RACSAM
*Rev. R. Acad. Cien. Serie A. Mat.*
VOL. **103** (1), 2009, pp. 49–54
Análisis Matemático / Mathematical Analysis

# On copies of $c_0$ in some function spaces

**Juan Carlos Ferrando**

**Abstract.** If $(\Omega, \Sigma, \mu)$ is a probability space and $X$ a Banach space, a theorem concerning sequences of $X$-valued random elements which do not converge to zero is applied to show from a common point of view that the $F$-normed space $L_0(\mu, X)$ of all classes of $X$-valued random variables, as well as the $p$-normed space $L_p(\mu, X)$ of all $X$-valued $p$-integrable random variables with $0 < p < 1$ and the space $P_1(\mu, X)$ of the $\mu$-measurable $X$-valued Pettis integrable functions, all contain a copy of $c_0$ if and only if $X$ does. We also show that if $\Omega$ is a noncompact hemicompact topological space, then the Banach space $C_0(\Omega)$ of all scalarly valued continuous functions defined on $\Omega$ vanishing at infinity, equipped with the supremum-norm, contains a norm-one complemented copy of $c_0$.

## Sobre copias de $c_0$ en algunos espacios de funciones

**Resumen.** Si $(\Omega, \Sigma, \mu)$ es un espacio de probabilidad y $X$ un espacio de Banach, aplicamos un teorema sobre sucesiones de variables aleatorias con valores en $X$ que no convergen a cero para demostrar, desde un punto de vista común, que el espacio $F$-normado $L_0(\mu, X)$ de todas las clases de variables aleatorias $X$-valoradas, así como el espacio $p$-normado $L_p(\mu, X)$ de las variables aleatorias $p$-integrables $X$-valoradas, con $0 < p < 1$, y el espacio $P_1(\mu, X)$ de las funciones $X$-valoradas $\mu$-medibles y Pettis integrables, todos ellos contienen una copia de $c_0$ si y sólo si $X$ también la contiene. Asimismo probamos que si $\Omega$ es un espacio topológico hemicompacto no compacto, el espacio de Banach $C_0(\Omega)$ de las funciones continuas con valores escalares definidas en $\Omega$ que se anulan en el infinito, equipado con la norma supremo, contiene una copia de $c_0$ norma-uno complementada.

## 1 Preliminaries

Throughout $\Delta$ denotes the product $\{-1,1\}^{\mathbb{N}}$, $\Gamma$ the $\sigma$-algebra of subsets of $\Delta$ generated by the $n$-cylinders of $\Delta$ for each $n \in \mathbb{N}$, and $\nu$ the Borel probability $\otimes_{i=1}^{\infty} \nu_i$ on $\Gamma$, where $\nu_i \colon 2^{\{-1,1\}} \to [0,1]$ is defined by $\nu_i(\emptyset) = 0$, $\nu_i(\{-1\}) = \nu_i(\{1\}) = 1/2$ and $\nu_i(\{-1,1\}) = 1$ for each $i \in \mathbb{N}$. Coordinate mappings on $\Delta$ are denoted by $\varepsilon_i$, as well as its values when considered as mappings into $\{-1,1\}$. In the sequel $(\Omega, \Sigma, \mu)$ will be a nontrivial complete probability space, $X$ a Banach space over the field $\mathbb{K}$ of real or complex numbers and $L_0(\mu, X)$ will stand for the $F$-space of all [classes of] $\mu$-measurable functions $f \colon \Omega \to X$ equipped with the (continuous) $F$-norm

$$\|f\|_0 = \int_\Omega \frac{\|f(\omega)\|}{1 + \|f(\omega)\|} \, \mathrm{d}\mu(\omega)$$

Presentado por / Submitted by Manuel Valdivia Ureña.
*Recibido / Received*: 11 de enero de 2009. *Aceptado / Accepted*: 14 de enero de 2009.
*Palabras clave / Keywords*: Banach space, copy of $c_0$, vector-valued random element, $F$-norm, $p$-norm, space $L_p(\mu, X)$ with $0 \leq p < 1$, space $P_1(\mu, X)$, hemicompact space, space $C_0(\Omega)$.
*Mathematics Subject Classifications*: 46E40, 46B09.

of the convergence in probability. We shall represent by $L_p(\mu, X)$ with $0 < p < 1$ the complete $p$-normed topological vector space of all $\mu$-measurable $X$-valued [classes of] $p$-Bochner integrable functions equipped with the (continuous) $p$-norm

$$\|f\|_p = \int_\Omega \|f(\omega)\|^p \, \mathrm{d}\mu(\omega).$$

We shall shorten by wuC the sentence 'weakly unconditionally Cauchy'. We stand for $P_1(\mu, X)$ the normed space of all those [classes of] $\mu$-measurable $X$-valued Pettis integrable functions $f$ defined on $\Omega$ provided with the semivariation norm

$$\|f\|_{P_1(\mu,X)} = \sup\left\{ \int_\Omega |x^* f(\omega)| \, \mathrm{d}\mu(\omega) : x^* \in X^*, \; \|x^*\| \le 1 \right\}.$$

If $A$ is a subset of a Banach space $X$ then $[A]$ will represent the closed linear span of $A$ in $X$. If $\Omega$ is endowed with a Hausdorff topology, the Banach space over $\mathbb{K}$ of all continuous functions $f\colon \Omega \to \mathbb{K}$ vanishing at infinity (that is, for each $\epsilon > 0$ there is a compact set $K_{f,\epsilon} \subseteq \Omega$ such that $|f(\omega)| < \epsilon$ for $\omega \in \Omega \setminus K_{f,\epsilon}$) equipped with the supremum-norm will be denoted by $C_0(\Omega)$ and the linear subspace of $C_0(\Omega)$ consisting of all those functions $f$ of compact support ($\operatorname{supp} f$) will be represented by $C_C(\Omega)$.

## 2 Copies of $c_0$ in $L_p(\mu, X)$ with $0 \le p < 1$

As is well known, a classic conjecture of Hoffmann-Jørgensen [7] on the $X$-inheritance of copies of $c_0$ in the Banach space $L_p(\mu, X)$ with $1 \le p < \infty$ was established by Kwapień in [9]. Later on Hoffmann-Jørgensen conjecture was shown also to be true for $p = 0$ [10, Theorem 2.11] (see [2, Theorem 9.1]) and for $0 < p < 1$ [2, Theorem 10.4]. In this section a theorem concerning certain sequences of $X$-valued random elements which do not converge to zero (Theorem 2 below) is used to provide an independent proof of these facts. The following result is contained in the proof of [9, Proposition].

**Theorem 1 (Kwapień)** *Let $X$ be a Banach space and $\{x_n\}$ a sequence in $X$. If*

$$\nu\left(\left\{\varepsilon \in \Delta : \sup_{n\in\mathbb{N}} \left\| \sum_{i=1}^{n} \varepsilon_i x_i \right\| = \infty \right\}\right) = 0,$$

*then the series $\sum_{n=1}^{\infty} x_n$ has a wuC subseries $\sum_{i=1}^{\infty} x_{n_i}$.*

**Lemma 1** *Let $\{f_n\}$ be a sequence of $X$-valued random variables for $(\Omega, \Sigma, \mu)$. If the series $\sum_{n=1}^{\infty} \zeta_n f_n$ converges in $L_0(\mu, X)$ for every $\zeta \in c_0$, then for each $A \in \Sigma$ with $\mu(A) > 0$ there is $\omega_A \in A$ such that the series $\sum_{n=1}^{\infty} f_n(\omega_A)$ has a wuC subseries.*

PROOF. If we set $\varphi_n(\varepsilon, \omega) = \varepsilon_n f_n(\omega)$ for each $(\varepsilon, \omega, n) \in \Delta \times \Omega \times \mathbb{N}$ and put $S_n = \sum_{i=1}^{n} \varphi_i$ for each $n \in \mathbb{N}$, it suffices to show that the sequence $\{S_n\}_{n=1}^{\infty}$ is stochastically bounded, that is, that $\{S_n : n \in \mathbb{N}\}$ is a bounded set in $L_0(\nu \otimes \mu, X)$. In fact, since $\{\varphi_i\}_{i=1}^{\infty}$ is a symmetric sequence of $(\nu \otimes \mu)$-measurable functions, if this property holds the sequence $\{S_n\}_{n=1}^{\infty}$ is $(\nu \otimes \mu)$-almost surely bounded (see for instance [7, Theorem 2.4]). This means that the set

$$Q = \left\{ (\varepsilon, \omega) \in \Delta \times \Omega : \sup_{n\in\mathbb{N}} \left\| \sum_{i=1}^{n} \varepsilon_i f_i(\omega) \right\| = \infty \right\},$$

verifies that $(\nu \otimes \mu)(Q) = 0$. Hence, if $Q_\omega$ denotes the $\omega$-slice of $Q$, the fact that

$$(\nu \otimes \mu)(Q) = \int_\Omega \nu(Q_\omega) \, \mathrm{d}\mu(\omega)$$

implies that $\nu(Q_\omega)=0$ for $\mu$-almost all $\omega\in\Omega$. Since $\mu(A)>0$, there is an $\omega_A\in A$ with $\nu(Q_{\omega_A})=0$, that is

$$\nu\left(\left\{\varepsilon\in\Delta:\sup_{n\in\mathbb{N}}\left\|\sum_{i=1}^{n}\varepsilon_i f_i(\omega_A)\right\|=\infty\right\}\right)=0.$$

So by Theorem 1 there exists in $X$ a wuC subseries of $\sum_{n=1}^{\infty} f_n(\omega_A)$.

Let us show that the sequence $\{S_n\}_{n=1}^{\infty}$ of the partial sums of the sequence $\{\varphi_i\}_{i=1}^{\infty}$ is bounded in $L_0(\nu\otimes\mu,X)$. Since $\sum_{i=1}^{\infty}\zeta_i\varepsilon_i f_i$ converges in $L_0(\mu,X)$ for each $(\zeta,\varepsilon)\in c_0\times\Delta$, setting

$$h_k^{\zeta}(\varepsilon)=\sup_{m,n\geq k}\left\|\sum_{i=1}^{m}\zeta_i\varepsilon_i f_i-\sum_{i=1}^{n}\zeta_i\varepsilon_i f_i\right\|_0$$

for $(\zeta,k)\in c_0\times\mathbb{N}$, then $h_k^{\zeta}(\varepsilon)\to 0$ for every $\varepsilon\in\Delta$ with fixed $\zeta\in c_0$ as $k\to\infty$. Due to the fact that $h_k^{\zeta}$ is $\nu$-measurable and $h_k^{\zeta}\leq 1$ for every $(\zeta,k)\in c_0\times\mathbb{N}$, Lebesgue's dominated convergence theorem implies that $\int_\Delta h_k^{\zeta}(\varepsilon)\,d\nu(\varepsilon)\to 0$ for each $\zeta\in c_0$. Hence given $\epsilon>0$ there is $k_0(\zeta)\in\mathbb{N}$ such that

$$\int_\Delta\left\|\sum_{i=1}^{m}\zeta_i\varepsilon_i f_i-\sum_{i=1}^{n}\zeta_i\varepsilon_i f_i\right\|_0 \mathrm{d}\nu(\varepsilon)<\epsilon$$

for every $m,n\geq k_0(\zeta)$ and Fubini's theorem shows that $\{\sum_{i=1}^{n}\zeta_i\varphi_i\}_{n=1}^{\infty}$ is a Cauchy sequence in $L_0(\nu\otimes\mu,X)$ for all $\zeta\in c_0$. Since $L_0(\nu\otimes\mu,X)$ is a complete linear space, we conclude that the series $\sum_{i=1}^{\infty}\zeta_i\varphi_i$ converges in $L_0(\nu\otimes\mu,X)$ for every $\zeta\in c_0$. Therefore an application of Banach-Steinhaus' theorem [8, 15.13.(3)] shows that the linear mapping $T\colon c_0\to L_0(\nu\otimes\mu,X)$ defined by $T\zeta=\sum_{i=1}^{\infty}\zeta_i\varphi_i$ is continuous. So $T(\{\sum_{i=1}^{n}e_i:n\in\mathbb{N}\})$ is a bounded subset of $L_0(\nu\otimes\mu,X)$ and consequently the sequence $\{S_n\}_{n=1}^{\infty}$ is bounded in $L_0(\nu\otimes\mu,X)$, as stated. ■

**Theorem 2** *Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of $X$-valued random variables for $(\Omega,\Sigma,\mu)$. If the two following conditions hold*

1. *$\sum_{n=1}^{\infty}\zeta_n f_n$ converges in $L_0(\mu,X)$ for every $\zeta\in c_0$*
2. *$\{f_n\}_{n=1}^{\infty}$ does not converge almost surely to zero*

*then $X$ contains a copy of $c_0$.*

PROOF. Let $A=\{\omega\in\Omega:f_n(\omega)\nrightarrow 0\}$. If $\mu(A)>0$, Lemma 1 yields an $\omega_A\in A$ and a strictly increasing sequence $\{n_i\}\subseteq\mathbb{N}$ such that the series $\sum_{i=1}^{\infty}f_{n_i}(\omega_A)$ is wuC in $X$. So there exists $\epsilon>0$ and a subsequence $\{x_i\}_{i=1}^{\infty}$ of $\{f_{n_i}(\omega_A)\}_{i=1}^{\infty}$ with $\|x_i\|\geq\epsilon$ for all $i\in\mathbb{N}$. Since $\inf_{i\in\mathbb{N}}\|x_i\|>0$ and $\sum_{i=1}^{\infty}x_i$ is wuC in $X$, the selection principle along with [1, Chapter V, Corollary 7] assures that $X$ contains a copy of $c_0$. On the other hand, if $\mu(A)=0$ then $f_n\to 0$ almost surely, contradicting condition 2 above. ■

**Corollary 1** *$L_0(\mu,X)$ contains a copy of $c_0$ if and only if $X$ does.*

PROOF. If $J$ is an isomorphism from $c_0$ into $L_0(\mu,X)$ and $f_n=Je_n$ for each $n\in\mathbb{N}$ then $\inf_{n\in\mathbb{N}}\|f_n\|_0>0$. Otherwise there is a subsequence $\{f_{n_k}\}$ of $\{f_n\}$ such that $f_{n_k}\to 0$ in $J(c_0)$, considered as a subspace of $L_0(\mu,X)$. So $e_{n_k}\to 0$ in $c_0$, a contradiction.

If a sequence $\{f_n\}$ of representatives of the original sequence of classes of functions verifies that $f_n\to 0$ almost surely, then $f_n\to 0$ in $L_0(\mu,X)$, contradicting the fact that $\inf_{n\in\mathbb{N}}\|f_n\|_0>0$. Thus condition 2 of the preceding theorem holds for $\{f_n\}$. On the other hand, if $\zeta\in c_0$, since $\sum_{n=0}^{\infty}\zeta_n e_n$ converges (to $\zeta$) in $c_0$ then $\sum_{n=0}^{\infty}\zeta_n f_n$ converges (to $J\zeta$) in $L_0(\mu,X)$. Consequently the sequence $\{f_n\}_{n=1}^{\infty}$ satisfies condition 1 of the preceding theorem as well. Hence Theorem 2 guarantees that $X$ contains a copy of $c_0$. For the converse note that $X$ is isomorphic to a linear subspace of $L_0(\mu,X)$. ■

**Lemma 2** *Let $J$ be an isomorphism from $c_0$ into $L_p(\mu, X)$ with $0 < p < 1$. If $f_n = Je_n$ for each $n \in \mathbb{N}$, where $\{e_n\}_{n=1}^{\infty}$ stands for the unit vector basis of $c_0$, then the sequence $\{\|f_n(\cdot)\|^p\}_{n=1}^{\infty}$ is uniformly integrable.*

PROOF. Assuming by contradiction that the sequence $\{\|f_n(\cdot)\|^p\}_{n=1}^{\infty}$ is not uniformly integrable, there are $\epsilon > 0$, a sequence $\{A_n\}_{n=1}^{\infty}$ of pairwise disjoint elements of $\Sigma$ and a subsequence of $\{f_n\}$, which we denote in the same way, with $\int_{A_n} \|f_n(\omega)\|^p \, d\mu(\omega) > \epsilon$ for every $n \in \mathbb{N}$. Since $\left\|\sum_{k=1}^{n} \xi_k e_k\right\|_{\infty} \leq 1$ for each $n \in \mathbb{N}$ and $\xi \in \ell_{\infty}$ with $\|\xi\|_{\infty} \leq 1$, there is $K > 0$ such that $\left\|\sum_{i=1}^{n} \xi_i f_i\right\|_p \leq K$ for each $n \in \mathbb{N}$ and $\xi \in \ell_{\infty}$ with $\|\xi\|_{\infty} \leq 1$. Hence, choosing $m \in \mathbb{N}$ such that $(m-1)\epsilon > K$, classic Rosenthal's lemma provides a strictly increasing sequence $\{n_i\}_{i=1}^{\infty}$ of positive integers satisfying that

$$\int_{\bigcup_{j \in \mathbb{N}, j \neq i} A_{n_j}} \|f_{n_i}(\omega)\|^p \, d\mu(\omega) < \frac{\epsilon}{m}$$

for each $i \in \mathbb{N}$. Since $(a+b)^p \leq a^p + b^p$ for $a, b \geq 0$ then $|x+y|^p \geq \big|\, |x| - |y| \,\big|^p \geq \big|\, |x|^p - |y|^p \big|$, which applies to get

$$K \geq \left\| \sum_{j=1}^{m} f_{n_j} \right\|_p \geq \sum_{i=1}^{m} \int_{A_{n_i}} \|f_{n_i}(\omega)\|^p \, d\mu(\omega) - \sum_{i=1}^{m} \int_{A_{n_i}} \left\| \sum_{j=1, j \neq i}^{m} f_{n_j}(\omega) \right\|^p d\mu(\omega).$$

So the fact that $|x+y|^p \leq |x|^p + |y|^p$ yields

$$\begin{aligned} K &\geq \sum_{i=1}^{m} \int_{A_{n_i}} \|f_{n_i}(\omega)\|^p \, d\mu(\omega) - \sum_{i=1}^{m} \sum_{\substack{j=1 \\ j \neq i}}^{m} \int_{A_{n_i}} \left\|f_{n_j}(\omega)\right\|^p d\mu(\omega) \\ &\geq \sum_{i=1}^{m} \int_{A_{n_i}} \|f_{n_i}(\omega)\|^p \, d\mu(\omega) - \sum_{j=1}^{m} \int_{\bigcup_{i \in \mathbb{N}, i \neq j} A_{n_i}} \left\|f_{n_j}(\omega)\right\|^p d\mu(\omega), \end{aligned}$$

which implies that $K \geq (m-1)\epsilon > K$, a contradiction. ■

**Corollary 2** *The space $L_p(\mu, X)$ with $0 < p < 1$ contains a copy of $c_0$ if and only if $X$ does.*

PROOF. Assume that $J$ is an isomorphism from $c_0$ into $L_p(\mu, X)$ and set $f_n = Je_n$ for each $n \in \mathbb{N}$. As in the proof of Corollary 1 one may show that $\inf_{n \in \mathbb{N}} \|f_n\|_p > 0$. By Lemma 2 the sequence $\{\|f_n(\cdot)\|^p\}_{n=1}^{\infty}$ is uniformly integrable. Hence if a sequence of representatives verifies that $f_n \to 0$ almost surely, an application of Vitali's lemma [3, Theorem 4.10.9] yields $\|f_n\|_p \to 0$, a contradiction. Thus $\{f_n\}_{n=1}^{\infty}$ does not converge almost surely to zero. As in addition the series $\sum_{n=1}^{\infty} \zeta_n f_n$ converges in $L_p(\mu, X)$, so in $L_0(\mu, X)$, for each $\zeta \in c_0$, Theorem 2 assures that $X$ contains a copy of $c_0$. ■

**Remark 1** *The same argument can be applied to show that $L_p(\mu, X)$, $1 \leq p < \infty$, contains a copy of $c_0$ if and only if $X$ does since, if $\{f_n\}_{n=1}^{\infty}$ is a normalized basic sequence in $L_p(\mu, X)$ equivalent to the unit vector basis of $c_0$, it can be easily shown that the sequence $\{\|f_n(\cdot)\|^p\}_{n=1}^{\infty}$ of $L_1(\mu)$ is also uniformly integrable. So Vitali's lemma together with Theorem 2 assure that $X$ contains a copy of $c_0$.*

# 3 Copies of $c_0$ in $P_1(\mu, X)$

Theorem 2 also applies to get the following well known result concerning copies of $c_0$ in the space of $\mu$-measurable $X$-valued Pettis integrable functions.

**Theorem 3 (Ferrando [4], Freniche [6])** *$P_1(\mu, X)$ contains a copy of $c_0$ if and only if $X$ does.*

PROOF. Assume by contradiction that $X$ contains no copy of $c_0$ and let $\{f_n\}_{n=1}^{\infty}$ be a normalized basic sequence in $P_1(\mu, X)$ equivalent to the unit vector basis of $c_0$. Since for each $x^* \in X^*$ and $E \in \Sigma$ the mapping $f \mapsto \int_E x^* f \, \mathrm{d}\mu$ is a bounded linear functional on $P_1(\mu, X)$ and the series $\sum_{n=1}^{\infty} f_n$ is wuC in $P_1(\mu, X)$, then the series $\sum_{n=1}^{\infty} \int_E f_n \, \mathrm{d}\mu$ is wuC in $X$ for all $E \in \Sigma$. If $\int_F f_n \, \mathrm{d}\mu \nrightarrow 0$ for some $F \in \Sigma$ then for sure that $X$ contains a copy of $c_0$, a contradiction. Hence $\int_E f_n \, \mathrm{d}\mu \to 0$ for all $E \in \Sigma$. Consequently, if we assume that there is a sequence of representatives verifying that $f_n \to 0$ almost surely, then [6, Lemma 3] assures that $\|f_n\|_{P_1(\mu,X)} \to 0$, another contradiction. So we must conclude that no sequence of representatives $\{f_n\}$ converges almost surely to zero.

Since the canonical inclusion map $T$ from $P_1(\mu, X)$ into $L_0(\mu, X)$ has closed graph [6, Lemma 4], the closed graph theorem for topological vector spaces [8, 15.12.(3)] guarantees that the restriction $S$ of $T$ to the copy $[f_n]$ of $c_0$ in $P_1(\mu, X)$ into $L_0(\mu, X)$ is continuous. Since $\sum_{n=1}^{\infty} \zeta_n f_n$ converges in $P_1(\mu, X)$ for every $\zeta \in c_0$, it follows that $\sum_{n=1}^{\infty} \zeta_n f_n$ converges in $L_0(\mu, X)$ for every $\zeta \in c_0$.

Since $\{f_n\}$ does not converge almost surely to zero and $\sum_{n=1}^{\infty} \zeta_n f_n$ converges in $L_0(\mu, X)$ for every $\zeta \in c_0$, Theorem 2 applies to show that $X$ contains a copy of $c_0$. This last contradiction completes the proof. ■

## 4 Complemented copies of $c_0$ in $C_0(\Omega)$

Throughout this section $\Omega$ will stand for a non-empty Hausdorff topological space. The following result adapts some ideas of [5] to show that if $\Omega$ is a noncompact hemicompact topological space then $C_0(\Omega)$ contains a norm-one complemented copy of $c_0$. Let us recall that for compact $\Omega$ then $C_0(\Omega) = C(\Omega)$ may contain no complemented copy of $c_0$, which happens for instance if $\Omega$ is extremally disconnected.

**Theorem 4** *If $\Omega$ is a non compact hemicompact space, then the Banach space $C_0(\Omega)$ contains a norm-one complemented copy of $c_0$*

PROOF. Since $\Omega$ is a noncompact hemicompact space there is a strictly increasing sequence $\{K_n\}$ of compact subsets satisfying the property that every compact subset $K$ of $\Omega$ is contained in a member of this sequence. For every $n \in \mathbb{N}$, set $\Delta_n := K_{n+1} \setminus K_n$.

Given that $\Delta_n$ is a relative open subset of the compact space $K_{n+1}$, choose $\omega_n \in \Delta_n$ and apply Urysohn's lemma in $K_{n+1}$ to get $f_n \in C_C(K_{n+1})$ with $0 \le f_n \le 1$ such that $f_n(\omega_n) = 1$ and $\operatorname{supp} f_n \subseteq \Delta_n$. Assuming each $f_n$ extended to the whole space $\Omega$ in a trivial way, then $\{f_n\}$ is a normalized basic sequence in $C_0(\Omega)$ equivalent to the unit vector basis $\{e_n\}$ of $c_0$. In fact, if $f = \sum_{n=1}^{\infty} a_n f_n \in [f_i]$ with $a_n \to 0$, given $\epsilon > 0$ choose $m \in \mathbb{N}$ such that $|a_n| < \epsilon$ for each $n > m$. Then $K := \bigcup_{i=1}^{m} \operatorname{supp} f_i$ is a compact set and $|f(\omega)| < \epsilon$ for each $\omega \in \Omega \setminus K$, hence $f \in C_0(\Omega)$.

If $\mu := \sum_{i=1}^{\infty} \frac{1}{2^i} \delta_{\omega_i}$, since each $f \in C_0(\Omega)$ is $\mu$-integrable, the conditional expectation value $\mathbf{E}_{\Delta_n}(f)$ of the random variable $f$ relative to the event $\Delta_n$, given by

$$\mathbf{E}_{\Delta_n}(f) = \frac{1}{\mu(\Delta_n)} \int_{\Delta_n} f \, \mathrm{d}\mu,$$

is well-defined for each $n \in \mathbb{N}$. Moreover, it must be pointed out that

$$\int_{\Delta_n} f_n \, \mathrm{d}\mu = \mu(\Delta_n) = \frac{1}{2^n} \tag{1}$$

for every $n \in \mathbb{N}$. Given $f \in C_0(\Omega)$, for each $\epsilon > 0$ there exists a compact set $K_{f,\epsilon} \subseteq \Omega$ such that $|f(\omega)| < \epsilon$ for all $\omega \in \Omega \setminus K_{f,\epsilon}$. Since $\Omega$ is hemicompact, there is $p \in \mathbb{N}$ such that $K_{f,\epsilon} \subseteq K_p$. Hence $|f(\omega)| < \epsilon$ for each $\omega \in \Delta_n$ with $n \ge p$, which implies that

$$|\mathbf{E}_{\Delta_n}(f)| \le \frac{1}{\mu(\Delta_n)} \int_{\Delta_n} |f(\omega)| \, \mathrm{d}\mu(\omega) \le \sup_{\omega \in \Delta_n} |f(\omega)| \le \epsilon$$

whenever $n \geq p$. This assures that $\mathbf{E}_{\Delta_n}(f) \to 0$ and, consequently, that the linear operator $P\colon C_0(\Omega) \to C_0(\Omega)$ given by $Pf = \sum_{n=1}^{\infty} \mathbf{E}_{\Delta_n}(f)\, f_n$ is well-defined as well.

Given $\omega \in \Omega$, since (i) $f_n(\omega) = 0$ if $\omega \notin \Delta_n$, (ii) $\|f_n\|_\infty = 1$ and (iii) $|\mathbf{E}_{\Delta_n}(f)| \leq \|f\|_\infty$ for each $f \in C_0(\Omega)$, if $\omega \in \Delta_m$ then

$$|(Pf)(\omega)| = \left| \left( \sum_{n=1}^{\infty} \mathbf{E}_{\Delta_n}(f)\, f_n \right)(\omega) \right| \leq |\mathbf{E}_{\Delta_m}(f)| \leq \|f\|_\infty .$$

This shows that $P$ is a bounded linear operator with $\|P\| \leq 1$. Finally, as a consequence of (1) and of the fact that the $f_n$ are disjointly supported it follows that $\mathbf{E}_{\Delta_i}(f_j) = \delta_{ij}$, so that

$$Pf_i = \sum_{n=1}^{\infty} \mathbf{E}_{\Delta_n}(f_i)\; f_n = f_i$$

for every $i \in \mathbb{N}$. So we conclude that $P$ is a norm-one bounded linear projection operator from $C_0(\Omega)$ onto $[f_i]$. ■

**Acknowledgement.** This research has been supported by the project MTM2008-01502 of the Spanish Ministry of Science and Innovation.

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**Juan Carlos Ferrando**
Centro de Investigación Operativa,
Universidad Miguel Hernández.
03202 Elche (Alicante). Spain.
jc.ferrando@umh.es