} M_{|\alpha|}, \qquad x \in \Omega, \quad \alpha \in \mathbb{N}_0^k.$$

We have that, for each $\alpha \in \mathbb{N}_0^k$,

$$\begin{aligned}
\|D^\alpha(g\varphi)\|_p &\le \sum_{\beta\le\alpha} \frac{\alpha!}{\beta!(\alpha-\beta)!} \|D^\beta g \cdot D^{\alpha-\beta}\varphi\|_p \\
&\le \sum_{\beta\le\alpha} \frac{\alpha!}{\beta!(\alpha-\beta)!} C_r (2r)^{-|\beta|} M_{|\beta|} \|D^{\alpha-\beta}\varphi\|_p \\
&\le C_r \sum_{\beta\le\alpha} \frac{\alpha!}{\beta!(\alpha-\beta)!} (2r)^{-|\beta|} M_{|\beta|} \|\varphi\|_{p,1/2r} M_{|\alpha-\beta|} (2r)^{-|\alpha-\beta|} \\
&\le C_r \sum_{\beta\le\alpha} \frac{\alpha!}{\beta!(\alpha-\beta)!} (2r)^{-|\alpha|} M_{|\alpha|} \|\varphi\|_{p,1/2r} = C_r \|\varphi\|_{p,1/2r} r^{-|\alpha|} M_{|\alpha|}
\end{aligned}$$

and thus

$$\sup_{\alpha\in\mathbb{N}_0^k} \frac{r^{|\alpha|}\|D^\alpha(g\varphi)\|_p}{M_{|\alpha|}} \le C_r \|\varphi\|_{p,1/2r}$$

and the conclusion follows. ■

In what follows, before stating our next lemma, we shall give the details of a previous construction. We take a bounded subset $\{ S_j : j \in J \}$ of $\mathcal{D}_{(L^p)}^{(M_n)'}(\Omega)$ so that there is a compact subset H of Ω with

$$\operatorname{supp} S_j \subset H, \qquad j \in J.$$

Let K be a compact subset of Ω with $H \subset \overset{\circ}{K}$. We choose an element η of $\mathcal{D}^{(M_n)'}(\Omega)$ which takes value one in a neighborhood of K and whose support is compact. For each $\varphi \in \mathcal{B}_{L^p}^{(M_n)}(\Omega)$, we have that, after the previous proposition, $\eta\varphi$ is in $\mathcal{D}_{(L^p)}^{(M_n)}(\Omega)$. We put

$$\langle \varphi, W_j \rangle := \langle \eta\varphi, S_j \rangle, \qquad j \in J, \quad \varphi \in \mathcal{B}_{(L^p)}^{(M_n)}(\Omega).$$

It is easy to see that $\{ W_j : j \in J \}$ is a bounded subset of $\mathcal{B}_{L^p}^{(M_n)'}(\Omega)$. We apply Proposition 4 to obtain $h > 0$ and, for each $j \in J$, a family $(f_{\alpha,j} : \alpha \in \mathbb{N}_0^k)$ of elements of $\mathcal{L}^q(\Omega)$ such that

$$\sup_{\substack{\alpha \in \mathbb{N}_0^k \\ j \in J}} h^{|\alpha|} M_{|\alpha|} \| f_{\alpha,j} \|_q < \infty$$

and

$$\langle \varphi, W_j \rangle = \sum_{\alpha \in \mathbb{N}_0^k} \int_\Omega D^\alpha \varphi \cdot f_{\alpha,j} \, \mathrm{d}x, \qquad j \in J, \quad \varphi \in \mathcal{B}_{L^p}^{(M_n)}(\Omega).$$

Let g be an element of $\mathcal{D}^{(M_n)}(\Omega)$ which takes value one in a neighborhood of H and whose support is contained in $\overset{\circ}{K}$. Then, on account of the previous proposition, we have

$$\begin{aligned} \langle g\varphi, S_j \rangle = \langle \eta g \varphi, S_j \rangle = \langle g\varphi, W_j \rangle &= \sum_{\alpha \in \mathbb{N}_0^k} \int_\Omega D^\alpha (g\varphi) \cdot f_{\alpha,j} \, \mathrm{d}x \\ &= \sum_{\alpha \in \mathbb{N}_0^k} \int_\Omega \Big(\sum_{\beta \leq \alpha} \frac{\alpha!}{\beta!(\alpha-\beta)!} D^\beta g \cdot D^{\alpha-\beta} \varphi \Big) \cdot f_{\alpha,j} \, \mathrm{d}x, \qquad j \in J, \quad \varphi \in \mathcal{B}_{L^p}^{(M_n)}(\Omega). \end{aligned} \tag{4}$$

We now take a positive integer $r > \frac{4k}{h}$. Let C_r be a positive constant such that

$$|D^\beta g(x)| \leq C_r r^{-|\beta|} M_{|\beta|}, \qquad x \in \Omega, \quad \beta \in \mathbb{N}_0^k.$$

We then have that

$$\begin{aligned} &\sum_{\beta \leq \alpha} \frac{\alpha!}{\beta!(\alpha-\beta)!} \int_\Omega |D^\beta g| \cdot |D^{\alpha-\beta}\varphi| \cdot |f_{\alpha,j}| \, \mathrm{d}x \\ &\qquad \leq \sum_{\beta \leq \alpha} \frac{\alpha!}{\beta!(\alpha-\beta)!} C_r r^{-|\beta|} M_{|\beta|} \int_\Omega |D^{\alpha-\beta}\varphi| \cdot |f_{\alpha,j}| \, \mathrm{d}x \\ &\qquad \leq \sum_{\beta \leq \alpha} \frac{\alpha!}{\beta!(\alpha-\beta)!} C_r r^{-|\beta|} M_{|\beta|} \| D^{\alpha-\beta}\varphi \|_p \cdot \| f_{\alpha,j} \|_q \\ &\qquad \leq \sum_{\beta \leq \alpha} \frac{\alpha!}{\beta!(\alpha-\beta)!} C_r r^{-|\beta|} M_{|\beta|} \| \varphi \|_{p,1/r} r^{-|\alpha-\beta|} M_{|\alpha-\beta|} \| f_{\alpha,j} \|_q \\ &\qquad \leq \sum_{\beta \leq \alpha} \frac{\alpha!}{\beta!(\alpha-\beta)!} C_r r^{-|\alpha|} M_{|\alpha|} \| \varphi \|_{p,1/r} \| f_{\alpha,j} \|_q \\ &\qquad = C_r \| \varphi \|_{p,1/r} \left(\frac{r}{2} \right)^{-|\alpha|} M_{|\alpha|} \| f_{\alpha,j} \|_q \end{aligned}$$

$$\leq C_r\|\varphi\|_{p,1/r}\frac{1}{(2k)^{|\alpha|}}h^{|\alpha|}M_{|\alpha|}\|f_{\alpha,j}\|_q$$
$$\leq C_r\|\varphi\|_{p,1/r}\frac{1}{(2k)^{|\alpha|}}\sup_{\substack{\delta\in\mathbb{N}_0^k\\ j\in J}}h^{|\delta|}M_{|\delta|}\|f_{\alpha,j}\|_q$$

and, noticing that

$$\sum_{\alpha\in\mathbb{N}_0^k}\frac{1}{(2k)^{|\alpha|}}=2,$$

we have that the series (4) converges absolutely and so we may write, putting $\gamma := \alpha-\beta$,

$$\sum_{\alpha\in\mathbb{N}_0^k}\sum_{\beta\leq\alpha}\frac{\alpha!}{\beta!(\alpha-\beta)!}\int_\Omega D^\beta g\cdot D^{\alpha-\beta}\varphi\cdot f_{\alpha,j}\,\mathrm{d}x$$
$$=\sum_{\gamma\in\mathbb{N}_0^k}\sum_{\beta\in\mathbb{N}_0^k}\frac{(\beta+\gamma)!}{\beta!\gamma!}\int_\Omega D^\beta g\cdot D^\gamma\varphi\cdot f_{\beta+\gamma,j}\,\mathrm{d}x. \tag{5}$$

Lemma 1. *Let $\{ S_j : j\in J \}$ be a bounded subset of $\mathcal{D}^{(M_n)'}_{(L^p)}(\Omega)$ such that there is a compact subset H in Ω with*

$$\operatorname{supp} S_j\subset H,\qquad j\in J.$$

Let K be a compact subset of Ω such that $H\subset\overset{\circ}{K}$. Then there are $h>0$ and, for each $j\in J$, a family $(g_{\alpha,j} : \alpha\in\mathbb{N}_0^k)$ of elements of $\mathcal{L}^q(\Omega)$ such that

$$\sup_{\substack{\alpha\in\mathbb{N}_0^k\\ j\in J}}h^{|\alpha|}M_{|\alpha|}\|g_{\alpha,j}\|_q<\infty$$

$$\operatorname{supp} g_{\alpha,j}\subset\overset{\circ}{K},\qquad j\in J,\quad \alpha\in\mathbb{N}_0^k,$$

and

$$\langle\varphi,S_j\rangle=\sum_{\alpha\in\mathbb{N}_0^k}\int_\Omega D^\alpha\varphi\cdot g_{\alpha,j}\,\mathrm{d}x,\qquad j\in J,\quad \varphi\in\mathcal{D}^{(M_n)}_{(L^p)}(\Omega),$$

where the series converges absolutely and uniformly when j varies in J and φ varies in any given bounded subset of $\mathcal{D}^{(M_n)}_{(L^p)}(\Omega)$.

PROOF. We choose $h>0$ and, for each $j\in J$, the family $(f_{\alpha,j} : \alpha\in\mathbb{N}_0^k)$ of elements of $\mathcal{L}^q(\Omega)$ with the properties above cited. We fix $\gamma\in\mathbb{N}_0^k$ and take $\rho\in\tilde{\rho}\in L^p(\Omega)$. We choose $r\in\mathbb{N}$, $r>4k/h$. Then

$$\left|\sum_{\beta\in\mathbb{N}_0^k}\frac{(\beta+\gamma)!}{\beta!\,\gamma!}\int_\Omega D^\beta g\cdot\rho\cdot f_{\beta+\gamma,j}\,\mathrm{d}x\right|$$
$$\leq\sum_{\beta\in\mathbb{N}_0^k}\frac{(\beta+\gamma)!}{\beta!\,\gamma!}\int_\Omega|D^\beta g|\cdot|\rho|\cdot|f_{\beta+\gamma,j}|\,\mathrm{d}x$$
$$\leq|\sum_{\beta\in\mathbb{N}_0^k}\frac{(\beta+\gamma)!}{\beta!\,\gamma!}C_r r^{-|\beta|}M_{|\beta|}\|\rho\|_p\cdot\|f_{\beta+\gamma,j}\|_q$$
$$\leq\frac{C_r\|\rho\|_p}{M_{|\gamma|}}\sum_{\beta\in\mathbb{N}_0^k}2^{|\beta+\gamma|}r^{-|\beta|}M_{|\beta+\gamma|}\|f_{\beta+\gamma,j}\|_q$$

$$\leq \frac{C_r\|\rho\|_p r^{|\gamma|}}{M_{|\gamma|}} \sum_{\beta\in\mathbb{N}_0^k} (r/2)^{-|\beta+\gamma|} M_{|\beta+\gamma|} \|f_{\beta+\gamma,j}\|_q$$
$$\leq \frac{C_r\|\rho\|_p}{M_{|\gamma|}} r^{|\gamma|} \sum_{\beta\in\mathbb{N}_0^k} (2k/h)^{-|\beta+\gamma|} M_{|\beta+\gamma|} \|f_{\beta+\gamma,j}\|_q$$
$$\leq \frac{C_r\|\rho\|_p}{M_{|\gamma|}} r^{|\gamma|} \left(\sup_{\substack{\alpha\in\mathbb{N}_0^k \\ j\in J}} h^{|\alpha|} M_{|\alpha|} \|f_{\alpha,j}\|_q \right) \sum_{\beta\in\mathbb{N}_0^k} \frac{1}{(2k)^{|\beta|}}$$

from where we get that there is $A_\gamma > 0$ such that

$$\left| \sum_{\beta\in\mathbb{N}_0^k} \frac{(\beta+\gamma)!}{\beta!\,\gamma!} \int_\Omega D^\beta g \cdot \rho \cdot f_{\beta+\gamma,j}\,\mathrm{d}x \right| \leq A_\gamma \|\tilde{\rho}\|_p. \tag{6}$$

If we put, for each $\rho \in \tilde{\rho} \in L^p(\Omega)$,

$$v_{\gamma,j}(\tilde{\rho}) := \sum_{\beta\in\mathbb{N}_0^k} \frac{(\beta+\gamma)!}{\beta!\,\gamma!} \int_\Omega D^\beta g \cdot \rho \cdot f_{\beta+\gamma,j}\,\mathrm{d}x,$$

we then have that $v_{\gamma,j}$ is a complex function defined in $L^p(\Omega)$, clearly linear, such that after (6) is also continuous. Then there is $g_{\gamma,j}$ in $\mathcal{L}^q(\Omega)$ such that

$$v_{\gamma,j}(\tilde{\rho}) = \int_\Omega \rho \cdot g_{\gamma,j}\,\mathrm{d}x, \qquad \rho \in \tilde{\rho} \in L^p(\Omega).$$

If M is the support of g, then it is clear that

$$\operatorname{supp} g_{\alpha,j} \subset M \subset \overset{\circ}{K}, \qquad j \in J, \quad \gamma \in \mathbb{N}_0^k.$$

For each $\varphi \in \mathcal{B}_{L^p}^{(M_n}(\Omega)$, we have

$$\sum_{\beta\in\mathbb{N}_0^k} \frac{(\beta+\gamma)!}{\beta!\,\gamma!} \int_\Omega D^\beta g \cdot D^\gamma\varphi \cdot f_{\beta+\gamma,j}\,\mathrm{d}x = \int_\Omega D^\gamma\varphi \cdot g_{\gamma,j}\,\mathrm{d}x$$

and, by (4) and (5),

$$\langle g\varphi, S_j\rangle = \sum_{\gamma\in\mathbb{N}_0^k} \int_\Omega D^\gamma\varphi \cdot g_{\alpha,j}\,\mathrm{d}x, \qquad j \in J, \quad \varphi \in \mathcal{B}_{L^p}^{(M_n)}(\Omega). \tag{7}$$

We now fix $\gamma \in \mathbb{N}_0^k$ and $j \in J$. We choose $\tilde{\rho} \in L^p(\Omega)$ such that $\|\tilde{\rho}\|_p < 2$ and $v_{\gamma,j}(\tilde{\rho}) = \|g_{\gamma,j}\|_q$. We take $r \in \mathbb{N}$ with $r > 4k/h$. If we put

$$C := 2 \sup_{\alpha\in\mathbb{N}_0^k} h^{|\alpha|} M_{|\alpha|} \|f_{\alpha,j}\|_q,$$

we have obtained above

$$\left| \sum_{\beta\in\mathbb{N}_0^k} \frac{(\beta+\gamma)!}{\beta!\,\gamma!} \int_\Omega D^\beta g \cdot \rho \cdot f_{\beta+\gamma,j}\,\mathrm{d}x \right| \leq \frac{C_r\|\tilde{\rho}\|_p\, r^{|\gamma|}}{M_{|\gamma|}} \cdot C.$$

Consequently,

$$\begin{aligned} r^{-|\gamma|}M_{|\gamma|}\|g_{\alpha,j}\|_q &= r^{-|\gamma|}M_{|\gamma|}v_{\gamma,j}(\tilde{\rho}) \\ &= r^{-|\gamma|}M_{|\gamma|}\left|\sum_{\beta\in\mathbb{N}_0^k}\frac{(\beta+\gamma)!}{\beta!\,\gamma!}\int_\Omega D^\beta g\cdot\rho\cdot f_{\beta+\gamma,j}\,\mathrm{d}x\right| \\ &\le r^{-|\gamma|}M_{|\gamma|}\frac{C_r\|\rho\|_p r^{|\gamma|}}{M_{|\gamma|}} \\ &= 2C_rC \end{aligned}$$

and so

$$\sup_{\substack{\gamma\in\mathbb{N}_0^k\\ j\in J}} r^{-|\gamma|}M_{|\gamma|}\|g_{\alpha,j}\|_q \le 2C_rC.$$

We apply now Proposition 3 to the families $(\, g_{\alpha,j} : \alpha\in\mathbb{N}_0^k\,)$, $j\in J$, and so obtain, for each $j\in J$, an element T_j in $\mathcal{B}_{L^p}^{(M_n)'}(\Omega)$ such that

$$\langle\varphi, T_j\rangle = \sum_{\alpha\in\mathbb{N}_0^K}\int_\Omega D^\alpha\varphi\cdot g_{\alpha,j}\,\mathrm{d}x, \qquad j\in J, \quad \varphi\in\mathcal{B}_{L^p}^{(M_n)}(\Omega),$$

where the series converges absolutely and uniformly when j varies in J and φ varies in any given bounded subset of $\mathcal{B}_{L^p}^{(M_n)}(\Omega)$. On the other hand, we have

$$\langle\varphi, S_j\rangle = \langle g\varphi, S_j\rangle = \sum_{\alpha\in\mathbb{N}_0^K}\int_\Omega D^\alpha\varphi\cdot g_{\alpha,j}\,\mathrm{d}x = \langle\varphi, T_j\rangle.$$

The conclusion is now obvious. ■

We now put $\mathcal{L}^q_{\text{loc}}(\Omega)$ for the linear space over $\mathbb{C}$ of the complex functions defined in Ω such that, for each compact subset K of Ω, $f_{|K}$ belongs to $\mathcal{L}^q(K)$. We write $|f|_{K,q} := \|f_{|K}\|_q$.

Theorem 1. *For each j in a set J, let $(\, f_{\alpha,j} : \alpha\in\mathbb{N}_0^k\,)$ be a family of elements of $\mathcal{L}^q_{\text{loc}}(\Omega)$ such that, given any compact subset K of Ω, there is $h>0$ such that*

$$\sup_{\substack{\alpha\in\mathbb{N}_0^k\\ j\in J}} h^{|\alpha|}M_{|\alpha|}|f_{\alpha,j}|_{K,q} < \infty.$$

Then, there is a bounded subset $\{\, S_j : j\in J\,\}$ in $\mathcal{D}_{(L^p)}^{(M_n)'}(\Omega)$ such that

$$\langle\varphi, S_j\rangle = \sum_{\alpha\in\mathbb{N}_0^K}\int_\Omega D^\alpha\varphi\cdot f_{\alpha,j}\,\mathrm{d}x, \qquad j\in J, \quad \varphi\in\mathcal{D}_{(L^p)}^{(M_n)}(\Omega),$$

where the series converges absolutely and uniformly when j varies in J and φ varies in any given bounded subset of $\mathcal{D}_{(L^p)}^{(M_n)}(\Omega)$.

PROOF. For each $m\in\mathbb{N}$, we put

$$f^m_{\alpha,j} := f_{\alpha,j|\overset{\circ}{K}_m}, \qquad \alpha\in\mathbb{N}_0^k, \quad j\in J.$$

We apply Proposition 3 and thus obtain a bounded subset $\{ S_j^m : j \in J \}$ of $\mathcal{B}_{L^p}^{(M_n)'}(\overset{\circ}{K}_m)$ such that

$$\langle \varphi, S_j^m \rangle = \sum_{\alpha \in \mathbb{N}_0^K} \int_\Omega D^\alpha \varphi \cdot f_{\alpha,j}^m \, \mathrm{d}x, \qquad j \in J, \quad \varphi \in \mathcal{B}_{L^p}^{(M_n)}(\overset{\circ}{K}_m),$$

where the series converges absolutely and uniformly when j varies in J and φ varies in any given bounded subset of $\mathcal{D}_{(L^p)}^{(M_n)}(\overset{\circ}{K}_m)$.

For a given element φ of $\mathcal{D}_{(L^p)}^{(M_n)}(\Omega)$, we find $m \in \mathbb{N}$ such that

$$\operatorname{supp} \varphi \subset \overset{\circ}{K}_m$$

and set

$$\langle \varphi, S_j \rangle := \langle \varphi, S_j^m \rangle.$$

It is easy to see that S_j is well defined and that $\{ S_j : J \in J \}$ is a bounded subset of $\mathcal{D}_{(L^p)}^{(M_n)'}(\Omega)$, which leads us to the desired result. ■

Theorem 2. *If $\{ S_j : j \in J \}$ is a bounded subset of $\mathcal{D}_{(L^p)}^{(M_n)'}(\Omega)$, then there is, for each $j \in J$, a family $(f_{\alpha,j} : \alpha \in \mathbb{N}_0^k)$ in $\mathcal{L}_{\mathrm{loc}}^p(\Omega)$ such that, given any compact subset K of Ω, there is $h > 0$ with*

$$\sup_{\substack{\alpha \in \mathbb{N}_0^k \\ j \in J}} h^{|\alpha|} M_{|\alpha|} |f_{\alpha,j}|_{K,q} < \infty$$

and

$$\langle \varphi, S_j \rangle = \sum_{\alpha \in \mathbb{N}_0^K} \int_\Omega D^\alpha \varphi \cdot f_{\alpha,j}^m \, \mathrm{d}x, \qquad j \in J, \quad \varphi \in \mathcal{D}_{(L^p)}^{(M_n)}(\Omega),$$

where the series converges absolutely and uniformly when j varies in J and φ varies in any given bounded subset of $\mathcal{D}_{(L^p)}^{(M_n)}(\Omega)$.

PROOF. Let $\{ O_m : m \in \mathbb{N} \}$ be a locally finite open covering of Ω such that O_m is relatively compact in Ω, $m \in \mathbb{N}$. Let $\{ g_m : m \in \mathbb{N} \}$ be a partition of unity of class (M_n) subordinated to that covering. It follows that $\{ g_m S_j : j \in J \}$ is a bounded subset of $\mathcal{D}_{(L^p)}^{(M_n)'}(\Omega)$ whose elements have their supports contained in a compact subset of O_m. Applying the previous lemma, we obtain, for each $j \in J$, a family $(f_{\alpha,j}^m : \alpha \in \mathbb{N}_0^k)$ of elements of $\mathcal{L}^q(\Omega)$ such that there is $h_m > 0$ with

$$\sup_{\substack{\alpha \in \mathbb{N}_0^k \\ j \in J}} h_m^{|\alpha|} M_{|\alpha|} \| f_{\alpha,j}^m \|_q < \infty,$$

$$\operatorname{supp} f_{\alpha,j}^m \subset O_m, \qquad j \in J, \quad \alpha \in \mathbb{N}_0^k,$$

and

$$\langle \varphi, g_m S_j \rangle = \sum_{\alpha \in \mathbb{N}_0^K} \int_\Omega D^\alpha \varphi \cdot f_{\alpha,j}^m \, \mathrm{d}x, \qquad j \in J, \quad \varphi \in \mathcal{D}_{(L^p)}^{(M_n)}(\Omega),$$

where the series converges absolutely and uniformly when j varies in J and φ varies in any given bounded subset of $\mathcal{D}_{(L^p)}^{(M_n)}(\Omega)$. We put, for each $x \in \Omega$, $\alpha \in \mathbb{N}_0^k$, $j \in J$,

$$f_{\alpha,j}(x) := \sum_{m=1}^{\infty} f_{\alpha,j}^m(x).$$

Given any compact subset K of Ω, there is a positive integer m_0 such that

$$K \cap O_m = \emptyset, \qquad m \geq m_0,$$

and thus $f_{\alpha,j}$ is well defined and belongs to $\mathcal{L}^q_{\mathrm{loc}}(\Omega)$. Besides, we have

$$|f_{\alpha,j}|_{K,q} \leq \sum_{m=1}^{m_0} |f^m_{\alpha,j}|_{K,q} \leq \sum_{m=1}^{m_0} \|f^m_{\alpha,j}\|_q$$

and so, if

$$h := \inf\{ h_m : m = 1, 2, \ldots, m_0 \},$$

we have that

$$\sup_{\substack{\alpha\in\mathbb{N}_0^k \\ j\in J}} h^{|\alpha|} M_{|\alpha|} |f_{\alpha,j}|_{K,q} \leq \sum_{m=1}^{m_0} \sup_{\substack{\alpha\in\mathbb{N}_0^k \\ j\in J}} h_m^{|\alpha|} M_{|\alpha|} \|f^m_{\alpha,j}\|_q < \infty.$$

We now apply the previous theorem to obtain a bounded subset $\{ T_j : j \in J \}$ of $\mathcal{D}^{(M_n)'}_{(L^p)}(\Omega)$ such that

$$\langle \varphi, T_j \rangle = \sum_{\alpha\in\mathbb{N}_0^K} \int_\Omega D^\alpha \varphi \cdot f_{\alpha,j} \, \mathrm{d}x, \qquad j \in J, \quad \varphi \in \mathcal{D}^{(M_n)}_{(L^p)}(\Omega),$$

where the series converges absolutely and uniformly when j varies in J and φ varies in any given bounded subset of $\mathcal{D}^{(M_n)}_{(L^p)}(\Omega)$.

We next choose φ in $\mathcal{D}^{(M_n)}_{(L^p)}(\Omega)$. We find $m_0 \in \mathbb{N}$ for which

$$O_m \cap \operatorname{supp} \varphi = \emptyset, \qquad m \geq m_0.$$

Then

$$\begin{aligned}
\langle \varphi, T_j \rangle &= \sum_{\alpha\in\mathbb{N}_0^K} \int_\Omega D^\alpha \varphi \cdot f_{\alpha,j} \, \mathrm{d}x \\
&= \sum_{\alpha\in\mathbb{N}_0^k} \int_\Omega \left(\sum_{m=1}^{m_0} D^\alpha \varphi \cdot f^m_{\alpha,hj} \right) \mathrm{d}x \\
&= \sum_{m=1}^{m_0} \sum_{\alpha\in\mathbb{N}_0^k} \int_\Omega D^\alpha \varphi \cdot f^m_{\alpha,j} \, \mathrm{d}x \\
&= \sum_{m=1}^{m_0} \langle \varphi, g_m S_j \rangle \\
&= \langle \sum_{m=1}^{m_0} \varphi \cdot g_m, S_j \rangle \\
&= \langle \varphi, S_j \rangle.
\end{aligned}$$

Consequently, $S_j = T_j$, $j \in J$, and the conclusion follows. ■

Proposition 7. *If M_n, $n = 0, 1, \ldots$, satisfies condition (1), then the canonical injection ζ from $\mathcal{D}^{(M_n)}(\Omega)$ into $\mathcal{D}^{(M_n)}_{(L^p)}(\Omega)$ is a topological isomorphism.*

PROOF. Clearly ζ is well defined linear and continuous. It is also plain that there exist $b > 0$ and $l > 0$ such that

$$M_{n+k} \leq b\, l^n M_n, \qquad n \in \mathbb{N}_0.$$

We take now an arbitrary element φ of $\mathcal{D}^{(M_n)}_{(L^1)}(\Omega)$ and $r \in \mathbb{N}$. Let s be an integer greater that rl.We extend φ to $\mathbb{R}^k$ such that $\varphi(x) = 0$, $x \in \mathbb{R}^k \setminus \Omega$. Given $\alpha \in \mathbb{N}_0^k$ and $x \in \mathbb{R}^k$, we have that

$$D^\alpha \varphi(x) = \int_{-\infty}^{x_1} \int_{-\infty}^{x_2} \cdots \int_{-\infty}^{x_k} \frac{\partial^{|\alpha|+k} \varphi(t)}{\partial^{\alpha_1+1} t_1 \partial^{\alpha_2+1} t_2 \ldots \partial^{\alpha_k+1} t_k} \,\mathrm{d}t_1\,\mathrm{d}t_2 \ldots \mathrm{d}t_k$$

and hence

$$\begin{aligned} |D^\alpha \varphi(x)| &\leq \int_\Omega \left| \frac{\partial^{|\alpha|+k} \varphi(t)}{\partial^{\alpha_1+1} t_1 \partial^{\alpha_2+1} t_2 \ldots \partial^{\alpha_k+1} t_k} \right| \mathrm{d}t_1\,\mathrm{d}t_2 \ldots \mathrm{d}t_k \\ &\leq \|\varphi\|_{1,1/s} s^{-|\alpha|-k} M_{|\alpha|+k} \\ &\leq \|\varphi\|_{1,1/s} s^{-|\alpha|-k} b\, l^{|\alpha|} M_{|\alpha|} \\ &\leq \|\varphi\|_{1,1/s} b\, s^{-k} \left(\frac{s}{l}\right)^{-|\alpha|} M_{|\alpha|} \\ &\leq b\, s^{-k} \|\varphi\|_{1,1/s} r^{-|\alpha|} M_{|\alpha|} \end{aligned}$$

and so

$$\varphi \in \mathcal{D}^{(M_n)}(\Omega).$$

Thus ζ is onto. The conclusion now follows by applying a theorem of Grothendieck's, [1, p. 17]. ■

Theorem 3. *If M_n, $n = 0, 1, \ldots$ satisfies condition (1) and $\{ S_j : j \in J \}$ is a bounded subset of $\mathcal{D}^{(M_n)'}(\Omega)$, then there is, for each $j \in J$, a family $(f_{\alpha,j} : \alpha \in \mathbb{N}_0^k)$ of elements of $\mathcal{L}^\infty_{\mathrm{loc}}(\Omega)$ such that, given a compact subset K of Ω, there is $h > 0$ such that*

$$\sup_{\substack{\alpha \in \mathbb{N}_0^k \\ j \in J}} h^{|\alpha|} M_{|\alpha|} |f_{\alpha,j}|_{K,\infty} < \infty$$

and

$$\langle \varphi, S_j \rangle = \sum_{\alpha \in \mathbb{N}_0^k} \int_\Omega D^\alpha \varphi \cdot f_{\alpha,j} \,\mathrm{d}x, \qquad j \in J, \quad \varphi \in \mathcal{D}‘(M_n)(\Omega),$$

where the series converges absolutely and uniformly when j varies in J and φ varies in any given bounded subset of $\mathcal{D}^{(M_n)}(\Omega)$.

PROOF. It is an immediate consequence of the previous proposition and Theorem 2. ■

We put now $\mathcal{D}^{(M_n)}_{L^p}(\Omega)$ for the subspace of $\mathcal{B}^{(M_n)}_{L^p}(\Omega)$ given by the closure of $\mathcal{D}^{(M_n)}(\Omega)$ in that space. $\mathcal{D}^{(M_n)'}_{L^p}(\Omega)$ will be the strong dual of $\mathcal{D}^{(M_n)}_{L^p}(\Omega)$. The two theorems that follow next are not difficult to prove by following a similar procedure to those in the proofs of Proposition 3 and Proposition 4, respectively. Those theorems constitute characterizations of certain ultradistributions of Beurling type in Ω.

Theorem 4. *For each j in a set J, let $(f_{\alpha,j} : \alpha \in \mathbb{N}_0^k)$ be a family of elements of $\mathcal{L}^q(\Omega)$ such that there is $h > 0$ with*

$$\sup_{\substack{\alpha \in \mathbb{N}_0^k \\ j \in J}} h^{|\alpha|} M_{|\alpha|} \|f_{\alpha,j}\|_q < \infty.$$

Then there exists a bounded subset $\{ S_j : j \in J \}$ *of* $\mathcal{D}_{L^p}^{(M_n)'}(\Omega)$ *such that*

$$\langle \varphi, S_j \rangle = \sum_{\alpha \in \mathbb{N}_0^k} \int_\Omega D^\alpha \varphi \cdot f_{\alpha,j} \, \mathrm{d}x, \qquad j \in J, \quad \varphi \in \mathcal{D}_{L^p}^{(M_n)}(\Omega),$$

where the series converges absolutely and uniformly when j *varies in* J *and* φ *varies in any given bounded subset of* $\mathcal{D}_{L^p}^{(M_n)}(\Omega)$.

Theorem 5. *If* $\{ S_j : j \in J \}$ *is a bounded subset of* $\mathcal{D}_{L^p}^{(M_n)}(\Omega)$*, there are* $h > 0$ *and, for each* $j \in J$*, a family* $(f_{\alpha,j} : \alpha \in \mathbb{N}_0^k)$ *of elements of* $\mathcal{L}^q(\Omega)$ *such that*

$$\sup_{\substack{\alpha \in \mathbb{N}_0^k \\ j \in J}} h^{|\alpha|} M_{|\alpha|} \| f_{\alpha,j} \|_q < \infty$$

and

$$\langle \varphi, S_j \rangle = \sum_{\alpha \in \mathbb{N}_0^k} \int_\Omega D^\alpha \varphi \cdot f_{\alpha,j} \, \mathrm{d}x, \qquad j \in J, \quad \varphi \in \mathcal{D}_{L^p}^{(M_n)}(\Omega),$$

where the series converges absolutely and uniformly when j *varies in* J *and* φ *varies in any given bounded subset of* $\mathcal{D}_{L^p}^{(M_n)}(\Omega)$.

4 Structure of the ultradistributions of Beurling type

Given $h > 0$, we put $\mathcal{E}_0^{(M_n),h}(\Omega)$ to denote the space over $\mathbb{C}$ of the complex functions f, defined and infinitely differentiable in Ω which vanish at infinity, as well as each of their derivatives of any order, that is, given $\epsilon > 0$, and $\beta \in \mathbb{N}_0^k$, there is a compact subset K in Ω for which

$$|D^\beta f(x)| < \varepsilon, \qquad x \in \Omega \setminus K,$$

satisfying also that there is $C > 0$, depending only on f, such that

$$|D^\alpha f| \leq C h^{|\alpha|} M_{|\alpha|}, \qquad x \in \Omega, \quad \alpha \in \mathbb{N}_0^k.$$

We put

$$|f|_h := \sup_{\alpha \in \mathbb{N}_0^k} \sup_{x \in \Omega} \frac{D^\alpha f(x)}{h^{|\alpha|} M_{|\alpha|}}$$

and assume that $\mathcal{E}_0^{(M_n),h}(\Omega)$ is endowed with the norm $|\cdot|_h$. We set

$$\mathcal{E}_0^{(M_n)}(\Omega) := \bigcap_{m=1}^{\infty} \mathcal{E}_0^{(M_n),1/m}(\Omega)$$

and consider $\mathcal{E}_0^{(M_n)}(\Omega)$ as the projective limit of the sequence $(\mathcal{E}_0^{(M_n),1/m}(\Omega))$ of Banach spaces. $\mathcal{E}_0^{(M_n)'}(\Omega)$ will be the strong dual of $\mathcal{E}_0^{(M_n)}(\Omega)$. By $C_0(\Omega)$ we represent the linear space over $\mathbb{C}$ of the complex functions f defined and continuous in Ω which vanish at infinity. We put

$$|f|_\infty := \sup_{x \in \Omega} |f(x)|$$

and assume that $C_0(\Omega)$ is provided with this norm.

If we replace the Banach space X of Section 2 by $C_0(\Omega)$, following a argument similar to that of the previous section, and also using results of [4], we may obtain the next two theorems, which are a generalization of result b) .

Theorem 6. *For each j in a set J, let $(\, u_{\alpha,j} . \alpha \in \mathbb{N}_0^k \,)$ be a family of Radon measures in Ω. If, given an arbitrary compact subset K of Ω, there is $h > 0$ such that*

$$\sup_{\substack{\alpha\in\mathbb{N}_0^k\\ j\in J}} h^{|\alpha|} M_{|\alpha|} \|u_{\alpha,j}\|(K) < \infty,$$

then there exists a bounded subset $\{\, S_j : j \in J \,\}$ of $\mathcal{D}^{(M_n)'}(\Omega)$ such that

$$\langle \varphi, S_j \rangle = \sum_{\alpha\in\mathbb{N}_0^k} \langle D^\alpha \varphi, u_{\alpha,j} \rangle, \qquad j \in J, \quad \varphi \in \mathcal{D}^{(M_n)}(\Omega),$$

where the series converges absolutely and uniformly when j varies in J and φ varies in any given bounded subset of $\mathcal{D}^{(M_n)}(\Omega)$.

Theorem 7. *If $\{\, S_j : j \in J \,\}$ is a bounded subset of $\mathcal{D}^{(M_n)'}(\Omega)$, there is, for each $j \in J$, a family $(\, u_{\alpha,j} : \alpha \in \mathbb{N}_0^k \,)$ of Radon measures in Ω such that, given a compact subset K of Ω, there is $h > 0$ with*

$$\sup_{\substack{\alpha\in\mathbb{N}_0^k\\ j\in J}} h^{|\alpha|} M_{|\alpha|} \|u_{\alpha,j}\|(K) < \infty,$$

and

$$\langle \varphi, S_j \rangle = \sum_{\alpha\in\mathbb{N}_0^k} \langle D^\alpha \varphi, u_{\alpha,j} \rangle, \qquad j \in J, \quad \varphi \in \mathcal{D}^{(M_n)}(\Omega),$$

where the series converges absolutely and uniformly when j varies in J and φ varies in any given bounded subset of $\mathcal{D}^{(M_n)}(\Omega)$.

We put $\overset{\circ}{\mathcal{E}}{}^{(M_n)}(\Omega)$ for the subspace of $\mathcal{E}_0^{(M_n)}(\Omega)$ given by the closure of $\mathcal{D}^{(M_n)}(\Omega)$. $\overset{\circ}{\mathcal{E}}{}^{(M_n)\prime}(\Omega)$ will denote its strong dual. The elements of this last space may be considered as Beurling ultradistributions in Ω. We characterize those ultradistributions in the following two theorems. Their proofs may be obtained by conveniently adapting the proofs of Proposition 3 and Proposition 4, respectively.

Theorem 8. *For each j in a set J, let $(\, \mu_{\alpha,j} . \alpha \in \mathbb{N}_0^k \,)$ be a family of complex Borel measures in Ω such that there is $h > 0$ with*

$$\sup_{\substack{\alpha\in\mathbb{N}_0^k\\ j\in J}} h^{|\alpha|} M_{|\alpha|} |\mu_{\alpha,j}|(\Omega) < \infty.$$

Then there exists a bounded subset $\{\, S_j : j \in J \,\}$ of $\overset{\circ}{\mathcal{E}}{}^{(M_n)\prime}(\Omega)$ such that

$$\langle \varphi, S_j \rangle = \sum_{\alpha\in\mathbb{N}_0^k} \int_\Omega D^\alpha \varphi \cdot \mathrm{d}\mu_{\alpha,j}, \qquad j \in J, \quad \varphi \in \overset{\circ}{\mathcal{E}}{}^{(M_n)}(\Omega),$$

where the series converges absolutely and uniformly when j varies in J and φ varies in any given bounded subset of $\overset{\circ}{\mathcal{E}}{}^{(M_n)}(\Omega)$.

Theorem 9. *If $\{\, S_j : j \in J \,\}$ is a bounded subset of $\overset{\circ}{\mathcal{E}}{}^{(M_n)\prime}(\Omega)$, there is $h > 0$ and, for each $j \in J$, a family $(\, \mu_{\alpha,j} : \alpha \in \mathbb{N}_0^k \,)$ of complex Borel measures in Ω with*

$$\sup_{\substack{\alpha\in\mathbb{N}_0^k\\ j\in J}} h^{|\alpha|} M_{|\alpha|} |\mu_{\alpha,j}|(\Omega) < \infty,$$

and

$$\langle \varphi, S_j \rangle = \sum_{\alpha \in \mathbb{N}_0^k} \int_\Omega D^\alpha \varphi \cdot \mathrm{d}\mu_{\alpha,j}, \qquad j \in J, \quad \varphi \in \mathcal{E}^{(\overset{\circ}{M_n})}(\Omega),$$

where the series converges absolutely and uniformly when j varies in J and φ varies in any given bounded subset of $\mathcal{E}^{(\overset{\circ}{M_n})}(\Omega)$.

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