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On the structure of certain ultradistributions

Manuel Valdivia

Abstract. Let Ω be a nonempty open subset of the k-dimensional euclidean space $\mathbb{R}^k$. In this paper we show that, if S is an ultradistribution in Ω, belonging to a class of Roumieu type stable under differential operators, then there is a family f_α, $\alpha \in \mathbb{N}_0^k$, of elements of $\mathcal{L}^\infty_{\mathrm{loc}}(\Omega)$ such that S is represented in the form $\sum_{\alpha\in\mathbb{N}_0^k} D^\alpha f_\alpha$. Some other results on the structure of certain ultradistributions of Roumieu type are also given.

Sobre la estructura de ciertas ultradistribuciones

Resumen. Sea Ω un subconjunto abierto no vacío del espacio euclídeo k-dimensional $\mathbb{R}^k$. En este trabajo demostramos que si S es una ultradistribución en Ω, perteneciente a una clase de tipo Roumieu estable bajo operadores diferenciales, entonces existe una familia f_α, $\alpha \in \mathbb{N}_0^k$, de elementos de $\mathcal{L}^\infty_{\mathrm{loc}}(\Omega)$ tal que S se representa en la forma $\sum_{\alpha\in\mathbb{N}_0^k} D^\alpha f_\alpha$. También se dan otros resultados sobre la estructura de ciertas ultradistribuciones de tipo Roumieu.

1 Introduction and notation

Throughout this paper all linear spaces are assumed to be defined over the field $\mathbb{C}$ of complex numbers. We write $\mathbb{N}$ for the set of positive integers and by $\mathbb{N}_0$ we mean the set of nonnegative integers.

If E is a locally convex space, E' will be its topological dual and $\langle\cdot,\cdot\rangle$ will denote the standard duality between E and E'. $\sigma(E',E)$ denotes the weak topology in E' and $\beta(E',E)$ is the strong topology in E'. E'' stands for the topological dual of E' $[\beta(E',E)]$. We identify in the usual manner E with a linear subspace of E''. We represent by $\rho(E,E')$ the topology in E given by the uniform convergence on every compact absolutely convex subset of E' $[\beta(E',E)]$. If A is a closed bounded absolutely convex subset of E, we write E_A to denote the normed space given by the linear span of A in E, with A as closed unit ball.We say that a subset B of E is locally compact (weakly compact) whenever there is a closed bounded absolutely convex subset A of E such that B is contained in E_A and it is a compact (weakly compact) subset in this space.

Given a Banach space X, $B(X)$ denotes its closed unit ball and X^* is the Banach space conjugate of X.

Given a positive integer k, if $\alpha := (\alpha_1,\alpha_2,\dots,\alpha_k)$ is a multiindex of order k, i.e., an element of $\mathbb{N}_0^k$, we put $|\alpha|$ for its length, that is, $|\alpha| = \alpha_1+\alpha_2+\dots+\alpha_k$, and $\alpha! := \alpha_1!\alpha_2!\cdots\alpha_k!$.

Given a complex function f, defined in the points $x = (x_1,x_2,\dots,x_k)$ of an open subset O of the k-dimensional euclidean space, which is infinitely differentiable, we write

$$D^\alpha f(x) := \frac{\partial^{|\alpha|} f(x)}{\partial x_1^{\alpha_1}\partial x_2^{\alpha_2}\dots\partial x_k^{\alpha_k}}, \qquad x\in O, \quad \alpha\in\mathbb{N}_0^k.$$

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We take a sequence of positive numbers $M_0, M_1, \ldots, M_n, \ldots$ satisfying the following conditions:

1. $M_0 = 1$.
2. Logarithmic convexity:
$$M_n^2 \leq M_{n-1}M_{n+1}, \qquad n \in \mathbb{N}.$$
3. Non quasi-analiticity:
$$\sum_{n=1}^{\infty} \frac{M_{n-1}}{M_n} < \infty.$$

We consider an open subset Ω of $\mathbb{R}^k$. We shall say that a complex function f defined and infinitely differentiable in Ω is *ultradifferentiable* of class $\{M_n\}$ if, given an arbitrary compact subset K of Ω, there exist $C > 0$ and $h > 0$ such that

$$|D^\alpha f(x)| \leq C\, h^{|\alpha|} M_{|\alpha|}, \qquad x \in K,\ \alpha \in \mathbb{N}_0^k.$$

We put $\mathcal{E}^{\{M_n\}}(\Omega)$ to denote the linear space over $\mathbb{C}$ of all the ultradifferentiable complex functions of class $\{M_n\}$. We write $\mathcal{D}^{\{M_n\}}(\Omega)$ to mean the linear subspace of $\mathcal{E}^{\{M_n\}}(\Omega)$ formed by those functions that have compact support.

Given $h > 0$ and a compact subset K of Ω, by $\mathcal{D}^{(M_n),h}(K)$ we denote the linear space over $\mathbb{C}$ of the complex functions f, defined and infinitely differentiable in Ω, with support in K, such that

$$|f|_h := \sup_{\alpha \in \mathbb{N}_0^k} \sup_{x \in \Omega} \frac{|D^\alpha f(x)|}{h^{|\alpha|} M_{|\alpha|}} < \infty.$$

We assume that $\mathcal{D}^{(M_n),h}(K)$ is endowed with the norm $|\cdot|_h$.

We take now a fundamental system of compact subsets of Ω:

$$K_1 \subset K_2 \subset \cdots \subset K_m \subset \cdots$$

We have that

$$\mathcal{D}^{\{M_n\}}(\Omega) = \bigcup_{m=1}^{\infty} \mathcal{D}^{(M_n),m}(K_m).$$

We consider $\mathcal{D}^{\{M_n\}}(\Omega)$ as the inductive limit of the sequence $(\mathcal{D}^{(M_n),m}(K_m))$ of Banach spaces. The elements of the topological dual $\mathcal{D}^{\{M_n\}'}(\Omega)$ of $\mathcal{D}^{\{M_n\}}(\Omega)$ are called *ultradistributions* in Ω of the Roumieu type. We assume that $\mathcal{D}^{\{M_n\}'}(\Omega)$ is endowed with the strong topology.

By $\mathcal{K}(\Omega)$ we represent the linear space over $\mathbb{C}$ of the complex functions defined continuous and with compact support in Ω. If K is a compact subset of Ω, we use $\mathcal{K}(K)$ to denote the linear subspace of $\mathcal{K}(\Omega)$ formed by those functions with support contained in K. For f in $\mathcal{K}(K)$, we put

$$|f|_\infty := \sup_{x \in \Omega} |f(x)|.$$

We assume $\mathcal{K}(K)$ is provided with the norm $|\cdot|_\infty$. We shall consider $\mathcal{K}(\Omega)$ as the inductive limit of the sequence of Banach spaces $(\mathcal{K}(K_m))$. A Radon measure u in Ω is an element of the topological dual $\mathcal{K}'(\Omega)$ of $\mathcal{K}(\Omega)$. Given a Radon measure u in Ω and a compact subset K of Ω, we write $\|u\|(K)$ to indicate the norm of the restriction of u to the Banach space $\mathcal{K}(K)$.

We use $\mathcal{L}^\infty_{\rm loc}(\Omega)$ to denote the linear space over $\mathbb{C}$ formed by the complex functions defined and Lebesgue measurable in Ω which are essentially bounded in each compact subset of Ω. Given a compact subset K of Ω and $f \in \mathcal{L}^\infty_{\rm loc}(\Omega)$, we put $|f|_{K,\infty}$ to denote the essential supremum of f in K, that is,

$$|f|_{K,\infty} := \operatorname{supess}\{\, |f(x)| : \ x \in K \,\}.$$

In the usual way, we shall consider the elements of $\mathcal{L}^{\infty}_{\rm loc}(\Omega)$ as distributions in Ω.

In [4] and [5], a theorem on the structure of ultradistributions is given which we can state in the following way:

Result a) *Let u_α, $\alpha \in \mathbb{N}_0^k$, be a family of Radon measures in $\mathbb{R}^k$ such that, for each $h > 0$ and each compact subset K of $\mathbb{R}^k$, we have that*

$$\sup_{\alpha\in\mathbb{N}_0^k} h^{|\alpha|} M_{|\alpha|} \|u_\alpha\|(K) < \infty. \tag{1}$$

Then the formula

$$S := \sum_{\alpha\in\mathbb{N}_0^k} D^\alpha u_\alpha \tag{2}$$

defines an element of $\mathcal{D}^{\{M_n\}'}(\mathbb{R}^k)$. Conversely, each ultradistribution S of $\mathcal{D}^{\{M_n\}'}(\mathbb{R}^k)$ may be written in the form of (2) for a family u_α, $\alpha \in \mathbb{N}_0^k$, of Radon measures in $\mathbb{R}^k$ satisfying condition (1).

In [3], certain objections to the method of proof used by Roumieu to obtain result a) are indicated and therefore the result is left as an open question. This led Komatsu to obtain result a) for ultradistributions in Ω under the additional assumption for the converse part that the class $\{M_n\}$ of ultradifferentiable functions in Ω be stable for differential operators, that is, there exist $A > 0$ and $L > 0$ such that

$$M_{n+1} \le A\, L^n M_n, \qquad n \in \mathbb{N}_0. \tag{3}$$

In this paper, in the first place, we give proof of the Roumieu-Komatsu result with no need of condition (3), i.e., we recover result a) . On the other hand, the method used to achieve this result will be used later to obtain the main result of this paper, which has as a particular case the following:

Result b) *If M_n, $n \in \mathbb{N}_0$, satisfies condition (3), then, given S in $\mathcal{D}^{\{M_n\}'}(\Omega)$, there is a family f_α, $\alpha \in \mathbb{N}_0^k$, of elements of $\mathcal{L}^{\infty}_{\rm loc}(\Omega)$ such that, for each compact subset K of Ω and $h > 0$, we have that*

$$\sup_{\alpha\in\mathbb{N}_0^k} h^{|\alpha|} M_{|\alpha|} |f_\alpha|_{K,\infty} < \infty$$

and

$$S = \sum_{\alpha\in\mathbb{N}_0^k} D^\alpha f_\alpha,$$

where the series is absolutely and uniformly convergent on every bounded subset of $\mathcal{D}^{\{M_n\}}(\Omega)$.

2 Basic constructions

Let X be a Banach space. We use $\|\cdot\|$ to denote its norm and also for the norm of X^*. Given $r \in \mathbb{N}$ and $\alpha \in \mathbb{N}_0^k$, we put, for each $x \in X$,

$$|x|_{r,\alpha} := \frac{\|x\|}{r^{|\alpha|} M_{|\alpha|}}.$$

We write $X_{r,\alpha}$ for the linear space X with the norm $|\cdot|_{r,\alpha}$, and $X^*_{r,\alpha}$ for the conjugate of $X_{r,\alpha}$. The norm of $X^*_{r,\alpha}$ will be denoted by $|\cdot|_{r,\alpha}$. Clearly, if u belongs to X^*, then

$$|u|_{r,\alpha} = r^{|\alpha|} M_{|\alpha|} \|u\|.$$

We represent by Y_r the linear space over $\mathbb{C}$ formed by the families $(\, x_\alpha : \alpha \in \mathbb{N}_0^k \,)$ of elements of X, which we shall simply denote by (x_α), such that

$$\|(x_\alpha)\|_r := \sup_{\alpha\in\mathbb{N}_0^k} \frac{\|x_\alpha\|}{r^{|\alpha|} M_{|\alpha|}} < \infty.$$

We endow Y_r with the norm of $\|\cdot\|_r$. It follows that $Y_r \subset Y_{r+1}$ and that the canonical injection from Y_r into Y_{r+1} is continuous. We write Y for the inductive limit of the sequence of Banach spaces (Y_r).

Let (x_α) be an element of Y and let β be in $\mathbb{N}_0^k$, we define

$$x_\alpha^\beta := \begin{cases} x_\beta, & \text{if } \alpha = \beta, \\ 0, & \text{if } \alpha \neq \beta \end{cases}$$

Clearly, if (x_α) is in Y_r, then (x_α^β) belongs to Y_r and

$$\|(x_\alpha^\beta)\|_r \leq \|(x_\alpha)\|_r.$$

For fixed $r \in \mathbb{N}$ and $\beta \in \mathbb{N}_0^k$, we put Y_r^β to denote the subspace of Y_r formed by those elements (x_α) which satisfy that $x_\alpha = 0$ when $\alpha \neq \beta$. Then

$$\|(x_\alpha)\|_r = \sup_{\alpha \in \mathbb{N}_0^k} \frac{\|x_\alpha\|}{r^{|\alpha|} M_{|\alpha|}} = \frac{\|x_\beta\|}{r^{|\beta|} M_{|\beta|}} = |x|_{r,\beta},$$

and thus Y_r^β is isometric to $X_{r,\beta}$. On the other hand, if we denote by Y^β the subspace of Y whose elements (x_α) satisfy that $x_\alpha = 0$ when $\alpha \neq \beta$, then Y^β is topologically isomorphic to X.

We assume that Y' is provided with the strong topology. If Y_r^* is the Banach space given by the conjugate of Y_r, then the projective limit of the sequence of Banach spaces (Y_r^*) is a Fréchet space which coincides with Y'. Setting U_r to be the polar set of $rB(Y_r)$ in Y', it follows that U_r, $r \in \mathbb{N}$, is a fundamental system of zero neighborhoods in Y'. If we consider a bounded subset B of Y, its polar set B° in Y' is a zero neighborhood in this space and so there is $s \in \mathbb{N}$ such that $B^\circ \supset U_s$, hence B is contained in the closure of $sB(Y_s)$ in Y.

Proposition 1. *The following properties hold:*

1. *For each r in $\mathbb{N}$, $B(Y_r)$ is a closed subset of Y.*
2. *If B is a bounded subset of Y, there is $r \in \mathbb{N}$ such that B is contained in Y_r and it is a bounded subset of this space.*

PROOF. 1. We take $v \in X^*$ and $\beta \in \mathbb{N}_0^k$. Let u be the linear functional on Y such that

$$u((x_\alpha)) = \langle x_\beta, v \rangle, \qquad (x_\alpha) \in Y.$$

Given $s \in \mathbb{N}$ and $(x_\alpha) \in Y_s$, we have

$$|u((x_\alpha))| = |\langle x_\beta, v \rangle| \leq \|x_\beta\| \cdot \|v\| = \frac{\|x_\beta\|}{s^{|\beta|} M_{|\beta|}} s^{|\beta|} M_{|\beta|} \|v\| \leq (s^{|\beta|} M_{|\beta|} \|v\|) \|(x_\alpha)\|_s$$

and so u belongs to Y'.

We take a net $\{ x_{\alpha,j} : j \in J, \geq \}$ in $B(Y_r)$ which converges to (x_α) in Y. We fix $\beta \in \mathbb{N}_0^k$ and take $v \in X^*$. Let w be the element of Y' such that

$$\langle (y_\alpha), w \rangle = \langle y_\beta, v \rangle, \qquad (y_\alpha) \in Y.$$

Then

$$0 = \lim_j \langle (x_\alpha - x_{\alpha,j}), w \rangle = \lim_j \langle x_\beta - x_{\beta,j}, v \rangle$$

and thus the net $\{ x_{\beta,j} : j \in J, \geq \}$ converges weakly to x_β in X. Given that

$$|x_{\beta,j}|_{r,\beta} = \frac{\|x_{\beta,j}\|}{r^{|\beta|} M_{|\beta|}} \leq \sup_{\alpha \in \mathbb{N}_0^k} \frac{\|x_{\alpha,j}\|}{r^{|\alpha|} M_{|\alpha|}} = \|(x_{\alpha,j})\|_r \leq 1,$$

it follows that $|x_\beta|_{r,\beta} \leq 1$, from where we deduce that

$$\sup_{\alpha\in\mathbb{N}_0^k} |x_\alpha|_{r,\alpha} = \sup_{\alpha\in\mathbb{N}_0^k} \frac{\|x_\alpha\|}{r^{|\alpha|}M_{|\alpha|}} \leq 1$$

and hence (x_α) is an element of Y_r that belongs to $B(Y_r)$.
2. We know there is $r \in \mathbb{N}$ such that B is contained in the closure of $rB(Y_r)$ in Y. Making use of part 1 the result follows. ■

If u is an arbitrary element of Y', we put, for each $r \in \mathbb{N}$,

$$\|u\|_{(r)} := \sup\{\, |\langle (x_\alpha), u\rangle| : (x_\alpha) \in B(Y_r)\,\}.$$

For every $u \in Y'$ and every $\beta \in \mathbb{N}_0^k$, we identify in a natural manner the restriction of u to Y^β with an element u_β of X^*.

Proposition 2. *For $u \in Y'$ and $r \in \mathbb{N}$, we have*

$$\sup_{\alpha\in\mathbb{N}_0^k} r^{|\alpha|}M_{|\alpha|}\|u_\alpha\| \leq \|u\|_{(r)}$$

and

$$\langle (x_\alpha), u\rangle = \sum_{\alpha\in\mathbb{N}_0^k} \langle x_\alpha, u_\alpha\rangle, \qquad (x_\alpha) \in Y.$$

PROOF. We fix $\beta \in \mathbb{N}_0^k$. We then have that

$$\begin{aligned}\|u\|_{(r)} &= \sup\{\, |\langle (x_\alpha), u\rangle| : (x_\alpha) \in B(Y_r)\,\}\\ &\geq \sup\{\, |\langle (x_\alpha^\beta), u\rangle| : (x_\alpha) \in B(Y_r)\,\}\\ &= \sup\{\, |\langle x_\beta, u_\beta\rangle| : (x_\alpha) \in B(Y_r)\,\}\\ &= |u_\beta|_{r,\beta}\\ &= r^{|\beta|}M_{|\beta|}\|u_\beta\|\end{aligned}$$

from where

$$\sup_{\alpha\in\mathbb{N}_0^k} r^{|\alpha|}M_{|\alpha|}\|u_\alpha\| \leq \|u\|_{(r)}.$$

Let us now take (x_α) in Y_r and we see that the family $\big((x_\alpha^\beta) : \beta \in \mathbb{N}_0^k\big)$ is summable to (x_α) in Y_s for every integer $s \geq 2r$. Given an arbitrary $q \in \mathbb{N}$, we have that

$$\begin{aligned}\Big\|(x_\alpha) - \sum_{|\beta|\leq q}(x_\alpha^\beta)\Big\|_s &= \sup_{|\alpha|>q} \frac{\|x_\alpha\|}{s^{|\alpha|}M_{|\alpha|}}\\ &\leq \sup_{|\alpha|>q} \frac{\|x_\alpha\|}{(2r)^{|\alpha|}M_{|\alpha|}} = \sup_{|\alpha|>q} \frac{1}{2^{|\alpha|}}\frac{\|x_\alpha\|}{r^{|\alpha|}M_{|\alpha|}}\\ &\leq \frac{1}{2^q}\sup_{\alpha\in\mathbb{N}_0^k} \frac{\|x_\alpha\|}{r^{|\alpha|}M_{|\alpha|}} = \frac{1}{2^q}\|(x_\alpha)\|_r,\end{aligned}$$

and the conclusion is obtained. It then follows that, if (x_α) is any element of Y, we have that, in this space,

$$(x_\alpha) = \sum_{\beta\in\mathbb{N}_0^k} (x_\alpha^\beta)$$

and thus

$$\langle (x_\alpha), u\rangle = \sum_{\alpha\in\mathbb{N}_0^k} \langle (x_\alpha^\beta), u\rangle = \sum_{\beta\in\mathbb{N}_0^k} \langle x_\beta, u_\beta\rangle.$$ ■

Proposition 3. *If M is a bounded subset of Y' and $r \in \mathbb{N}$, then*

$$\sup_{\substack{\alpha \in \mathbb{N}_0^k \\ u \in M}} r^{|\alpha|} M_{|\alpha|} \|u_\alpha\| < \infty.$$

PROOF. We have that

$$U := \{ v \in Y' : \|v\|_{(r)} \leq 1 \}$$

is a zero neighborhood in Y', hence there is $b > 0$ such that $bM \subset U$. If $u \in M$, then $\|u\|_{(r)} \leq b^{-1}$ and, after the previous proposition, the result follows. ■

Proposition 4. *If $(z_\alpha : \alpha \in \mathbb{N}_0^k)$ is a family of elements of X^* such that, for each $h > 0$,*

$$\sup_{\alpha \in \mathbb{N}_0^k} h^{|\alpha|} M_{|\alpha|} \|z_\alpha\| < \infty,$$

then there is a unique element u of Y' for which $u_\alpha = z_\alpha$, $\alpha \in \mathbb{N}_0^k$.

PROOF. We fix $\beta \in \mathbb{N}_0^k$, $r \in \mathbb{N}$ and $(x_\alpha) \in Y_r$. Then

$$\begin{aligned} |\langle x_\beta, z_\beta \rangle| \leq \|x_\beta\| \cdot \|z_\beta\| &= \frac{\|x_\beta\|}{(2kr)^{|\beta|} M_{|\beta|}} (2kr)^{|\beta|} M_{|\beta|} \|z_\beta\| \\ &\leq \frac{1}{(2k)^{|\beta|}} \|(x_\alpha)\|_r \sup_{\alpha \in \mathbb{N}_0^k} (2kr)^{|\alpha|} M_{|\alpha|} \|z_\alpha\| \end{aligned}$$

and consequently

$$\begin{aligned} \sum_{\beta \in \mathbb{N}_0^k} |\langle x_\beta, z_\beta \rangle| &\leq \|(x_\alpha)\|_r \sup_{\alpha \in \mathbb{N}_0^k} (2kr)^{|\alpha|} M_{|\alpha|} \|z_\alpha\| \sum_{\beta \in \mathbb{N}_0^k} \frac{1}{(2k)^{|\beta|}} \\ &= \Big(2 \sup_{\alpha \in \mathbb{N}_0^k} (2kr)^{|\alpha|} M_{|\alpha|} \|z_\alpha\| \Big) \cdot \|(x_\alpha)\|_r, \end{aligned}$$

from where it follows that the complex function u defined in Y such that

$$u\big((x_\alpha)\big) := \sum_{\alpha \in \mathbb{N}_0^k} \langle x_\alpha, z_\alpha \rangle, \qquad (x_\alpha) \in Y,$$

which is clearly linear, is also continuous. After Proposition 2, we have that

$$\langle (x_\alpha), u \rangle = \sum_{\alpha \in \mathbb{N}_0^k} \langle x_\alpha, u_\alpha \rangle, \qquad (x_\alpha) \in Y.$$

We fix $\beta \in \mathbb{N}_0^k$ and take an arbitrary element $x_\beta \in X$. Then, if $x_\alpha^\beta := 0$, $\alpha \neq \beta$, and $x_\beta^\beta := x_\beta$, we have that

$$\langle x_\beta, u_\beta \rangle = \langle (x_\alpha^\beta), u \rangle = \sum_{\alpha \in \mathbb{N}_0^k} \langle x_\alpha^\beta, z_\alpha \rangle = \langle x_\beta, z_\beta \rangle$$

and so $u_\beta = z_\beta$, $\beta \in \mathbb{N}_0^k$. The uniqueness of u follows from the density in Y of the linear span of $\cup \{ Y^\beta : \beta \in \mathbb{N}_0^k \}$. ■

Proposition 5. *Let E be a locally convex space such that $E'\,[\beta(E', E)]$ is a Fréchet space. Let F be a linear subspace of E such that each closed bounded absolutely convex subset of F is locally weakly compact, then F is closed in $E''\,[\sigma(E'', E')]$ and, provided with the Mackey topology $\mu(F, F')$, it is an (LB)-space.*

PROOF. Let v be a linear functional on F, which is bounded on every bounded subset of F. Let A be a closed bounded absolutely convex subset of $E''\,[\sigma(E'',E')]$. We find a subset M of F, closed bounded and absolutely convex, such that $A\cap F$ is contained in F_M and it is weakly compact in this space. Consequently, $A\cap F$ is weakly compact in $E''\,[\sigma(E'',E')]$ and so $v^{-1}(0)\cap A$ is weakly compact in $E''\,[\sigma(E'',E')]$. We apply Krein-Smulyan's theorem, [2, p. 246], and we have that $v^{-1}(0)$ is closed in $E''\,[\sigma(E'',E')]$. Thus F is closed in $E''\,[\sigma(E'',E')]$ and v is continuous in F. The result now is clear. ■

Proposition 6. *Let E be a locally convex space such that $E'\,[\beta(E',E)]$ is a Fréchet space. If F is a subspace of E such that every closed bounded absolutely convex subset of F is locally compact, then F, with the topology induced by $\rho(E,E')$ is an (LB)-space.*

PROOF. We apply the former proposition and obtain that $F\,[\mu(F,F')]$ is an (LB)-space and F is closed in $E''\,[\sigma(E'',E')]$. Let $F^{\perp}$ stand for the subspace of E' orthogonal to F. Let ψ be the canonical mapping from E' onto $E'/F^{\perp}$. Every closed bounded absolutely convex subset of F is locally compact in $E''\,[\sigma(E'',E')]$, from where we have that $E'\,[\beta(E',E)]/F^{\perp}$ is a Fréchet-Montel space and thus $\mu(F,F')=\rho(F,E'/F^{\perp})$. We now consider a closed absolutely convex zero neighborhood U in F. Let U° be the polar of U in $E'/F^{\perp}$. It follows that U° is compact in $E'\,[\beta(E',E)]/F^{\perp}$, hence we may find a compact absolutely convex subset P of $E'\,[\beta(E',E)]$ such that $\psi(P)=U^{\circ}$, [2, p. 274]. If P° is the polar set of P in E, we have that $P^{\circ}\cap F=U$, and the result follows. ■

Proposition 7. *If X is reflexive, then Y is weakly locally compact.*

PROOF. Given a positive integer r, we take a sequence $(x_{\alpha,m})$, $m=1,2,\ldots$ in $B(Y_r)$. For each α of $\mathbb{N}_0^k$, the sequence $x_{\alpha,m}$, $m=1,2,\ldots$, belongs to $B(X_{r,\alpha})$, hence we may find by means of a diagonal process a subsequence $(y_{\alpha,m})$, $m=1,2,\ldots$, of $(x_{\alpha,m})$ such that, for each $\alpha\in\mathbb{N}_0^k$, the sequence $y_{\alpha,m}$, $m=1$, $2,\ldots$, converges weakly in $X_{r,\alpha}$ to an element y_α which will clearly lie in $B(X_{r,\alpha})$. We deduce then that $(y_\alpha)\in B(Y_r)$. We take now an integer $s\geq 2r$ and we show that $(y_{\alpha,m})$ converges weakly to (y_α) in Y_s. In the proof of Proposition 2, we saw that, given (x_α) in Y_r and $q\in\mathbb{N}$, we have that

$$\Big\|(x_\alpha)-\sum_{|\beta|\leq q}(x_\alpha^\beta)\Big\|_s\leq\frac{1}{2^q}\|(x_\alpha)\|_r.$$

Thus, given $\varepsilon>0$, we find $q_0\in\mathbb{N}$ such that $1/2^{q_0}<\varepsilon/4$. Then

$$\Big\|(y_\alpha)-\sum_{|\beta|\leq q_0}(y_\alpha^\beta)\Big\|_s\leq\frac{1}{2^{q_0}}\|(y_\alpha)\|_r<\frac{\varepsilon}{4}$$

and, if $(y_{\alpha,m}^\beta)$ is the element of Y such that $y_{\alpha,m}^\beta=0$, if $\alpha\neq\beta$, and $y_{\beta,m}^\beta=y_{\beta,m}$, it follows that

$$\Big\|(y_{\alpha,m})-\sum_{|\beta|\leq q_0}(y_{\alpha,m}^\beta)\Big\|_s\leq\frac{1}{2^{q_0}}\|(y_{\alpha,m})\|_r<\frac{\varepsilon}{4}.$$

We now take u in $B(Y_s^*)$. We find $m_0\in\mathbb{N}$ such that

$$\sum_{|\beta|\leq q_0}\big|\langle(y_\alpha^\beta)-(y_{\alpha,m}^\beta),u\rangle\big|<\frac{\varepsilon}{2},\qquad m\geq m_0.$$

Then, for those values of m, we have that

$$\big|\langle(y_\alpha)-(y_{\alpha,m}),u\rangle\big|\left|\langle(y_\alpha)-(y_{\alpha,m})+\sum_{|\beta|\leq q_0}\langle(y_{\alpha,m}^\beta)-(y_\alpha^\beta),u\rangle\right|+\sum_{|\beta|\leq q_0}\big|\langle(y_\alpha^\beta)-(y_{\alpha,m}^\beta),u\rangle\big|$$

$$
\begin{aligned}
&\leq \Big|\langle (y_\alpha) - \sum_{|\beta|\leq q_0} (y_\alpha^\beta), u\rangle\Big| + \Big|\langle (y_{\alpha,m}) - \sum_{|\beta|\leq q_0} (y_{\alpha,m}^\beta), u\rangle\Big| + \frac{\varepsilon}{2} \\
&\leq \Big\|(y_\alpha) - \sum_{|\beta|\leq q_0} (y_\alpha^\beta)\Big\|_s + \Big\|(y_{\alpha,m}) - \sum_{|\beta|\leq q_0} (y_{\alpha,m}^\beta)\Big\|_s + \frac{\varepsilon}{2} \\
&< \varepsilon.
\end{aligned}
$$

We obtain from here that $B(Y_r)$ is a weakly compact subset of Y_s and the result follows. ■

3 The space $\mathcal{E}_0^{\{M_n\}}(\Omega)$

Given $h > 0$, we denote by $\mathcal{E}_0^{(M_n),h}(\Omega)$ the linear space over $\mathbb{C}$ of the complex functions f, defined and infinitely differentiable in Ω, which vanish at infinity and so do each of its derivatives of any order, that is, given $\beta \in \mathbb{N}_0^k$ and $\varepsilon > 0$, there is a compact subset K of Ω such that

$$|D^\beta f(x)| < \varepsilon, \qquad x \in \Omega \setminus K.$$

On the other hand, f also satisfies that there is $C > 0$ such that

$$|D^\alpha f(x)| \leq C\, h^{|\alpha|} M_{|\alpha|}, \qquad x \in \Omega, \quad \alpha \in \mathbb{N}_0^k.$$

We put

$$|f|_h := \sup_{\alpha \in \mathbb{N}_0^k} \sup_{x\in\Omega} \frac{|D^\alpha f(x)|}{h^{|\alpha|} M_{|\alpha|}}$$

and assume that $\mathcal{E}_0^{(M_n),h}(\Omega)$ is provided with the norm $|\cdot|_h$. We write

$$\mathcal{E}_0^{\{M_n\}}(\Omega) := \bigcup_{m=1}^\infty \mathcal{E}_0^{(M_n),m}(\Omega)$$

and consider $\mathcal{E}_0^{\{M_n\}}(\Omega)$ as the inductive limit of the sequence $(\mathcal{E}_0^{(M_n),m}(\Omega))$ of Banach spaces. We assume that the topological dual $\mathcal{E}_0^{\{M_n\}'}(\Omega)$ of $\mathcal{E}_0^{\{M_n\}}(\Omega)$ is endowed with the strong topology.

We put $C_0(\Omega)$ for the linear space over $\mathbb{C}$ of the complex functions f which are defined and continuous in Ω and vanish at infinity. We write

$$|f|_\infty := \sup_{x\in\Omega} |f(x)|$$

and assume that $C_0(\Omega)$ is provided with the norm $|\cdot|_\infty$.

In this section, we substitute the space X of the previous section by $C_0(\Omega)$. Then, each element of Y_r is a family $(\, f_\alpha : \alpha \in \mathbb{N}_0^k \,)$ of elements of $C_0(\Omega)$ such that

$$\|(f_\alpha)\|_r := \sup_{\alpha\in\mathbb{N}_0^k} \frac{|f_\alpha|_\infty}{r^{|\alpha|} M_{|\alpha|}} < \infty.$$

We denote by W_r the subspace of Y_r formed by those families $(\, D^\alpha f : \alpha \in \mathbb{N}_0^k \,)$ such that

$$f \in \mathcal{E}_0^{(M_n),r}(\Omega).$$

Let

$$\Phi_r \colon \mathcal{E}_0^{(M_n),r}(\Omega) \longrightarrow W_r$$

be such that

$$\Phi_r(f) = (D^\alpha f), \qquad f \in \mathcal{E}_0^{(M_n),r}(\Omega).$$

Then, Φ_r is an onto linear isometry. We put W for $\cup\{ W_r : r \in \mathbb{N} \}$ and we consider W as a subspace of Y. Let

$$\Phi\colon \mathcal{E}_0^{\{M_n\}}(\Omega) \longrightarrow W$$

be such that

$$\Phi(f) = (D^\alpha f), \qquad f \in \mathcal{E}_0^{\{M_n\}}(\Omega).$$

Clearly, Φ is a continuous one-to-one and onto linear map.

Theorem 1. *For each j of a certain set J, let $(\mu_{\alpha,j} : \alpha \in \mathbb{N}_0^k)$ a family of complex Borel measures in Ω such that, for each $h > 0$,*

$$\sup_{\substack{\alpha\in\mathbb{N}_0^k \\ j\in J}} h^{|\alpha|}\, M_{|\alpha|}\, |\mu_{\alpha,j}|(\Omega) < \infty.$$

Then, there is a bounded subset $\{ S_j : j \in J \}$ in $\mathcal{E}_0^{\{M_n\}'}(\Omega)$ such that

$$\langle \varphi, S_j\rangle = \sum_{\alpha\in\mathbb{N}_0^k} \int_\Omega D^\alpha \varphi \,\mathrm{d}\mu_{\alpha,j}, \qquad j \in J, \quad \varphi \in \mathcal{E}_0^{\{M_n\}}(\Omega),$$

where the series converges absolutely and uniformly when j varies in J and φ belongs to any given bounded subset of $\mathcal{E}_0^{\{M_n\}}(\Omega)$.

PROOF. We consider each $\mu_{\alpha,j}$ as a linear functional on $C_0(\Omega)$ by means of the duality

$$\langle \varphi, \mu_{\alpha,j}\rangle = \int_\Omega \varphi \,\mathrm{d}\mu_{\alpha,j}, \qquad \varphi \in C_0(\Omega).$$

Then, the norm of this linear functional is $|\mu_{\alpha,j}|(\Omega)$. We apply Proposition 4 and obtain, for every $j \in J$, an element u_j in Y' such that its restriction to Y_α coincides with $\mu_{\alpha,j}$, $\alpha \in \mathbb{N}_0^k$. Making use of Proposition 2 we obtain that

$$\langle (f_\alpha), u_j\rangle = \sum_{\alpha\in\mathbb{N}_0^k} \int_\Omega f_\alpha \,\mathrm{d}\mu_{\alpha,j}, \qquad (f_\alpha) \in Y. \tag{4}$$

We fix now a bounded subset B of Y. We find $r \in \mathbb{N}$ such that B is a bounded subset of Y_r. We take (f_α) in B. It follows that

$$\begin{aligned}\sum_{\alpha\in\mathbb{N}_0^k} \left| \int_\Omega f_\alpha \,\mathrm{d}\mu_{\alpha,j} \right| &\le \sum_{\alpha\in\mathbb{N}_0^k} \int_\Omega |f_\alpha| \,\mathrm{d}|\mu_{\alpha,j}| \\ &\le \sum_{\alpha\in\mathbb{N}_0^k} |f_\alpha|_\infty |\mu_{\alpha,j}|(\Omega) \\ &= \sum_{\alpha\in\mathbb{N}_0^k} \frac{1}{(2k)^{|\alpha|}} \frac{|f_\alpha|_\infty}{r^{|\alpha|} M_{|\alpha|}} (2kr)^{|\alpha|} |M_{|\alpha|}| \mu_{\alpha,j}|(\Omega) \\ &\le \|f_\alpha\|_r \left(\sup_{\substack{\beta\in\mathbb{N}_0^k \\ j\in J}} (2kr)^{|\beta|} M_{|\beta|} |\mu_{\beta,j}|(\Omega) \right) \sum_{\alpha\in\mathbb{N}_0^k} \frac{1}{(2k)^{|\alpha|}}.\end{aligned}$$

We deduce from this that the series in (4) converges absolutely and uniformly when j varies in J and (f_α) varies in B. Besides

$$\sup_{\substack{j\in J \\ (f_\alpha)\in B}} \left| \langle (f_\alpha), u_j\rangle \right| < \infty,$$

from where we get that $\{ u_j : j \in J \}$ is a bounded subset of Y'. If ψ is the map Φ considered from $\mathcal{E}_0^{\{M_n\}}(\Omega)$ into Y, and ${}^t\psi$ is the transpose of ψ, we put

$$S_j := {}^t\psi(u_j), \qquad j \in J.$$

Then $\{ S_j : j \in J \}$ is a bounded subset of $\mathcal{E}_0^{\{M_n\}'}(\Omega)$. On the other hand, for each $\varphi \in \mathcal{E}_0^{\{M_n\}}(\Omega)$, we have that

$$\langle (D^\alpha \varphi), u_j \rangle = \langle \psi(\varphi), u_j \rangle = \langle \varphi, {}^t\psi(u_j) \rangle = \langle \varphi, S_j \rangle.$$

Consequently, for each $\varphi \in \mathcal{E}_0^{\{M_n\}}(\Omega)$ and $j \in J$, making use of (4), we obtain

$$\langle \varphi, S_j \rangle = \langle (D^\alpha \varphi), u_j \rangle = \sum_{\alpha \in \mathbb{N}_0^k} \int_\Omega D^\alpha \varphi \, \mathrm{d}\mu_{\alpha,j}.$$

Finally, when φ varies in a bounded subset of $\mathcal{E}_0^{\{M_n\}}(\Omega)$, $(D^\alpha \varphi)$ varies in a bounded subset of Y, from where we deduce that the series above converges absolutely and uniformly when j varies in J and φ belongs to any given bounded subset of $\mathcal{E}_0^{\{M_n\}}(\Omega)$. ■

We shall need later the following result which is found in [3, p. 42]:

Result c) *Let K be a compact subset of Ω. If $0 < h < h' < \infty$, then the canonical injection from $\mathcal{D}^{(M_n),h}(K)$ into $\mathcal{D}^{(M_n),h'}(K)$ is a compact map.*

For each compact subset K of Ω, we put

$$\mathcal{D}^{\{M_n\}}(K) := \bigcup_{r=1}^{\infty} \mathcal{D}^{(M_n),r}(K)$$

and assume that $\mathcal{D}^{\{M_n\}}(K)$ is provided with the structure of (LB)-space as the inductive limit of the sequence $(\mathcal{D}^{(M_n),r}(K))$ of Banach spaces. By $\mathcal{D}^{\{M_n\}'}(K)$ we denote the strong dual of $\mathcal{D}^{\{M_n\}}(K)$.

For the two next propositions, we fix a compact subset K of Ω. Given $r \in \mathbb{N}$, let V_r be the subspace of Y whose elements have the form $(D^\alpha \varphi)$, with φ in $\mathcal{D}^{(M_n),r}(K)$. Let

$$\Lambda_r \colon \mathcal{D}^{(M_n),r}(K) \longrightarrow V_r$$

be such that

$$\Lambda_r(\varphi) = (D^\alpha \varphi), \qquad \varphi \in \mathcal{D}^{(M_n),r}(K).$$

It follows that Λ_r is an onto linear isometry. We put $V := \cup \{ V_r : r \in \mathbb{N} \}$ and we consider V as a subspace of Y. We write

$$\Lambda \colon \mathcal{D}^{\{M_n\}}(K) \longrightarrow V$$

such that

$$\Lambda(\varphi) = (D^\alpha \varphi), \qquad \varphi \in \mathcal{D}^{\{M_n\}}(K).$$

Clearly, Λ is linear continuous one-to-one and onto.

Proposition 8. *Λ is a topological isomorphism.*

PROOF. We take a closed bounded absolutely convex subset A of V. Applying Proposition 1 we obtain $r \in \mathbb{N}$ such that A is a bounded subset of V_r. Then $\Lambda_r^{-1}(A) = \Lambda^{-1}(A)$ is a bounded absolutely convex subset of $\mathcal{D}^{(M_n),r}(K)$, and is closed in $\mathcal{D}^{\{M_n\}}(K)$. We apply result 3 and obtain that $\Lambda_r^{-1}(A)$ is compact in $\mathcal{D}^{(M_n),r+1}(K)$, from where we deduce that A is compact in V_{r+1}. Applying now Proposition 6 we have that V is the inductive limit of the sequence (V_r) of Banach spaces. Consequently, Λ is a topological isomorphism. ■

We consider now $\mathcal{D}^{\{M_n\}}(K)$ as a subspace, clearly closed, of $\mathcal{D}^{\{M_n\}}(\Omega)$. If A is a closed bounded absolutely convex subset of $\mathcal{D}^{\{M_n\}}(K)$, then A is a closed bounded absolutely convex subset of $\mathcal{D}^{\{M_n\}}(\Omega)$, hence there is $m \in \mathbb{N}$ such that A is a compact subset of $\mathcal{D}^{(M_n),m}(K_m)$, [3, p. 44], therefore A is locally compact in $\mathcal{D}^{\{M_n\}}(K)$. Proposition 6 applies again to have that the topology induced by $\mathcal{D}^{\{M_n\}}(\Omega)$ in $\mathcal{D}^{\{M_n\}}(K)$ coincides with the original topology of $\mathcal{D}^{\{M_n\}}(K)$.

In what follows we put Γ for the mapping Λ considered from $\mathcal{D}^{\{M_n\}}(K)$ into Y. Then, if ${}^t\Gamma$ is the transpose of Γ, we have that

$$ {}^t\Gamma \colon Y' \longrightarrow \mathcal{D}^{\{M_n\}'}(K) $$

is onto.

Proposition 9. *If $\{ S_j : j \in J \}$ is a bounded subset of $\mathcal{D}^{\{M_n\}'}(\Omega)$, then there is, for each $j \in J$, a family $(\mu_{\alpha,j} : \alpha \in \mathbb{N}_0^k)$ of complex Borel measures in Ω such that, for each $h > 0$,*

$$ \sup_{\substack{\alpha\in\mathbb{N}_0^k \\ j\in J}} h^{|\alpha|} \, M_{|\alpha|} \, |\mu_{\alpha,j}|(\Omega) < \infty $$

and

$$ \langle \varphi, S_j \rangle = \sum_{\alpha\in\mathbb{N}_0^k} \int_\Omega D^\alpha \varphi \, \mathrm{d}\mu_{\alpha,j}, \qquad j \in J, \quad \varphi \in \mathcal{D}^{\{M_n\}}(K). $$

PROOF. Let S_j^* be the restriction of S_j to $\mathcal{D}^{\{M_n\}}(K)$. We then have that $\{ S_j^* : j \in J \}$ is a relatively compact subset of $\mathcal{D}^{\{M_n\}'}(K)$. Applying [2, p. 274], we obtain a relatively compact subset $\{ T_j : j \in J \}$ in Y' such that ${}^t\Gamma(T_j) = S_j^*$, $j \in J$. If $(T_j)_\alpha$ is the element of $C_0(\Omega)^*$ which identifies with the restriction of T_j to Y^α, $\alpha \in \mathbb{N}_0^k$, making use of Riesz's representation theorem, [6, p. 131], we have that there is a complex Borel measure $\mu_{\alpha,j}$ in Ω such that

$$ \langle \varphi, (T_j)_\alpha \rangle = \int_\Omega \varphi \, \mathrm{d}\mu_{\alpha,j}, \qquad \varphi \in C_0(\Omega), $$

and $|\mu_{\alpha,j}|(\Omega)$ is the norm of $(T_j)_\alpha$. After Proposition 3 we obtain

$$ \sup_{\alpha\in\mathbb{N}_0^k,\, j\in J} h^{|\alpha|} \, M_{|\alpha|} \, |\mu_{\alpha,j}(\Omega)| < \infty. $$

By using Proposition 2 we get, for (f_α) in Y and $j \in J$,

$$ \langle (f_\alpha), T_j \rangle = \sum_{\alpha\in\mathbb{N}_0^k} \int_\Omega f_\alpha \, \mathrm{d}\mu_{\alpha,j} $$

and, in particular, if φ belongs to $\mathcal{D}^{\{M_n\}}(K)$,

$$ \langle (D^\alpha\varphi), T_j \rangle = \sum_{\alpha\in\mathbb{N}_0^k} \int_\Omega D^\alpha \varphi \, \mathrm{d}\mu_{\alpha,j}. $$

On the other hand, if φ belongs to $\mathcal{D}^{\{M_n\}}(K)$, we have

$$ \langle (D^\alpha\varphi), T_j \rangle = \langle \Gamma(\varphi), T_j \rangle = \langle \varphi, {}^t\Gamma(T_j) \rangle = \langle \varphi, S_j^* \rangle = \langle \varphi, S_j \rangle $$

and the result follows. ■

Before giving the proof of the next theorem, we need the following construction. We take a bounded subset $\{ S_j : j \in J \}$ of $\mathcal{D}^{\{M_n\}'}(\Omega)$ so that there is a compact subset H of Ω which contains the support of S_j, $j \in J$, that is

$$ \operatorname{supp} S_j \subset H, \qquad j \in J. $$

Let K be a compact subset of Ω such that its interior $\overset{\circ}{K}$ contains H. Applying Proposition 9, we obtain, for each $j \in J$, a family $(\,\mu_{\alpha,j} : \alpha \in \mathbb{N}_0^k\,)$ of complex Borel measures in Ω such that, for each $h > 0$,

$$\sup_{\substack{\alpha\in\mathbb{N}_0^k\\ j\in J}} h^{|\alpha|}\, M_{|\alpha|}\, |\mu_{\alpha,j}|(\Omega) < \infty$$

and

$$\langle \varphi, S_j\rangle = \sum_{\alpha\in\mathbb{N}_0^k} \int_\Omega D^\alpha \varphi \,\mathrm{d}\mu_{\alpha,j}, \qquad j \in J, \quad \varphi \in \mathcal{D}^{\{M_n\}}(K).$$

We take an element g of $\mathcal{D}^{\{M_n\}}(\Omega)$ which has value 1 in a neighborhood of H and with support contained in $\overset{\circ}{K}$. We find $b > 0$ and a positive integer s such that

$$|D^\alpha g(x)| \le b\, s^{|\alpha|}\, M_{|\alpha|}, \qquad x \in \Omega, \quad \alpha \in \mathbb{N}_0^k.$$

We take $\varphi \in \mathcal{E}_0^{\{M_n\}}(\Omega)$. Since $g\varphi$ belongs to $\mathcal{D}^{\{M_n\}}(K)$, it follows that, for each $j \in J$,

$$\begin{aligned}\langle g\varphi, S_j\rangle &= \sum_{\alpha\in\mathbb{N}_0^k} \int_\Omega D^\alpha (g\varphi)\,\mathrm{d}\mu_{\alpha,j}\\ &= \sum_{\alpha\in\mathbb{N}_0^k} \int_\Omega \left(\sum_{\beta\le\alpha} \frac{\alpha!}{\beta!(\alpha-\beta)!} D^\beta g\cdot D^{\alpha-\beta}\varphi\right)\mathrm{d}\mu_{\alpha,j}. \end{aligned} \tag{5}$$

We take an integer $m > s$ such that φ is in $\mathcal{E}_0^{(M_n),m}(\Omega)$. It follows that, for each $x \in \Omega$,

$$\begin{aligned}&\sum_{\beta\le\alpha} \frac{\alpha!}{\beta!(\alpha-\beta)!} |D^\beta g(x)|\cdot|D^{\alpha-\beta}\varphi(x)|\\ &\qquad\le \sum_{\beta\le\alpha} \frac{\alpha!}{\beta!(\alpha-\beta)!} b\, s^{|\beta|}\, M_{|\beta|}\, m^{|\alpha-\beta|}\, |\varphi|_m\, M_{|\alpha-\beta|}\\ &\qquad\le b\, m^{|\alpha|}\, |\varphi|_m \sum_{\beta\le\alpha} \frac{\alpha!}{\beta!(\alpha-\beta)!}\, M_{|\beta|}\, M_{|\alpha-\beta|}\\ &\qquad\le b\, m^{|\alpha|}\, |\varphi|_m\, 2^{|\alpha|} M_{|\alpha|}\end{aligned}$$

and hence

$$\begin{aligned}&\sum_{\alpha\in\mathbb{N}_0^k}\sum_{\beta\le\alpha} \frac{\alpha!}{\beta!(\alpha-\beta)!} \int_\Omega |D^\beta g\cdot D^{\alpha-\beta}\varphi|\,\mathrm{d}|\mu_{\alpha,j}|\\ &\qquad\le b\,|\varphi|_m \sum_{\alpha\in\mathbb{N}_0^k} \frac{1}{(2k)^{|\alpha|}} (4km)^{|\alpha|}\, M_{|\alpha|}\, |\mu_{\alpha,j}|(\Omega)\\ &\qquad\le b\,|\varphi|_m \sum_{\alpha\in\mathbb{N}_0^k} \frac{1}{(2k)^{|\alpha|}} \sup_{\substack{\delta\in\mathbb{N}_0^k\\ j\in J}} (4km)^{|\delta|}\, M_{|\delta|}|\, \mu_{\delta,j}|(\Omega)\end{aligned}$$

from where we deduce that the series (5) is absolutely convergent, thus we may write, putting $\gamma := \alpha - \beta$,

$$\sum_{\alpha\in\mathbb{N}_0^k}\sum_{\beta\le\alpha} \frac{\alpha!}{\beta!(\alpha-\beta)!} \int_\Omega D^\beta g\cdot D^{\alpha-\beta}\varphi\,\mathrm{d}\mu_{\alpha,j} = \sum_{\gamma\in\mathbb{N}_0^k}\sum_{\beta\in\mathbb{N}_0^k} \frac{(\beta+\gamma)!}{\beta!\gamma!} \int_\Omega D^\beta g\cdot D^\gamma \varphi\,\mathrm{d}\mu_{\beta+\gamma,j}. \tag{6}$$

Theorem 2. *Let $\{ S_j : j \in J \}$ be a bounded subset of $\mathcal{D}^{\{M_n\}'}(\Omega)$ such that there is a compact subset H of Ω with*

$$\operatorname{supp} S_j \subset H, \qquad j \in J.$$

Let K be a compact subset of Ω such that $\overset{\circ}{K} \supset H$. Then, there is, for each $j \in J$, a family $(\nu_{\alpha,j} : \alpha \in \mathbb{N}_0^k)$ of complex Borel measures in Ω such that, for each $h > 0$,

$$\sup_{\substack{\alpha\in\mathbb{N}_0^k \\ j\in J}} h^{|\alpha|}\, M_{|\alpha|} |\nu_{\alpha,j}|(\Omega) < \infty$$

$$\operatorname{supp} \nu_{\alpha,j} \subset \overset{\circ}{K}, \qquad \alpha \in \mathbb{N}_0^k, \quad j \in J,$$

and

$$\langle \varphi, S_j \rangle = \sum_{\alpha\in\mathbb{N}_0^k} \int_\Omega D^\alpha \varphi \, \mathrm{d}\nu_{\alpha,j}, \qquad j \in J, \quad \varphi \in \mathcal{D}^{\{M_n\}}(\Omega),$$

where the series converges absolute and uniformly when j varies in J and φ varies in any given bounded subset of $\mathcal{D}^{\{M_n\}}(\Omega)$.

PROOF. For each $j \in J$, we find the family $(\mu_{\alpha,j} : \alpha \in \mathbb{N}_0^k)$ of complex Radon measures in Ω with the properties above cited. We now fix $\gamma \in \mathbb{N}_0^k$ and take an arbitrary element η of $C_0(\Omega)$. Then

$$\begin{aligned}
\left| \sum_{\beta\in\mathbb{N}_0^k} \frac{(\beta+\gamma)!}{\beta!\gamma!} \int_\Omega D^\beta g \cdot \eta \, \mathrm{d}\mu_{\beta+\gamma,j} \right| &\le \sum_{\beta\in\mathbb{N}_0^k} \frac{(\beta+\gamma)!}{\beta!\gamma!} \int_\Omega |D^\beta g| \cdot |\eta| \, \mathrm{d}|\mu_{\beta+\gamma,j}| \\
&\le b\, |\eta|_\infty \sum_{\beta\in\mathbb{N}_0^k} \frac{(\beta+\gamma)!}{\beta!\gamma!} s^{|\beta|}\, M_{|\beta|}\, |\mu_{\beta+\gamma,j}|(\Omega) \\
&\le b|\eta|_\infty \sum_{\beta\in\mathbb{N}_0^k} 2^{|\beta+\gamma|}\, s^{|\beta|}\, M_{|\beta|}\, |\mu_{\beta+\gamma,j}|(\Omega).
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\sum_{\beta\in\mathbb{N}_0^k} 2^{|\beta+\gamma|}\, s^{|\beta|}\, M_{|\beta|}\, |\mu_{\beta+\gamma,j}|(\Omega) &\le \sum_{\beta\in\mathbb{N}_0^k} (2s)^{|\beta+\gamma|}\, M_{|\beta+\gamma|}\, |\mu_{\beta+\gamma,j}|(\Omega) \\
&\le \sum_{\beta\in\mathbb{N}_0^k} \frac{1}{(2k)^{|\beta|}} \cdot (4ks)^{|\beta+\gamma|}\, M_{|\beta+\gamma|}\, |\mu_{\beta+\gamma,j}|(\Omega) \\
&\le 2 \sup_{\substack{\alpha\in\mathbb{N}_0^k \\ j\in J}} (4ks)^{|\alpha|}\, M_{|\alpha|}\, |\mu_{\alpha,j}|(\Omega).
\end{aligned}$$

Consequently, there is a constant $C > 0$ such that

$$\left| \sum_{\beta\in\mathbb{N}_0^k} \frac{(\beta+\gamma)!}{\beta!\gamma!} \int_\Omega D^\beta g \cdot \eta \, \mathrm{d}\mu_{\beta+\gamma,j} \right| \le C\, |\eta|_\infty. \tag{7}$$

If we set

$$v_{\gamma,j}(\eta) := \sum_{\beta\in\mathbb{N}_0^k} \frac{(\beta+\gamma)!}{\beta!\gamma!} \int_\Omega D^\beta g \cdot \eta \, \mathrm{d}\mu_{\beta+\gamma,j}, \qquad \eta \in C_0(\Omega),$$

we have that $v_{\gamma,j}$ is a complex function which is clearly linear and, after (7), belongs to $C_0(\Omega)^*$. We apply Riesz's representation theorem, [6, p. 131], and so obtain a complex Borel measure $\nu_{\gamma,j}$ in Ω such that

$$v_{\gamma,j}(\eta) = \int_\Omega \eta \cdot \mathrm{d}\nu_{\gamma,j}, \qquad \eta \in C_0(\Omega).$$

If M denotes the support of g, it is plain that

$$\operatorname{supp} \nu_{\gamma,j} \subset M \subset \mathring{K}, \qquad j \in J, \quad \gamma \in \mathbb{N}_0^k.$$

For each $\varphi \in \mathcal{E}_0^{\{M_n\}}(\Omega)$, we have that

$$\sum_{\beta \in \mathbb{N}_0^k} \frac{(\beta+\gamma)!}{\beta!\gamma!} \int_\Omega D^\beta g \cdot D^\gamma \varphi \cdot \mathrm{d}\mu_{\beta+\gamma,j} = \int_\Omega D^\gamma \varphi \cdot \mathrm{d}\nu_{\gamma,j}$$

and, having in mind (6),

$$\langle g\varphi, S_j \rangle = \sum_{\gamma \in \mathbb{N}_0^k} \int_\Omega D^\gamma \varphi \cdot \mathrm{d}\nu_{\gamma,j}, \qquad \varphi \in \mathcal{E}_0^{\{M_n\}}(\Omega). \tag{8}$$

Let us now fix γ in $\mathbb{N}_0^k$ and j in J. We choose η in $C_0(\Omega)$ such that $|\eta|_\infty < 2$ and $v_{\gamma,j}(\eta) = |\nu_{\gamma,j}|(\Omega)$. We take $h > 1$. Then

$$\begin{aligned} h^{|\gamma|} M_{|\gamma|} |\nu_{\gamma,j}|(\Omega) &= h^{|\gamma|} M_{|\gamma|} v_{\gamma,j}(\eta) \\ &\le h^{|\gamma|} M_{|\gamma|} \sum_{\beta \in \mathbb{N}_0^k} \frac{(\beta+\gamma)!}{\beta!\gamma!} b\, s^{|\beta|} M_{|\beta|} |\eta|_\infty |\mu_{\beta+\gamma,j}|(\Omega) \\ &\le 2b \sum_{\beta \in \mathbb{N}_0^k} (2hs)^{|\beta+\gamma|} M_{|\beta+\gamma|} |\mu_{\beta+\gamma,j}|(\Omega) \\ &\le 2b \sum_{\beta \in \mathbb{N}_0^k} \frac{1}{(2k)^{|\beta+\gamma|}} \sup_{\substack{\alpha \in \mathbb{N}_0^k \\ j \in J}} (4khs)^{|\alpha|} M_{|\alpha|} |\mu_{\alpha,j}|(\Omega) \\ &\le 4b \sup_{\substack{\alpha \in \mathbb{N}_0^k \\ j \in J}} (4ks)^{|\alpha|} M_{|\alpha|} |\mu_{\alpha,j}|(\Omega). \end{aligned}$$

It follows from above that

$$\sup_{\substack{\gamma \in \mathbb{N}_0^k \\ j \in J}} h^{|\gamma|} M_{|\gamma|} |\nu_{\gamma,j}|(\Omega) < \infty.$$

Theorem 1 now applies to obtain, for each $j \in J$, an element $T_j \in \mathcal{E}_0^{\{M_n\}'}(\Omega)$ such that

$$\langle \varphi, T_j \rangle = \sum_{\alpha \in \mathbb{N}_0^k} \int_\Omega D^\alpha \varphi \cdot \mathrm{d}\nu_{\alpha,j}, \qquad \varphi \in \mathcal{E}_0^{\{M_n\}}(\Omega)$$

where the series converges absolutely and uniformly when j varies in J and φ varies in any given bounded subset of $\mathcal{E}_0^{\{M_n\}}(\Omega)$. On the other hand, for each $\varphi \in \mathcal{D}^{\{M_n\}}(K)$ and each $j \in J$,

$$\langle \varphi, S_j \rangle = \langle g\varphi, S_j \rangle = \sum_{\alpha \in \mathbb{N}_0^k} \int_\Omega D^\alpha (g\varphi)\, \mathrm{d}\mu_{\alpha,j} = \sum_{\alpha \in \mathbb{N}_0^k} \int_\Omega D^\alpha \varphi \cdot \mathrm{d}\nu_{\alpha,j} = \langle \varphi, T_j \rangle.$$

Given an arbitrary element x in Ω, if x belongs to $\overset{\circ}{K}$, then this set is a neighborhood of x such that, if $\varphi \in \mathcal{D}^{\{M_n\}}(\overset{\circ}{K})$, then $\varphi \in \mathcal{D}^{\{M_n\}}(K)$, and so $\langle \varphi, S_j \rangle = \langle \varphi, T_j \rangle$. If x does not belong to $\overset{\circ}{K}$, we find an open neighborhood U_x of x such that $U_x \cap M = \emptyset$. We take φ in $\mathcal{D}^{\{M_n\}}(U_x)$. Then, $\int_\Omega D^\alpha \varphi \cdot \mathrm{d}\nu_{\alpha,j} = 0$, $\alpha \in \mathbb{N}_0^k$, thus $\langle \varphi, T_j \rangle = 0$. Besides, $U_x \cap H = \emptyset$, therefore $\langle \varphi, S_j \rangle = 0$.

We have thus proved that S_j and T_j coincide locally, from where it follows that S_j and T_j coincide in $\mathcal{D}^{\{M_n\}}(\Omega)$. The conclusion now follows. ■

4 Structure of the ultradistributions of Roumieu type

Theorem 3. *For each j in a set J, let $(\, u_{\alpha,j} : \ \alpha \in \mathbb{N}_0^k \,)$ be a family of Radon measures in Ω. If, given $h > 0$ and a compact subset $K \subset \Omega$, we have that*

$$\sup_{\substack{\alpha \in \mathbb{N}_0^k \\ j \in J}} h^{|\alpha|} \, M_{|\alpha|} \, \|u_{\alpha,j}\|(K) < \infty,$$

then there is a bounded subset $\{\, S_j : j \in J \,\}$ in $\mathcal{D}^{\{M_n\}'}(\Omega)$ such that

$$\langle \varphi, S_j \rangle = \sum_{\alpha \in \mathbb{N}_0^k} \langle D^\alpha \varphi, u_{\alpha,j} \rangle, \qquad \varphi \in \mathcal{D}^{\{M_n\}}(\Omega)$$

where the series converges absolutely and uniformly when j varies in J and φ varies in any given bounded subset of $\mathcal{D}^{\{M_n\}}(\Omega)$.

Proof. For each $m \in \mathbb{N}$, we identify $\mathcal{K}(K_m)$ with $C_0(\overset{\circ}{K}_m)$. We put $u^m_{\alpha,j}$ for the restriction of $u_{\alpha,j}$ to $\mathcal{K}(K_m)$. If $\mu^m_{\alpha,j}$ is the complex Borel measure in $\overset{\circ}{K}_m$ such that

$$\langle f, u^m_{\alpha,j} \rangle = \int_{\overset{\circ}{K}_m} f \, \mathrm{d}\mu^m_{\alpha,j}, \qquad f \in C_0(\overset{\circ}{K}_m),$$

we have that

$$\|u_{\alpha,j}\|(K_m) = |\mu^m_{\alpha,j}|(\overset{\circ}{K}_m).$$

Therefore, given $h > 0$, it follows that

$$\sup_{\alpha \in \mathbb{N}_0^k, \, j \in J} h^{|\alpha|} \, M_{|\alpha|} \, |\mu_{\alpha,j}|(\overset{\circ}{K}_m) < \infty,$$

from where we obtain, applying Theorem 1, that there is a bounded subset $\{\, S^m_j : j \in J \,\}$ of $\mathcal{E}_0^{\{M_n\}'}(\overset{\circ}{K}_m)$ such that

$$\langle \varphi, S^m_j \rangle = \sum_{\alpha \in \mathbb{N}_0^k} \int_{\overset{\circ}{K}_m} D^\alpha \varphi \cdot \mathrm{d}\mu^m_{\alpha,j}, \qquad \varphi \in \mathcal{E}_0^{\{M_n\}}(\overset{\circ}{K}_m)$$

where the series converges absolutely and uniformly when j varies in J and φ varies in any given bounded subset of $\mathcal{E}_0^{\{M_n\}}(\overset{\circ}{K}_m)$.

Given an arbitrary element φ of $\mathcal{D}^{\{M_n\}}(\Omega)$, we find $m \in \mathbb{N}$ such that

$$\operatorname{supp} \varphi \subset \overset{\circ}{K}_m,$$

we put

$$\langle \varphi, S_j \rangle := \langle \varphi, S^m_j \rangle.$$

It is easy to see that $\{\, S_j : \ j \in J \,\}$ is a bounded subset of $\mathcal{D}^{\{M_n\}'}(\Omega)$ satisfying the requirements of the statement. ■

Theorem 4. *If $\{ S_j : j \in J \}$ is a bounded subset of $\mathcal{D}^{\{M_n\}'}(\Omega)$, then there is, for each $j \in J$, a family $(u_{\alpha,j} : \alpha \in \mathbb{N}_0^k)$ of Radon measures in Ω such that, given $h > 0$ and a compact subset K of Ω, we have that*

$$\sup_{\substack{\alpha\in\mathbb{N}_0^k \\ j\in J}} h^{|\alpha|}\, M_{|\alpha|}\, \|u_{\alpha,j}\|(K) < \infty$$

and

$$\langle \varphi, S_j\rangle = \sum_{\alpha\in\mathbb{N}_0^k} \langle D^\alpha \varphi, u_{\alpha,j}\rangle, \qquad j \in J, \quad \varphi \in \mathcal{D}^{\{M_n\}}(\Omega),$$

where the series converges absolutely and uniformly when j varies in J and φ varies in any given bounded subset of $\mathcal{D}^{\{M_n\}}(\Omega)$.

PROOF. Let $\{ O_m : m \in \mathbb{N} \}$ be a locally finite open cover of Ω such that O_m is relatively compact in Ω, $m \in \mathbb{N}$. Let $\{ g_m : m \in \mathbb{N} \}$ be a partition of unity of class $\{M_n\}$ subordinated to such covering. It follows that $\{ g_m S_j : j \in J \}$ is a bounded subset of $\mathcal{D}^{\{M_n\}'}(\Omega)$ whose elements have their supports contained in a compact subset of O_m. Applying Theorem 2, we obtain, for each $j \in J$, a family $(\nu^m_{\alpha,j} : \alpha \in \mathbb{N}_0^k)$ of complex Borel measures in Ω such that, for each $h > 0$,

$$\sup_{\substack{\alpha\in\mathbb{N}_0^k \\ j\in J}} h^{|\alpha|}\, M_{|\alpha|}\, |\nu^m_{\alpha,j}|(\Omega) < \infty$$

$$\operatorname{supp} \nu^m_{\alpha,j} \subset O_m, \qquad j \in J, \quad \alpha \in \mathbb{N}_0^k,$$

and

$$\langle \varphi, g_m S_j\rangle = \sum_{\alpha\in\mathbb{N}_0^k} \int_\Omega D^\alpha\varphi \cdot \mathrm{d}\nu^m_{\alpha,j}, \qquad \varphi \in \mathcal{D}^{\{M_n\}}(\Omega)$$

where the series converges absolutely and uniformly when j varies in J and φ varies in any given bounded subset of $\mathcal{D}^{\{M_n\}}(\Omega)$.

Given an arbitrary element f of $\mathcal{K}(\Omega)$, there is a finite number of subindex m such that

$$O_m \cap \operatorname{supp} f \neq \emptyset.$$

Consequently, we may define, for each $\alpha \in \mathbb{N}_0^k$ and $j \in J$,

$$u_{\alpha,j}(f) := \sum_{m\in\mathbb{N}} \int_\Omega f \cdot \mathrm{d}\nu^m_{\alpha,j}.$$

We then have that $u_{\alpha,j}$ is a linear functional in $\mathcal{K}(\Omega)$. Given any compact subset K of Ω, there is a positive integer m_0 such that $K \cap O_m = \emptyset$, $m > m_0$. Hence, if f has its support contained in K, it follows that, for each $j \in J$,

$$|u_{\alpha,j}(f)| \leq \sum_{m=1}^{m_0} \int_\Omega |f|\, \mathrm{d}|\nu^m_{\alpha,j}| \leq \sum_{m=1}^{m_0} |\nu^m_{\alpha,j}|(\Omega) \cdot |f|_\infty,$$

from where we deduce that $u_{\alpha,j}$ is a Radon measure in Ω. Besides

$$\|u_{\alpha,j}\|(K) \leq \sum_{m=1}^{m_0} |\nu^m_{\alpha,j}|(\Omega), \qquad j \in J,$$

and, if h is an arbitrary positive number,

$$\sup_{\substack{\alpha\in\mathbb{N}_0^k \\ j\in J}} h^{|\alpha|}\, M_{|\alpha|} \|u_{\alpha,j}\|(K) \leq \sum_{m=1}^{m_0} \sum_{\substack{\alpha\in\mathbb{N}_0^k \\ j\in J}} h^{|\alpha|}\, M_{|\alpha|}\, |\nu^m_{\alpha,j}|(\Omega) < \infty.$$

We take now φ in $\mathcal{D}^{\{M_n\}}(\Omega)$ with support in K. Then

$$\begin{aligned}\langle\varphi, S_j\rangle &= \left\langle\varphi\sum_{m=1}^{\infty} g_m, S_j\right\rangle = \left\langle\varphi\sum_{m=1}^{m_0} g_m, S_j\right\rangle\\ &= \sum_{m=1}^{m_0}\langle\varphi, g_m S_j\rangle = \sum_{m=1}^{m_0}\sum_{\alpha\in\mathbb{N}_0^k}\int_\Omega D^\alpha\varphi\cdot \mathrm{d}\nu_{\alpha,j}^m\\ &= \sum_{\alpha\in\mathbb{N}_0^k}\sum_{m=1}^{m_0}\int_\Omega D^\alpha\varphi\cdot\mathrm{d}\nu_{\alpha,j}^m = \sum_{\alpha\in\mathbb{N}_0^k}\langle D^\alpha\varphi, u_{\alpha,j}\rangle.\end{aligned}$$

It is now easy to show that the last series converges absolute and uniformly when j varies in J and when φ varies in any given bounded subset of $\mathcal{D}^{\{M_n\}}(\Omega)$. ■

5 The space $\mathcal{D}_{(L_p)}^{\{M_n\}}(\Omega)$

We put $L^p(\Omega)$ and $\mathcal{L}^p(\Omega)$, $1\le p\le\infty$, to denote the classical Lebesgue spaces. If $f\in\tilde{f}\in L^p(\Omega)$, $1\le p<\infty$, we write

$$\|f\|_p = \|\tilde{f}\|_p = \left(\int_\Omega |f|\,\mathrm{d}x\right)^{\frac{1}{p}},$$

and if $f\in\tilde{f}\in L^\infty(\Omega)$, then

$$\|f\|_\infty = \|\tilde{f}\|_\infty = \mathrm{supess}\{\,|f(x)| : x\in\Omega\,\}.$$

By $\mathcal{D}_{L^p}(\mathbb{R}^k)$, $1\le p<\infty$, we represent the classical L. Schwartz space, [7, p. 199]. We put $\mathcal{B}_{L^p}(\Omega)$ for the linear space over $\mathbb{C}$ of the complex functions f, defined and infinitely differentiable in Ω, such that $D^\alpha f$ belongs to $\mathcal{L}^p(\Omega)$, $\alpha\in\mathbb{N}_0^k$. We assume that $\mathcal{B}_{L^p}(\Omega)$ is endowed with the metrizable locally convex topology such that a sequence (f_m) in $\mathcal{B}_{L^p}(\Omega)$ converges to the origin if and only if $(\|D^\alpha f_m\|_p)$ converges to zero for every $\alpha\in\mathbb{N}_0^k$. We then have that $\mathcal{B}_{L^p}(\Omega)$ is a Fréchet space. We have that $\mathcal{B}_{L^p}(\mathbb{R}^k)$ coincides with $\mathcal{D}_{L^p}(\mathbb{R}^k)$.

Given $r\in\mathbb{N}$ and $1\le p<\infty$, we use $\mathcal{B}_{L^p}^{(M_n),r}(\Omega)$ to denote the linear space over $\mathbb{C}$ of the functions f in $\mathcal{B}_{L^p}(\Omega)$ which satisfy:

$$\|f\|_{p,r} := \sup_{\alpha\in\mathbb{N}_0^k}\frac{\|D^\alpha f\|_p}{r^{|\alpha|}\,M_{|\alpha|}} < \infty.$$

We assume that $\mathcal{B}_{L^p}^{(M_n),r}(\Omega)$ is provided with the norm $\|\cdot\|_{p,r}$. Given a Cauchy sequence in $\mathcal{B}_{L^p}^{(M_n),r}(\Omega)$, it is immediate that (f_m) is also a Cauchy sequence in $\mathcal{B}_{L^p}(\Omega)$, thus it converges in this space to a function f. For a given $\varepsilon>0$, there is a positive integer m_0 such that

$$\|f_m - f_s\|_{p,r} < \varepsilon, \qquad m,s\ge m_0.$$

Then, for those values of m and s, and for each $\alpha\in\mathbb{N}_0^k$, we have

$$\frac{\|D^\alpha f_m - D^\alpha f_s\|_p}{r^{|\alpha|}\,M_{|\alpha|}} < \varepsilon$$

and so, for $m\ge m_0$,

$$\frac{\|D^\alpha f_m - D^\alpha f\|_p}{r^{|\alpha|}\,M_{|\alpha|}} \le \varepsilon,$$

from where we deduce that $f \in \mathcal{B}_{L^p}^{(M_n),r}(\Omega)$ and

$$\|f_m - f\|_{p,r} \leq \varepsilon, \qquad m > m_0.$$

Consequently, $\mathcal{B}_{L^p}^{(M_n),r}(\Omega)$ is a Banach space. We put

$$\mathcal{B}_{L^p}^{\{M_n\}}(\Omega) := \bigcup_{r=1}^{\infty} \mathcal{B}_{L^p}^{(M_n),r}(\Omega)$$

and assume that $\mathcal{B}_{L^p}^{\{M_n\}}(\Omega)$ is the inductive limit of the sequence $\big(\mathcal{B}_{L^p}^{(M_n),r}(\Omega)\big)$ of Banach spaces. We assume that the topological dual $\mathcal{B}_{L^p}^{\{M_n\}'}(\Omega)$ of $\mathcal{B}_{L^p}^{\{M_n\}}(\Omega)$ is endowed with the strong topology.

In this section, we substitute the space X of Section 2 by the space $L^p(\Omega)$, $1 \leq p < \infty$. Then, each element of Y_r is a family $(\,\tilde{f}_\alpha : \alpha \in \mathbb{N}_0^k\,)$ of elements of $L^p(\Omega)$ with

$$\|(\tilde{f}_\alpha)\|_r = \sup_{\alpha \in \mathbb{N}_0^k} \frac{\|\tilde{f}_\alpha\|_p}{r^{|\alpha|}\, M_{|\alpha|}} < \infty.$$

If f belongs to $\mathcal{B}_{L^p}^{\{M_n\}}(\Omega)$, we put $\tilde{D}^\alpha f$ for the element of $L^p(\Omega)$ to which $D^\alpha f$ belongs, $\alpha \in \mathbb{N}_0^k$.

By Z_r we denote the subspace of Y_r formed by those families $(\,\tilde{D}^\alpha f : \alpha \in \mathbb{N}_0^k\,)$ such that

$$f \in \mathcal{B}_{L^p}^{(M_n),r}(\Omega).$$

Let

$$\mathcal{X}_r \colon \mathcal{B}_{L^p}^{(M_n),r}(\Omega) \longrightarrow Z_r$$

be such that

$$\mathcal{X}_r(f) = (\tilde{D}^\alpha f), \qquad f \in \mathcal{B}_{L^p}^{(M_n),r}(\Omega).$$

Then, $\mathcal{X}_r$ is a linear onto isometry. By Z we mean $\cup\{\, Z_r : r \in \mathbb{N}\,\}$ considered as a subspace of Y. Let

$$\mathcal{X} \colon \mathcal{B}_{L^p}^{\{M_n\}}(\Omega) \longrightarrow Z$$

be such that

$$\mathcal{X}(f) = (\tilde{D}^\alpha f), \qquad f \in \mathcal{B}_{L^p}^{\{M_n\}}(\Omega).$$

Clearly, $\mathcal{X}$ is linear bijective and continuous.

We put W for the map $\mathcal{X}$ considered from $\mathcal{B}_{L^p}^{\{M_n\}}(\Omega)$ into Y. By ${}^tW$ we denote the map from Y' into $\mathcal{B}_{L^p}^{\{M_n\}'}(\Omega)$ given by the transpose of W.

Throughout what follows in this section, we fix $1 \leq p < \infty$ and write q for the conjugate value of p, i.e., $\frac{1}{p} + \frac{1}{q} = 1$.

Theorem 5. *For each j in a set J, let $(\,g_{\alpha,j} \colon \alpha \in \mathbb{N}_0^k\,)$ be a family of elements of $\mathcal{L}^q(\Omega)$ such that, for each $h > 0$,*

$$\sup_{\substack{\alpha \in \mathbb{N}_0^k \\ j \in J}} h^{|\alpha|}\, M_{|\alpha|}\, \|g_{\alpha,j}\|_q < \infty.$$

Then there is a bounded subset $\{\, S_j \colon j \in J\,\}$ in $\mathcal{B}_{L^p}^{\{M_n\}'}(\Omega)$ so that

$$\langle \varphi, S_j \rangle = \sum_{\alpha \in \mathbb{N}_0^k} \int_\Omega D^\alpha \varphi \cdot g_{\alpha,j}\, \mathrm{d}x, \qquad j \in J, \quad \varphi \in \mathcal{B}_{L^p}^{\{M_n\}}(\Omega),$$

where the series converges absolutely and uniformly when j varies in J and φ varies in any given bounded subset of $\mathcal{B}_{L^p}^{\{M_n\}}(\Omega)$.

PROOF. We identify in the usual manner $g_{\alpha,j}$ with a continuous linear functional on $L^p(\Omega)$, whose norm is $\|g_{\alpha,j}\|_q$. We apply Proposition 4 to obtain, for each $j \in J$, an element u_j in Y' such that its restriction to Y_α coincides with $g_{\alpha,j}$, $\alpha \in \mathbb{N}_0^k$. Applying now Proposition 2 we have that

$$\langle(\tilde{f}_\alpha), u_j\rangle = \sum_{\alpha\in\mathbb{N}_0^k}\int_\Omega f_\alpha \cdot g_{\alpha,j}\,\mathrm{d}x, \qquad f_\alpha \in \tilde{f}_\alpha, \quad (\tilde{f}_\alpha) \in Y. \tag{9}$$

Let us now fix a bounded subset B of Y. We find $r \in \mathbb{N}$ such that B is a bounded subset of Y_r. We take $(\tilde{f}_\alpha)$ in B. It follows that

$$\begin{aligned}\sum_{\alpha\in\mathbb{N}_0^k}\left|\int_\Omega f_\alpha \cdot g_{\alpha,j}\,\mathrm{d}x\right| &\le \sum_{\alpha\in\mathbb{N}_0^k}\int_\Omega |f_\alpha|\cdot|g_{\alpha,j}|\,\mathrm{d}x \\ &\le \sum_{\alpha\in\mathbb{N}_0^k}\|f_\alpha\|_p\cdot\|g_{\alpha,j}\|_q \\ &= \sum_{\alpha\in\mathbb{N}_0^k}\frac{1}{(2k)^{|\alpha|}}\frac{\|f_\alpha\|_p}{r^{|\alpha|}\,M_{|\alpha|}}(2kr)^{|\alpha|}\,M_{|\alpha|}\,\|g_{\alpha,j}\|_q \\ &\le \|(\tilde{f}_\alpha)\|_r\left(\sup_{\substack{\beta\in\mathbb{N}_0^k\\ j\in J}}(2kr)^{|\beta|}\,M_{|\beta|}\,\|g_{\beta,j}\|_q\right)\sum_{\alpha\in\mathbb{N}_0^k}\frac{1}{(2k)^{|\alpha|}}.\end{aligned}$$

We deduce from here that the series (9) converges absolutely and uniformly when j varies in J and $(\tilde{f}_\alpha)$ varies in B. Besides,

$$\sup_{\substack{j\in J\\ (\tilde{f}_\alpha)\in B}}\left|\langle(\tilde{f}_\alpha), u_j\rangle\right| < \infty,$$

from where it follows that $\{\, u_j : j \in J \,\}$ is a bounded subset of Y'. We now write

$$S_j := {}^tW(u_j), \qquad j \in J.$$

Then $\{\, S_j : j \in J \,\}$ is a bounded subset of $\mathcal{B}_{L^p}^{\{M_n\}'}(\Omega)$. On the other hand, for each $\varphi \in \mathcal{B}_{L^p}^{\{M_n\}}(\Omega)$, we have that

$$\langle(\tilde{D}^\alpha\varphi), u_j\rangle = \langle W(\varphi), u_j\rangle = \langle\varphi, {}^tW(u_j)\rangle = \langle\varphi, S_j\rangle.$$

Consequently, for each $\varphi \in \mathcal{B}_{L^p}^{\{M_n\}}(\Omega)$ and each $j \in J$, we obtain, making use of (9), that

$$\langle\varphi, S_j\rangle = \langle(\tilde{D}^\alpha\varphi), u_j\rangle = \sum_{\alpha\in\mathbb{N}_0^k}\int_\Omega D^\alpha\varphi\cdot g_{\alpha,j}\,\mathrm{d}x.$$

Finally, when φ varies in a bounded subset of $\mathcal{B}_{L^p}^{\{M_n\}}(\Omega)$, $(\tilde{D}^\alpha\varphi)$ varies in a bounded subset of Y, from where we deduce that the above series converges absolutely and uniformly when j varies in J and φ varies in any given bounded subset of $\mathcal{B}_{L^p}^{\{M_n\}}(\Omega)$. ■

Given a compact subset K of Ω and $r \in \mathbb{N}$, we put $\mathcal{D}_{(L^p)}^{(M_n),r}(K)$ for the subspace of $\mathcal{B}_{L^p}^{(M_n),r}(\Omega)$ whose elements have their support in K. If (f_m) is a sequence in $\mathcal{D}_{L^p}^{(M_n),r}(K)$ which converges to f in $\mathcal{B}_{L^p}^{(M_n),r}(\Omega)$, there is a subsequence (f_{m_i}) of (f_m) which converges to f almost everywhere. Since $f_{m_i}(x) = 0$, $x \in \Omega \setminus K$, $i \in \mathbb{N}$, it follows that f belongs to $\mathcal{D}_{L^p}^{(M_n),r}(K)$ and thus this space is complete. We write

$$\mathcal{D}_{(L^p)}^{\{M_n\}}(K) := \bigcup_{r=1}^\infty \mathcal{D}_{(L^p)}^{(M_n),r}(K)$$

and we assume that $\mathcal{D}_{L^p}^{\{M_n\}}(K)$ is endowed with the structure of (LB) space, as the inductive limit of the sequence $(\mathcal{D}_{(L^p)}^{(M_n),r}(K))$ of Banach spaces. We write $\mathcal{D}_{(L^p)}^{\{M_n\}'}(K)$ for the strong topological dual of $\mathcal{D}_{(L^p)}^{\{M_n\}}(K)$.

Proposition 10. *The closed unit ball B_r of $\mathcal{D}_{(L^p)}^{(M_n),r}(K)$ is a compact subset of $\mathcal{D}_{(L^p)}^{(M_n),r+1}(K)$.*

PROOF. Let μ be the Lebesgue measure in $\mathbb{R}^k$. We assume the elements f of $\mathcal{D}_{(L^p)}^{(M_n),r}(K)$ extended to $\mathbb{R}^k$ setting $f(x)=0$, $x \in \mathbb{R}^k \setminus K$. Given $\alpha \in \mathbb{N}_0^k$ and $f \in B_r$, we have that

$$D^\alpha f(x) = \int_{-\infty}^{x_1}\int_{-\infty}^{x_2}\dots\int_{-\infty}^{x_k} \frac{\partial^{|\alpha|+k} f(t)}{\partial^{\alpha_1+1}t_1\partial^{\alpha_2+1}t_2\dots\partial^{\alpha_k+1}t_k}\,\mathrm{d}t_1\,\mathrm{d}t_2\dots\mathrm{d}t_k$$

and hence, if $\beta_j := \alpha_j+1$, $j=1,2,\dots,k$, we have that

$$\begin{aligned}|D^\alpha f(x)| &\le \int_K |D^\beta f(t)|\,\mathrm{d}t \le \mu(K)^{1/q}\|D^\beta f\|_p \\ &\le \mu(K)^{1/q}\|f\|_{p,r} r^{|\beta|}\,M_{|\beta|} \le \mu(K)^{1/q}\,r^{|\beta|}\,M_{|\beta|},\end{aligned}$$

and so the set of functions $\{\, D^\alpha f : f \in B_r \,\}$ is uniformly bounded in $\mathbb{R}^k$. Consequently, for each $\gamma \in \mathbb{N}_0^k$, the set $\{\, D^\gamma f : f \in B_r \,\}$ is equicontinuous, therefore, applying Ascoli's theorem and a diagonal process, given an arbitrary sequence (f_m) in B_r, there is a complex function f, defined and infinitely differentiable in $\mathbb{R}^k$, and a subsequence (f_{m_i}) of (f_m) such that, for each $\alpha \in \mathbb{N}_0^k$, $(D^\alpha f_{m_i})$ converges uniformly a $D^\alpha f$ in $\mathbb{R}^k$. Since, for each $\alpha \in \mathbb{N}_0^k$,

$$\frac{\|D^\alpha f_{m_i}\|_p}{r^{|\alpha|}\,M_{|\alpha|}} \le 1, \qquad i \in \mathbb{N},$$

it follows that

$$\frac{\|D^\alpha f\|_p}{r^{|\alpha|}\,M_{|\alpha|}} \le 1,$$

and thus f is in B_r. We see next that (f_{m_i}) converges to f in $\mathcal{D}_{(L^p)}^{(M_n),r+1}(K)$. Given $\varepsilon > 0$, we find a positive integer s_0 such that

$$\left(\frac{r}{r+1}\right)^{s_0} < \frac{\varepsilon}{4}.$$

We determine a positive integer i_0 such that

$$\mu(K)^{1/p}\big|D^\alpha f_{m_i}(x) - D^\alpha f(x)\big| < \frac{\varepsilon}{2}, \qquad x \in \mathbb{R}^k, \quad i \ge i_0, \quad |\alpha| \le s_0.$$

Then, if $i \ge i_0$, we have

$$\begin{aligned}\|f_{m_i} - f\|_{p,r+1} &= \sup_{\alpha\in\mathbb{N}_0^k} \frac{\|D^\alpha f_{m_i} - D^\alpha f\|_p}{(r+1)^{|\alpha|}\,M_{|\alpha|}} \\ &\le \sup_{|\alpha|\le s_0} \frac{\|D^\alpha f_{m_i} - D^\alpha f\|_p}{(r+1)^{|\alpha|}\,M_{|\alpha|}} + \sup_{|\alpha|>s_0} \frac{\|D^\alpha f_{m_i} - D^\alpha f\|_p}{(r+1)^{|\alpha|}\,M_{|\alpha|}} \\ &\le \sup_{|\alpha|\le s_0}\sup_{x\in\mathbb{R}^k} \mu(K)^{1/p}|D^\alpha f_{m_i}(x) - D^\alpha f(x)| + \sup_{|\alpha|>s_0}\left(\frac{r}{r+1}\right)^{|\alpha|} \frac{\|D^\alpha f_{m_i} - D^\alpha f\|_p}{(r)^{|\alpha|}\,M_{|\alpha|}} \\ &\le \frac{\varepsilon}{2} + \left(\frac{r}{r+1}\right)^{s_0} \sup_{|\alpha|>s_0} \frac{\|D^\alpha f_{m_i}\|_p + \|D^\alpha f\|_p}{(r)^{|\alpha|}\,M_{|\alpha|}} \\ &\le \frac{\varepsilon}{2} + \frac{\varepsilon}{4}(\|f_{m_i}\|_{p,r} + \|f\|_{p,r}) \le \varepsilon.\end{aligned}$$

It then follows that (f_{m_i}) converges to f in $\mathcal{D}^{(M_n),r+1}_{(L^p)}(K)$ and the result follows. ■

For the next two propositions, we are going to fix a compact subset K of Ω. Given $r \in \mathbb{N}$, let P_r be the subspace of Y_r whose elements have the form $(\tilde{D}^\alpha \varphi)$, with $\varphi \in \mathcal{D}^{(M_n),r}_{(L^p)}(K)$. Let

$$\zeta_r \colon \mathcal{D}^{(M_n),r}_{(L^p)}(K) \longrightarrow P_r$$

be such that

$$\zeta_r(\varphi) = (\tilde{D}^\alpha \varphi), \qquad \varphi \in \mathcal{D}^{(M_n),r}_{(L^p)}(K).$$

We have that ζ_r is a linear onto isometry. We put $P := \cup\{\, P_r : \ r \in \mathbb{N} \,\}$ and we consider it as a subspace of Y. We then write

$$\zeta \colon \mathcal{D}^{\{M_n\}}_{(L^p)}(K) \longrightarrow P$$

such that

$$\zeta(\varphi) = (\tilde{D}^\alpha \varphi), \qquad \varphi \in \mathcal{D}^{\{M_n\}}_{(L^p)}(K).$$

Clearly, ζ is linear bijective and continuous.

Proposition 11. *ζ is a topological isomorphism.*

PROOF. We take an absolutely convex closed and bounded subset A of P. Applying Proposition 1, we obtain $r \in \mathbb{N}$ such that A is a bounded subset of P_r. Then $\zeta_r^{-1}(A) = \zeta^{-1}(A)$ is an absolutely convex bounded subset of $\mathcal{D}^{(M_n),r}_{(L^p)}(K)$ which is closed in $\mathcal{D}^{\{M_n\}}_{(L^p)}(K)$. Making use of the former proposition, we obtain that $\zeta_r^{-1}(A)$ is compact in $\mathcal{D}^{(M_n),r+1}_{(L^p)}(K)$, from where we have that A is a compact subset of P_{r+1}. We apply Proposition 6 to have that P is the inductive limit of the sequence (P_r) of Banach spaces. Consequently, ζ is a topological isomorphism. ■

We now put

$$\mathcal{D}^{\{M_n\}}_{(L^p)}(\Omega) := \bigcup_{r=1}^{\infty} \mathcal{D}^{(M_n),r}_{(L^p)}(K_r)$$

and assume that $\mathcal{D}^{\{M_n\}}_{(L^p)}(\Omega)$ is the inductive limit of the sequence $(\mathcal{D}^{(M_n),r}_{(L^p)}(K_r))$ of Banach spaces. We write $\mathcal{D}^{\{M_n\}'}_{(L^p)}(\Omega)$ for the strong topological dual of $\mathcal{D}^{\{M_n\}}_{(L^p)}(\Omega)$. It follows, from Proposition 10, that $\mathcal{D}^{\{M_n\}'}_{(L^p)}(\Omega)$ is Fréchet-Schwartz space.

We now consider $\mathcal{D}^{\{M_n\}}_{(L^p)}(K)$ as a subspace of $\mathcal{D}^{\{M_n\}}_{(L^p)}(\Omega)$. If A is an absolutely convex closed bounded subset of $\mathcal{D}^{\{M_n\}}_{(L^p)}(K)$, then A is a bounded subset of $\mathcal{D}^{\{M_n\}}_{(L^p)}(\Omega)$ and thus there is a positive integer m such that $K \subset K_r$ and A is a relatively compact subset of $\mathcal{D}^{(M_n),r}_{(L^p)}(K_r)$. Clearly, $\mathcal{D}^{(M_n),r}_{(L^p)}(K)$ is a closed subspace of $\mathcal{D}^{(M_n),r}_{(L^p)}(K_r)$ and A is closed in the Banach space $\mathcal{D}^{(M_n),r}_{(L^p)}(K)$, from where we conclude that A is compact in $\mathcal{D}^{(M_n),r}_{(L^p)}(K)$. We apply Proposition 6 and so we obtain that the topology induced by $\mathcal{D}^{\{M_n\}}_{(L^p)}(\Omega)$ in $\mathcal{D}^{\{M_n\}}_{(L^p)}(K)$ coincides with the original topology of this space.

In the following, we put η for the mapping ζ considered from $\mathcal{D}^{\{M_n\}}_{(L^p)}(K)$ into Y. Then, if ${}^t\eta$ is the transpose of η, we have that

$${}^t\eta \colon Y' \longrightarrow \mathcal{D}^{\{M_n\}'}_{(L^p)}(K)$$

is an onto map.

Proposition 12. *If $\{\, S_j : j \in J \,\}$ is a bounded subset of $\mathcal{D}^{\{M_n\}'}_{(L^p)}(\Omega)$, then there is, for each $j \in J$, a family $(\, g_{\alpha,j} : \alpha \in \mathbb{N}_0^k \,)$ of elements of $\mathcal{L}^q(\Omega)$ such that, for each $h > 0$,*

$$\sup_{\substack{\alpha \in \mathbb{N}_0^k \\ j \in J}} h^{|\alpha|}\, M_{|\alpha|} \|g_{\alpha,j}\|_q < \infty$$

and

$$\langle \varphi, S_j \rangle = \sum_{\alpha \in \mathbb{N}_0^k} \int_\Omega D^\alpha \varphi \cdot g_{\alpha,j} \, \mathrm{d}x, \qquad j \in J, \quad \varphi \in \mathcal{D}^{\{M_n\}}_{(L^p)}(K).$$

PROOF. Let S_j^* be the restriction of S_j to $\mathcal{D}^{\{M_n\}}_{(L^p)}(K)$. It follows that $\{ S_j^* : j \in J \}$ is a relatively compact subset of $\mathcal{D}^{\{M_n\}'}_{(L^p)}(K)$. Applying [2, p. 274], we obtain a relatively compact subset $\{ T_j : j \in J \}$ of Y' such that ${}^t\eta(T_j) = S_j^*$, $j \in J$. If $(T_j)_\alpha$ is the element of $L^q(\Omega)$ which identifies with the restriction of T_j to Y^α, $\alpha \in \mathbb{N}_0^k$, we obtain an element $g_{\alpha,j}$ in $\mathcal{L}^q(\Omega)$ such that

$$\langle \tilde{\varphi}, (T_j)_\alpha \rangle = \int_\Omega \varphi \cdot g_{\alpha,j} \, \mathrm{d}x, \qquad j \in J, \quad \varphi \in \tilde{\varphi} \in L^p(\Omega).$$

Given $h > 0$, we take $r \in \mathbb{N}$, $r > h$. Making use of Proposition 3 we obtain that

$$\sup_{\alpha \in \mathbb{N}_0^k,\, j \in J} h^{|\alpha|} \, M_{|\alpha|} \|g_{\alpha,j}\|_q < \infty.$$

Having in mind Proposition 2, we have, for each $(\tilde{f}_\alpha)$ in Y and $j \in J$,

$$\langle (\tilde{f}_\alpha), T_j \rangle = \sum_{\alpha \in \mathbb{N}_0^k} \int_\Omega f_\alpha \cdot g_{\alpha,j} \, \mathrm{d}x, \qquad f_\alpha \in \tilde{f}_\alpha, \quad \alpha \in \mathbb{N}_0^k,$$

and, in particular, if $\varphi \in \mathcal{D}^{\{M_n\}}_{(L^p)}(K)$, then

$$\langle (\tilde{D}^\alpha \varphi), T_j \rangle = \sum_{\alpha \in \mathbb{N}_0^k} \int_\Omega D^\alpha \varphi \cdot g_{\alpha,j} \, \mathrm{d}x$$

and besides

$$\langle (\tilde{D}^\alpha \varphi), T_j \rangle = \langle \eta(\varphi), T_j \rangle = \langle \varphi, {}^t\eta(T_j) \rangle = \langle \varphi, S_j^* \rangle = \langle \varphi, S_j \rangle,$$

from where the result follows. ∎

If $g \in \mathcal{L}^{p_1}(\mathbb{R}^k)$ and $l \in \mathcal{L}^{p_2}(\mathbb{R}^k)$, with $1 \le p_1 \le \infty$, $1 \le p_2 \le \infty$ and $\frac{1}{p_1} + \frac{1}{p_2} \ge 1$, then there exists almost everywhere the convolution $g * l \in \mathcal{L}^s(\mathbb{R}^k)$, being $\frac{1}{s} := \frac{1}{p_1} + \frac{1}{p_2} - 1$. Also having that

$$\|g * l\|_s \le \|g\|_{p_1} \cdot \|l\|_{p_2}. \tag{10}$$

This property will be used in the next proposition.

Proposition 13. *The linear space $\mathcal{D}^{\{M_n\}}(\Omega)$ is dense in $\mathcal{D}^{\{M_n\}}_{(L^p)}(\Omega)$.*

PROOF. We may assume that $\overset{\circ}{K}_1 \neq \emptyset$ and $K_s \subset \overset{\circ}{K}_{s+1}$, $s \in \mathbb{N}$.

Given $\delta > 0$, we put $B(\delta)$ for the closed ball in $\mathbb{R}^k$ with center in the origin and radius δ. We take $f \in \mathcal{D}^{\{M_n\}}_{(L^p)}(\Omega)$. We find $r \in \mathbb{N}$ such that $f \in \mathcal{D}^{(M_n),r}_{(L^p)}(K_r)$. We choose a sequence (ψ_i) in $\mathcal{E}^{\{M_n\}}(\mathbb{R}^k)$ satisfying

(i) $\psi_i(x) \ge 0$, $x \in \Omega$.

(ii) $\int_\Omega \psi_i(x) \, \mathrm{d}x = 1$.

(iii) $\psi_i \in \mathcal{D}^{(M_n),r}(B(\delta_i))$, $\delta_1 > \delta_2 > \cdots > \delta_i > \ldots$,

$$\lim_i \delta_i = 0 \text{ and } K_r + B(\delta_i) \subset \overset{\circ}{K}_{r+1}.$$

We consider f extended to $\mathbb{R}^k$ by setting $f(x) = 0$, $x \in \mathbb{R}^k \setminus \Omega$. We put $f_i := f * \psi_i$, $i \in \mathbb{N}$. We shall see next that (f_i) is a sequence in $\mathcal{D}^{(M_n),r+1}(K_{r+1})$ which converges to f in $\mathcal{D}^{(M_n),r+1}_{(L^p)}(K_{r+1})$. For each $\alpha \in \mathbb{N}_0^k$, we have

$$D^\alpha f_i(x) = \int_{\mathbb{R}^k} f(y)(D^\alpha \psi_i)(x-y)\,\mathrm{d}y, \qquad x \in \mathbb{R}^k,$$

from where we get that f_i belongs to $\mathcal{D}^{(M_n),r+1}(K_{r+1})$. We now take $\varepsilon > 0$, We find a positive integer s_0 such that

$$\left(\frac{r}{r+1}\right)^{s_0} \|f\|_{p,r} < \frac{\varepsilon}{4}.$$

Given $\alpha \in \mathbb{N}_0^k$, we have that, for each $x \in \mathbb{R}^k$,

$$\begin{aligned}\left|D^\alpha f_i(x) - D^\alpha f(x)\right| &\leq \int_{\mathbb{R}^k} \left|(D^\alpha f)(x-y) - D^\alpha f(x)\right|\psi_i(y)\,\mathrm{d}y \\ &\leq \sup\{\, |(D^\alpha f)(x-y) - D^\alpha f(x)| : y \in B(\delta_i)\,\}.\end{aligned}$$

We find $i_0 \in \mathbb{N}$ for which

$$|D^\alpha f_i(x) - D^\alpha f(x)| < \frac{\varepsilon}{2\mu(K_{r+1})}, \qquad i \geq i_0, \quad x \in \mathbb{R}^k, \quad |\alpha| \leq s_0.$$

Then

$$\|D^\alpha f_i - D^\alpha f\|_p \leq \frac{\varepsilon}{2}, \qquad i \geq i_0.$$

We now apply (10) for $p_1 = p$, $p_2 = 1$, $g = D^\alpha f$ and $l = \psi_i$. Then

$$\|D^\alpha f_i\|_p = \|(D^\alpha f) * \psi_i\|_p \leq \|D^\alpha f\|_p \cdot \|\psi_i\|_1 = \|D^\alpha f\|_p.$$

Consequently, for $i \geq i_0$, it follows that

$$\begin{aligned}\|f - f_i\|_{p,r+1} &= \sup_{\alpha \in \mathbb{N}_0^k} \frac{\|D^\alpha(f - f_i)\|_p}{(r+1)^{|\alpha|}\, M_{|\alpha|}} \\ &\leq \sup_{|\alpha| \leq s_0} \frac{\|D^\alpha(f - f_i)\|_p}{(r+1)^{|\alpha|} M_{|\alpha|}} + \sup_{|\alpha| \geq s_0} \frac{\|D^\alpha(f - f_i)\|_p}{(r+1)^{|\alpha|}\, M_{|\alpha|}} \\ &\leq \frac{\varepsilon}{2} + \sup_{|\alpha| \geq s_0} \left(\frac{r}{r+1}\right)^{|\alpha|} \frac{\|D^\alpha f\|_p + \|D^\alpha f_i\|_p}{(r)^{|\alpha|}\, M_{|\alpha|}} \\ &\leq \frac{\varepsilon}{2} + \left(\frac{r}{r+1}\right)^{s_0} \sup_{\alpha \in \mathbb{N}_0^k} \frac{2\|D^\alpha f\|_p}{r^{|\alpha|}\, M_{|\alpha|}} < \varepsilon. \quad \blacksquare\end{aligned}$$

The last proposition tells us that the elements of $\mathcal{D}^{\{M_n\}'}_{(L^p)}(\Omega)$ may be considered as ultradistributions. In theorems 7 and 8, we shall characterize those ultradistributions.

We proceed now in a similar way to the construction previous to Theorem 2. We take a bounded subset $\{\, S_j : j \in J \,\}$ in $\mathcal{D}^{\{M_n\}'}_{(L^p)}(\Omega)$ in such a way that there is a compact subset H of Ω with

$$\operatorname{supp} S_j \subset H, \qquad j \in J.$$

Let K be a compact subset of Ω with $H \subset \overset{\circ}{K}$. We apply Proposition 12 to obtain, for each $j \in J$, a family $(\, f_{\alpha,j} : \alpha \in \mathbb{N}_0^k \,)$ in $\mathcal{L}^q(\Omega)$ such that, for each $h > 0$,

$$\sup_{\alpha \in \mathbb{N}_0^k,\, j \in J} h^{|\alpha|} M_{|\alpha|} \cdot \|f_{\alpha,j}\|_q < \infty$$

and

$$\langle \varphi, S_j \rangle = \sum_{\alpha \in \mathbb{N}_0^k} \int_\Omega D^\alpha \varphi \cdot f_{\alpha,j} \, \mathrm{d}x, \qquad j \in J, \quad \varphi \in \mathcal{D}_{(L^p)}^{\{M_n\}}(K).$$

We take an element g of $\mathcal{D}^{\{M_n\}}(\Omega)$ which takes value one in a neighborhood of H and whose support is contained in $\overset{\circ}{K}$. We find $b > 0$ and a positive integer s such that

$$\left| D^\alpha g(x) \right| \leq b \, s^{|\alpha|} M_{|\alpha|}, \qquad x \in \Omega, \quad \alpha \in \mathbb{N}_0^k.$$

We take φ in $\mathcal{B}_{L^p}^{\{M_n\}}(\Omega)$. It means no difficulty to see that $g\varphi$ belongs to $\mathcal{D}_{(L^p)}^{\{M_n\}}(K)$ and thus we have, for each $j \in J$,

$$\begin{aligned} \langle g\varphi, S_j \rangle &= \sum_{\alpha \in \mathbb{N}_0^k} \int_\Omega D^\alpha (g\varphi) \cdot f_{\alpha,j} \, \mathrm{d}x \\ &= \sum_{\alpha \in \mathbb{N}_0^k} \int_\Omega \left(\sum_{\beta \leq \alpha} \frac{\alpha!}{\beta!(\alpha-\beta)!} D^\beta g \cdot D^{\alpha-\beta} \varphi \right) \cdot f_{\alpha,j} \, \mathrm{d}x. \end{aligned} \tag{11}$$

We take now a positive integer $m > s$ such that φ is in $\mathcal{B}_{L^p}^{(M_n),m}(\Omega)$. We then have

$$\begin{aligned} &\sum_{\beta \leq \alpha} \frac{\alpha!}{\beta!(\alpha-\beta)!} \int_\Omega |D^\beta g| \cdot |D^{\alpha-\beta} \varphi| \cdot |f_{\alpha,j}| \, \mathrm{d}x \\ &\qquad \leq \sum_{\beta \leq \alpha} \frac{\alpha!}{\beta!(\alpha-\beta)!} b \, s^{|\beta|} M_{|\beta|} \|D^{\alpha-\beta}\varphi\|_p \cdot \|f_{\alpha,j}\|_q \\ &\qquad \leq \sum_{\beta \leq \alpha} \frac{\alpha!}{\beta!(\alpha-\beta)!} b \, s^{|\beta|} M_{|\beta|} \|\varphi\|_{p,m} m^{|\alpha-\beta|} M_{|\alpha-\beta|} \|f_{\alpha,j}\|_q \\ &\qquad \leq \sum_{\beta \leq \alpha} \frac{\alpha!}{\beta!(\alpha-\beta)!} b \, m^{|\alpha|} M_{|\alpha|} \|\varphi\|_{p,m} \|f_{\alpha,j}\|_q \\ &\qquad = 2^{|\alpha|} b \, m^{|\alpha|} M_{|\alpha|} \|\varphi\|_{p,m} \|f_{\alpha,j}\|_q \\ &\qquad \leq b \, \|\varphi\|_{p,m} \frac{1}{(2k)^{|\alpha|}} \sup_{\delta \in \mathbb{N}_0^k, \, j \in J} (4km)^{|\delta|} M_{|\delta|} \|f_{\delta,j}\|_q \end{aligned}$$

from where we deduce that the series (11) is absolutely convergent, hence, putting $\gamma := \alpha - \beta$, we may write

$$\sum_{\alpha \in \mathbb{N}_0^k} \sum_{\beta \leq \alpha} \frac{\alpha!}{\beta!(\alpha-\beta)!} \int_\Omega D^\beta g \cdot D^{\alpha-\beta} \varphi \cdot f_{\alpha,j} \, \mathrm{d}x = \sum_{\gamma \in \mathbb{N}_0^k} \sum_{\beta \in \mathbb{N}_0^k} \frac{(\beta+\gamma)!}{\beta! \, \gamma!} \int_\Omega D^\beta g \cdot D^\gamma \varphi \cdot f_{\beta+\gamma,j} \, \mathrm{d}x. \tag{12}$$

Theorem 6. *Let $\{ S_j : j \in J \}$ be a bounded subset of $\mathcal{D}_{(L^p)}^{\{M_n\}'}(\Omega)$ such that there is a compact subset H of Ω with*

$$\operatorname{supp} S_j \subset H, \qquad j \in J.$$

Let K be a compact subset of Ω such that $H \subset \overset{\circ}{K}$. Then there is, for each $j \in J$, a family $(g_{\alpha,j} : \alpha \in \mathbb{N}_0^k)$ of elements of $\mathcal{L}^q(\Omega)$ such that, for each $h > 0$, we have

$$\sup_{\substack{\alpha \in \mathbb{N}_0^k \\ j \in J}} h^{|\alpha|} M_{|\alpha|} \|g_{\alpha,j}\|_q < \infty,$$

$$\operatorname{supp} g_{\alpha,j} \subset \overset{\circ}{K}, \qquad j \in J, \quad \alpha \in \mathbb{N}_0^k,$$

and

$$\langle \varphi, S_j \rangle = \sum_{\alpha \in \mathbb{N}_0^k} \int_\Omega D^\alpha \varphi \cdot g_{\alpha,j} \,\mathrm{d}x, \qquad j \in J, \quad \varphi \in \mathcal{D}_{(L^p)}^{\{M_n\}}(\Omega),$$

where the series converges absolutely and uniformly when j varies in J and φ varies in any given bounded subset of $\mathcal{D}_{(L^p)}^{\{M_n\}}(\Omega)$.

PROOF. For each $j \in J$, we obtain the family $(\, f_{\alpha,j} : \alpha \in \mathbb{N}_0^k\,)$ of elements of $\mathcal{L}^q(\Omega)$ with the properties cited above. We fix $\gamma \in \mathbb{N}_0^k$ and take $\rho \in \tilde{\rho} \in L^q(\Omega)$. Then

$$\begin{aligned}
\left| \sum_{\beta \in \mathbb{N}_0^k} \frac{(\beta+\gamma)!}{\beta!\gamma!} \int_\Omega D^\beta g \cdot \rho \cdot f_{\beta,j} \,\mathrm{d}x \right| &\leq \sum_{\beta \in \mathbb{N}_0^k} \frac{(\beta+\gamma)!}{\beta!\gamma!} \int_\Omega |D^\beta g| \cdot |\rho| \cdot |f_{\beta+\gamma,j}| \,\mathrm{d}x \\
&\leq \sum_{\beta \in \mathbb{N}_0^k} \frac{(\beta+\gamma)!}{\beta!\gamma!} b\, s^{|\beta|} M_{|\beta|} \|\rho\|_p \|f_{\beta+\gamma,j}\|_q \\
&\leq b \|\rho\|_p \sum_{\beta \in \mathbb{N}_0^k} 2^{|\beta+\gamma|} s^{|\beta|} M_{|\beta|} \|f_{\beta+\gamma,j}\|_q \\
&\leq b\, \|\rho\|_p \sum_{\beta \in \mathbb{N}_0^k} \frac{1}{(2k)^{|\beta|}} (4ks)^{|\beta+\gamma|} M_{|\beta+\gamma|} \|f_{\beta+\gamma,j}\|_q \\
&\leq b\, \|\rho\|_p \sum_{\beta \in \mathbb{N}_0^k} \frac{1}{(2k)^{|\beta|}} \sup_{\alpha \in \mathbb{N}_0^k,\, j \in J} (4ks)^{|\alpha|} M_{|\alpha|} \|f_{\alpha,j}\|_q \\
&= 2\, b\, \|\rho\|_p \sup_{\substack{\alpha \in \mathbb{N}_0^k \\ j \in J}} (4ks)^{|\alpha|} M_{|\alpha|} \|f_{\alpha,j}\|_q,
\end{aligned}$$

from where we deduce that there is $C > 0$ such that

$$\left| \sum_{\beta \in \mathbb{N}_0^k} \frac{(\beta+\gamma)!}{\beta!\gamma!} \int_\Omega D^\beta g \cdot \rho \cdot f_{\beta+\gamma,j} \,\mathrm{d}x \right| \leq C \|\rho\|_p. \tag{13}$$

If we put, for each $\rho \in \tilde{\rho} \in L^p(\Omega)$,

$$v_{\gamma,j}(\tilde{\rho}) := \sum_{\beta \in \mathbb{N}_0^k} \frac{(\beta+\gamma)!}{\beta!\gamma!} \int_\Omega D^\beta g \cdot \rho \cdot f_{\beta+\gamma,j} \,\mathrm{d}x,$$

we have that $v_{\gamma,j}$ is a complex function, clearly linear, such that, after (13), belongs to $L^q(\Omega)$. Then, there is $g_{\gamma,j} \in \mathcal{L}^q(\Omega)$ for which

$$v_{\gamma,j}(\tilde{\rho}) = \int_\Omega \rho \cdot g_{\gamma,j} \,\mathrm{d}x, \qquad \rho \in \tilde{\rho} \in L^p(\Omega).$$

If M is the support of g, it is plain that

$$\operatorname{supp} g_{\gamma,j} \subset M \subset \overset{\circ}{K}, \qquad j \in J, \quad \gamma \in \mathbb{N}_0^k.$$

It follows that, for each $\varphi \in \mathcal{B}_{L^p}^{\{M_n\}}(\Omega)$,

$$\sum_{\beta \in \mathbb{N}_0^k} \frac{(\beta+\gamma)!}{\beta!\,\gamma!} \int_\Omega D^\beta g \cdot D^\gamma \varphi \cdot f_{\beta+\gamma,j} \,\mathrm{d}x = \int_\Omega D^\gamma \varphi \cdot g_{\gamma,j} \,\mathrm{d}x$$

and, having in mind (11),

$$\langle g\varphi, S_j\rangle = \sum_{\gamma\in\mathbb{N}_0^k} \int_\Omega D^\gamma\varphi \cdot g_{\gamma,j}\,\mathrm{d}x, \qquad \varphi \in \mathcal{B}_{L^p}^{\{M_n\}}(\Omega). \tag{14}$$

We fix γ in $\mathbb{N}_0^k$ and j in J. We choose $\tilde\rho$ in $L^p(\Omega)$ such that $\|\tilde\rho\|_p < 2$ and $v_{\gamma,j}(\tilde\rho) = \|g_{\gamma,j}\|_q$. We take $h > 1$. Then, if $\rho \in \tilde\rho$,

$$\begin{aligned}
h^{|\gamma|}M_{|\gamma|}\|g_{\gamma,j}\|_q &= h^{|\gamma|}M_{|\gamma|}v_{\gamma,j}(\tilde\rho)\\
&\le h^{|\gamma|}M_{|\gamma|}\left|\sum_{\beta\in\mathbb{N}_0^k}\frac{(\beta+\gamma)!}{\beta!\,\gamma!}\int_\Omega D^\beta g\cdot\rho\cdot f_{\beta+\gamma,j}\,\mathrm{d}x\right|\\
&\le h^{|\gamma|}M_{|\gamma|}\|\tilde\rho\|_p\, b\sum_{\beta\in\mathbb{N}_0^k}\frac{(\beta+\gamma)!}{\beta!\,\gamma!}s^{|\beta|}M_{|\beta|}\|f_{\beta+\gamma,j}\|_q\\
&\le 2\,b\sum_{\beta\in\mathbb{N}_0^k}2^{|\beta+\gamma|}(sh)^{|\beta+\gamma|}M_{|\beta+\gamma|}\|f_{\beta+\gamma,j}\|_q\\
&\le 2\,b\sum_{\beta\in\mathbb{N}_0^k}\frac{1}{(2k)^{|\beta|}}\sup_{\substack{\alpha\in\mathbb{N}_0^k\\ j\in J}}(4ksh)^{|\alpha|}M_{|\alpha|}\|f_{\alpha,j}\|_q
\end{aligned}$$

from where we deduce that

$$\sup_{\substack{\gamma\in\mathbb{N}_0^k\\ j\in J}} h^{|\gamma|}M_{|\gamma|}\|g_{\gamma,j}\|_q < \infty.$$

We apply Theorem 5 to obtain, for each $j \in J$, an element T_j in $\mathcal{B}_{L^p}^{\{M_n\}'}(\Omega)$ such that

$$\langle\varphi, T_j\rangle = \sum_{\alpha\in\mathbb{N}_0^k}\int_\Omega D^\alpha\varphi\cdot g_{\alpha,j}\,\mathrm{d}x, \qquad \varphi\in\mathcal{B}_{L^p}^{\{M_n\}}(\Omega),$$

where the series converges absolutely and uniformly when j varies in J and φ varies in any given bounded subset of $\mathcal{B}_{L^p}^{\{M_n\}}(\Omega)$. On the other hand, for each $\varphi \in \mathcal{D}_{(L^p)}^{\{M_n\}}(K)$ and each $j \in J$,

$$\begin{aligned}
\langle\varphi, S_j\rangle = \langle g\,\varphi, S_j\rangle &= \sum_{\alpha\in\mathbb{N}_0^k}\int_\Omega D^\alpha(g\varphi)\cdot f_{\alpha,j}\,\mathrm{d}x\\
&= \sum_{\alpha\in\mathbb{N}_0^k}\int_\Omega D^\alpha\varphi\cdot g_{\alpha,j}\,\mathrm{d}x = \langle\varphi, T_j\rangle.
\end{aligned}$$

Finally, it can be shown in the same way that it was done in the proof of Theorem 2 that S_j and T_j coincide in $\mathcal{D}_{(L^p)}^{\{M_n\}}(\Omega)$ for every $j \in J$, so the result follows. ■

We put $\mathcal{L}_{\mathrm{loc}}^p(\Omega)$ for the linear space over $\mathbb{C}$ of the complex functions f defined in Ω such that, for each compact subset K of Ω, $f_{|K}$ belongs to $\mathcal{L}^q(K)$.

Theorem 7. *For each j in a set J, let $(\, f_{\alpha,j} : \alpha\in\mathbb{N}_0^k\,)$ be a family of elements of $\mathcal{L}_{\mathrm{loc}}^q(\Omega)$ such that, for each $h > 0$ and each compact subset K of Ω, we have that*

$$\sup_{\substack{\alpha\in\mathbb{N}_0^k\\ j\in J}} h^{|\alpha|}M_{|\alpha|}\|f_{\alpha,j|K}\|_q < \infty.$$

Then, there is a bounded subset $\{ S_j : j \in J \}$ *of* $\mathcal{D}^{\{M_n\}'}_{(L^p)}(\Omega)$ *such that*

$$\langle \varphi, S_j \rangle = \sum_{\alpha \in \mathbb{N}_0^k} \int_\Omega D^\alpha \varphi \cdot f_{\alpha,j}\, \mathrm{d}x, \qquad j \in J, \quad \varphi \in \mathcal{D}^{\{M_n\}}_{(L^p)}(\Omega),$$

where the series converges absolutely and uniformly when j *varies in* J *and* φ *varies in any given bounded subset of* $\mathcal{D}^{\{M_n\}}_{(L^p)}(\Omega)$.

PROOF. For each $m \in \mathbb{N}$, we put

$$f^m_{\alpha,j} := f_{\alpha,j|\overset{\circ}{K}_m}, \qquad \alpha \in \mathbb{N}_0^k, \quad j \in J.$$

It follows that, for each $h > 0$,

$$\sup_{\substack{\alpha \in \mathbb{N}_0^k \\ j \in J}} h^{|\alpha|} M_{|\alpha|} \| f^m_{\alpha,j} \|_q < \infty.$$

We apply now Theorem 5 to obtain a bounded subset $\{ S^m_j : j \in J \}$ of $\mathcal{B}^{\{M_n\}'}_{L^p}(\overset{\circ}{K}_m)$ such that

$$\langle \varphi, S^m_j \rangle = \sum_{\alpha \in \mathbb{N}_0^k} \int_{\overset{\circ}{K}_m} D^\alpha \varphi \cdot f^m_{\alpha,j}\, \mathrm{d}x, \qquad j \in J, \quad \varphi \in \mathcal{B}^{\{M_n\}}_{L^p}(\overset{\circ}{K}_m),$$

where the series converges absolutely and uniformly when j varies in J and φ varies in any given bounded subset of $\mathcal{B}^{\{M_n\}}_{L^p}(\overset{\circ}{K}_m)$.

Given an arbitrary element φ in $\mathcal{D}^{\{M_n\}}_{(L^p)}(\Omega)$, we find $m \in \mathbb{N}$ such that

$$\operatorname{supp} \varphi \subset \overset{\circ}{K}_m$$

and we put

$$\langle \varphi, S_j \rangle := \langle \varphi, S^m_j \rangle.$$

It is easy to verify that S_j is well defined, $j \in J$, and that $\{ S_j : j \in J \}$ is a bounded subset of $\mathcal{D}^{\{M_n\}'}(\Omega)$, which satisfies the statement of our theorem. ■

Theorem 8. *If* $\{ S_j : j \in J \}$ *is a bounded subset of* $\mathcal{D}^{\{M_n\}'}_{(L^p)}(\Omega)$, *then there is, for each* $j \in J$, *a family* $(f_{\alpha,j} : \alpha \in \mathbb{N}_0^k)$ *in* $\mathcal{L}^q_{\mathrm{loc}}(\Omega)$ *such that, for each* $h > 0$ *and each compact subset* K *of* Ω, *we have that*

$$\sup_{\substack{\alpha \in \mathbb{N}_0^k \\ j \in J}} h^{|\alpha|} M_{|\alpha|} \| f_{\alpha,j|K} \|_q < \infty$$

and

$$\langle \varphi, S_j \rangle = \sum_{\alpha \in \mathbb{N}_0^k} \int_\Omega D^\alpha \varphi \cdot f_{\alpha,j}\, \mathrm{d}x, \qquad j \in J, \quad \varphi \in \mathcal{D}^{\{M_n\}}_{(L^p)}(\Omega),$$

where the series converges absolutely and uniformly when j *varies in* J *and* φ *varies in any given bounded subset of* $\mathcal{D}^{\{M_n\}}_{(L^p)}(\Omega)$.

PROOF. Let $\{ O_m : m \in \mathbb{N} \}$ be a locally finite open cover of Ω such that O_m is relatively compact in Ω, $m \in \mathbb{N}$. Let $\{ g_m : m \in \mathbb{N} \}$ be a partition of unity of class $\{M_n\}$ subordinated to that open cover. We then have that $\{ g_m S_j : j \in J \}$ is a bounded subset of $\mathcal{D}^{\{M_n\}'}_{(L^p)}(\Omega)$ whose elements have their supports contained in a compact subset of O_m. We apply Theorem 6 to obtain, for each $j \in J$, a family $(\, f^m_{\alpha,j} : \alpha \in \mathbb{N}_0^k \,)$ of elements of $\mathcal{L}^q(\Omega)$ such that, for each $h > 0$,

$$\sup_{\substack{\alpha\in\mathbb{N}_0^k \\ j\in J}} h^{|\alpha|} M_{|\alpha|} \|f^m_{\alpha,j}\|_q < \infty,$$

$$\operatorname{supp} f^m_{\alpha,j} \subset O_m, \qquad j \in J, \quad \alpha \in \mathbb{N}_0^k,$$

and

$$\big\langle \varphi, g_m S_j \big\rangle = \sum_{\alpha\in\mathbb{N}_k^0} \int_\Omega D^\alpha \varphi \cdot f^m_{\alpha,j}\, \mathrm{d}x, \qquad j \in J, \quad \varphi \in \mathcal{D}^{\{M_n\}}_{(L^p)}(\Omega),$$

where the series converges absolutely and uniformly when j varies in J and φ varies in any given bounded subset of $\mathcal{D}^{\{M_n\}}_{(L^p)}(\Omega)$. We put, for each $x \in \Omega$, $\alpha \in \mathbb{N}_0^k$ and $j \in J$,

$$f_{\alpha,j}(x) := \sum_{m=1}^\infty f^m_{\alpha,j}(x).$$

If K is a compact subset of Ω, there is a positive integer m_0 such that

$$K \cap O_m = \emptyset, \qquad m \geq m_0,$$

and hence $f_{\alpha,j}$ is well defined and belongs to $\mathcal{L}^q_{\mathrm{loc}}(\Omega)$. Besides, we have

$$\|f_{\alpha,j|K}\|_q \leq \sum_{m=1}^{m_0} \|f^m_{\alpha,j|K}\|_q \leq \sum_{m=1}^{m_0} \|f^m_{\alpha,j}\|_q$$

and thus, given $h > 0$, it follows that

$$\sup_{\alpha\in\mathbb{N}_0^k,\, j\in J} h^{|\alpha|} M_{|\alpha|} \|f_{\alpha,j|K}\|_q < \infty.$$

Applying now Theorem 7, we obtain a bounded subset $\{\, T_j : j \in J \,\}$ of $\mathcal{D}^{\{M_n\}'}_{(L^p)}(\Omega)$ such that

$$\big\langle \varphi, T_j \big\rangle = \sum_{\alpha\in\mathbb{N}_0^k} \int_\Omega D^\alpha \varphi \cdot f_{\alpha,j}\, \mathrm{d}x, \qquad j \in J, \quad \varphi \in \mathcal{D}^{\{M_n\}}_{(L^p)}(\Omega),$$

where the series converges absolutely and uniformly when j varies in J and φ varies in any given bounded subset of $\mathcal{D}^{\{M_n\}}_{(L^p)}(\Omega)$.

We now choose $\varphi \in \mathcal{D}^{\{M_n\}}_{(L^p)}(\Omega)$. We find $m_0 \in \mathbb{N}$ such that

$$O_m \cap \operatorname{supp} \varphi = \emptyset, \qquad m > m_0.$$

Then

$$\begin{aligned}
\big\langle \varphi, T_j \big\rangle &= \sum_{\alpha\in\mathbb{N}_0^k} \int_\Omega D^\alpha \varphi \cdot f_{\alpha,j}\, \mathrm{d}x = \sum_{\alpha\in\mathbb{N}_0^k} \sum_{m=1}^{m_0} \int_\Omega D^\alpha \varphi \cdot f^m_{\alpha,j}\, \mathrm{d}x \\
&= \sum_{m=1}^{m_0} \sum_{\alpha\in\mathbb{N}_0^k} \int_\Omega D^\alpha \varphi \cdot f^m_{\alpha,j}\, \mathrm{d}x = \sum_{m=1}^{m_0} \langle \varphi, g_m S_j \rangle = \Big\langle \sum_{m=1}^{m_0} \varphi \cdot g_m, S_j \Big\rangle \\
&= \langle \varphi, S_j \rangle.
\end{aligned}$$

Consequently, $S_j = T_j$, $j \in J$, and the result now follows. ■

From this and up to the end of this section we shall assume that the sequence M_n, $n \in \mathbb{N}$, satisfies condition (3), that is, it is stable for differential operators.

Proposition 14. *The canonical injection from $\mathcal{D}^{\{M_n\}}(\Omega)$ into $\mathcal{D}^{\{M_n\}}_{(L^1)}(\Omega)$ is a topological isomorphism.*

PROOF. Clearly,

$$\zeta\colon \mathcal{D}^{\{M_n\}}(\Omega) \longrightarrow \mathcal{D}^{\{M_n\}}_{(L^1)}(\Omega)$$

such that

$$\zeta(f) = f, \qquad f \in \mathcal{D}^{\{M_n\}}(\Omega)$$

is well defined, linear and continuous.

It is immediate that there are $b > 0$ and $l > 0$ for which

$$M_{n+k} \le b\, l^n\, M_n, \qquad n \in \mathbb{N}_0.$$

We now take an arbitrary element $\varphi \in \mathcal{D}^{\{M_n\}}_{(L^1)}(\Omega)$. We extend φ to $\mathbb{R}^k$ by putting $\varphi(x) = 0$, $x \in \mathbb{R}^k \setminus \Omega$. We find $r \in \mathbb{N}$ such that φ is in $\mathcal{D}^{\{M_n\},r}_{(L^1)}(\mathbb{R}^k)$. Given $\alpha \in \mathbb{N}_0^k$ and $x \in \mathbb{R}^k$, we have

$$D^\alpha \varphi(x) = \int_{-\infty}^{x_1} \int_{-\infty}^{x_2} \cdots \int_{-\infty}^{x_k} \frac{\partial^{|\alpha|+k} \varphi(t)}{\partial^{\alpha_1+1} t_1\, \partial^{\alpha_2+1} t_2 \ldots \partial^{\alpha_k+1} t_k}\, \mathrm{d}t_1\, \mathrm{d}t_2\, \ldots \mathrm{d}t_k$$

and thus

$$\begin{aligned}
\left|D^\alpha \varphi(x)\right| &\le \int_{K_r} \left| \frac{\partial^{|\alpha|+k} \varphi(t)}{\partial t_1^{\alpha_1+1} \partial t_2^{\alpha_2+1} \ldots \partial t_k^{\alpha_k+1}} \right| \mathrm{d}t \\
&\le \|\varphi\|_{1,r} \cdot r^{|\alpha|+k} M_{|\alpha|+k} \\
&\le \|\varphi\|_{1,r} r^{|\alpha|+k}\, b\, l^{|\alpha|} M_{|\alpha|},
\end{aligned}$$

from where we deduce that, if s is an integer greater than rl,

$$\sup_{\alpha \in \mathbb{N}_0^k} \sup_{x \in K_r} \frac{|D^\alpha \varphi(x)|}{s^{|\alpha|} M_{|\alpha|}} \le \|\varphi\|_{1,r}\, r^k\, b$$

and so

$$\varphi \in \mathcal{D}^{(M_n),s}(K_r) \subset \mathcal{D}^{\{M_n\}}(\Omega).$$

It follows that ζ is onto. Applying now Grothendieck's theorem, [1, p. 17], the result follows. ■

Theorem 9. *If $\{\, S_j : j \in J \,\}$ is a bounded subset of $\mathcal{D}^{\{M_n\}'}(\Omega)$, then there is, for each $j \in J$, a family $(\, f_{\alpha,j} : \alpha \in \mathbb{N}_0^k \,)$ of elements of $\mathcal{L}^\infty_{\mathrm{loc}}(\Omega)$ such that, for each $h > 0$ and each compact subset K of Ω, we have that*

$$\sup_{\alpha \in \mathbb{N}_0^k,\, j \in J} h^{|\alpha|} M_{|\alpha|} |f_{\alpha,j}|_{K,\infty} < \infty$$

and

$$\langle \varphi, S_j \rangle = \sum_{\alpha \in \mathbb{N}_0^k} \int_\Omega D^\alpha \varphi \cdot f_{\alpha,j}\, \mathrm{d}x, \qquad j \in J, \quad \varphi \in \mathcal{D}^{\{M_n\}}(\Omega),$$

where the series converges absolutely and uniformly when j varies in J and φ varies in any given bounded subset of $\mathcal{D}^{\{M_n\}}(\Omega)$.

PROOF. It is an immediate consequence of the former proposition and Theorem 8. ■

6 The space $\mathcal{D}_{L^p}^{\{M_n\}}(\Omega)$

We shall use in this section the same notation as in the previous one. In particular, $1 \le p < \infty$ and q is the conjugate element of p.

For each $r \in \mathbb{N}$, we put $\mathcal{D}_{L^p}^{(M_n),r}(\Omega)$ for the Banach space given by the closure of $\cup_{m=1}^{\infty}\mathcal{D}_{L^p}^{(M_n),r}(K_m)$ in $\mathcal{B}_{L^p}^{(M_n),r}(\Omega)$. We put

$$\mathcal{D}_{L^p}^{\{M_n\}}(\Omega) := \bigcup_{r=1}^{\infty} \mathcal{D}_{L^p}^{(M_n),r}(\Omega)$$

and we assume that $\mathcal{D}_{L^p}^{\{M_n\}}(\Omega)$ is provided with the structure of (LB)-space as the inductive limit of the sequence $\big(\mathcal{D}_{L^p}^{(M_n),r}(\Omega)\big)$ of Banach spaces. We put $\mathcal{D}_{L^p}^{\{M_n\}'}(\Omega)$ for the strong topological dual of $\mathcal{D}_{L^p}^{\{M_n\}}(\Omega)$. It is immediate that the canonical injection from $\mathcal{D}^{\{M_n\}}(\Omega)$ into $\mathcal{D}_{L^p}^{\{M_n\}}(\Omega)$ is continuous, therefore we may consider the elements of $\mathcal{D}_{L^p}^{\{M_n\}'}(\Omega)$ as ultradistributions. We shall characterize later these ultradistributions for the case $p > 1$.

We write Q_r for the subspace of Z_r formed by those families $(\, \tilde{D}^{\alpha} f \, : \, \alpha \in \mathbb{N}_0^k \,)$ such that $f \in \mathcal{D}_{L^p}^{\{M_n\}}(\Omega)$. Let

$$\tau_r \colon \mathcal{D}_{L^p}^{(M_n),r}(\Omega) \longrightarrow Q_r$$

be such that

$$\tau_r(f) = (\tilde{D}^{\alpha} f), \qquad f \in \mathcal{D}^{(M_n),r}(\Omega).$$

Then τ_r is an onto linear isometry. We put Q to denote $\cup\{\, Q_r : r \in \mathbb{N} \,\}$ considering it as a subspace of Y. Let

$$\tau \colon \mathcal{D}_{L^p}^{\{M_n\}}(\Omega) \longrightarrow Q$$

be such that

$$\tau(f) = (\tilde{D}^{\alpha} f), \qquad f \in \mathcal{D}_{L^p}^{\{M_n\}}(\Omega).$$

Clearly, τ is linear bijective and continuous. We put λ for the map τ considered from $\mathcal{D}_{L^p}^{\{M_n\}}(\Omega)$ into Y. By ${}^t\lambda$ we mean as usual the transpose map of λ.

Theorem 10. *For each j of a set J, let $(\, g_{\alpha,j} : \alpha \in \mathbb{N}_0^k \,)$ be a family of elements of $\mathcal{L}^q(\Omega)$ such that, for each $h > 0$,*

$$\sup_{\substack{\alpha \in \mathbb{N}_0^k \\ j \in J}} h^{|\alpha|} M_{|\alpha|} \|g_{\alpha,j}\|_q < \infty.$$

Then there is a bounded subset $\{\, S_j : j \in J \,\}$ of $\mathcal{D}_{L^p}^{\{M_n\}'}(\Omega)$ such that

$$\langle \varphi, S_j \rangle = \sum_{\alpha \in \mathbb{N}_0^k} \int_{\Omega} D^{\alpha}\varphi \cdot g_{\alpha,j} \, \mathrm{d}x, \qquad j \in J, \quad \varphi \in \mathcal{D}_{L^p}^{\{M_n\}}(\Omega),$$

where the series converges absolutely and uniformly when j varies in J and φ varies in any given bounded subset of $\mathcal{D}_{L^p}^{\{M_n\}}(\Omega)$.

Proof. It is analogous to the proof of Theorem 5, just replacing $\mathcal{B}_{L^p}^{\{M_n\}}(\Omega)$ by $\mathcal{D}_{L^p}^{\{M_n\}}(\Omega)$. ■

If $1 < p$, we apply Proposition 7 to obtain that Q_r is reflexive, $r \in \mathbb{N}$. After Proposition 5, Q with the Mackey topology is an (LB)-space. It follows from this that

$${}^t\lambda \colon Y' \longrightarrow \mathcal{D}_{L^p}^{\{M_n\}'}(\Omega)$$

is an onto map.

Theorem 11. *If* $p > 1$ *and* $\{ S_j : j \in J \}$ *is a bounded subset of* $\mathcal{D}_{L^p}^{\{M_n\}'}(\Omega)$*, then there is, for each* $j \in J$*, a family* $\left(g_{\alpha,j} : \alpha \in \mathbb{N}_0^k \right)$ *of elements of* $\mathcal{L}^q(\Omega)$ *such that, for each* $h > 0$*,*

$$\sup_{\substack{\alpha\in\mathbb{N}_0^k \\ j\in J}} h^{|\alpha|} M_{|\alpha|} \|g_{\alpha,j}\|_q < \infty$$

and

$$\langle \varphi, S_j \rangle = \sum_{\alpha\in\mathbb{N}_0^k} \int_\Omega D^\alpha \varphi \cdot g_{\alpha,j} \, \mathrm{d}x, \qquad j \in J, \quad \varphi \in \mathcal{D}_{L^p}^{\{M_n\}}(\Omega).$$

where the series converges absolutely and uniformly when j *varies in* J *and* φ *varies in any given bounded subset of* $\mathcal{D}_{L^p}^{\{M_n\}}(\Omega)$*.*

PROOF. We apply [2, p. 274] and so obtain a relatively compact infinite subset $\{ T_j : j \in J \}$ of Y' such that ${}^t\lambda(T_j) = S_j$. If $(T_j)_\alpha$ is the element of $\mathcal{L}^q(\Omega)$ which identifies with the restriction of T_j to Y^α, $\alpha \in \mathbb{N}_0^k$, we obtain an element $g_{\alpha,j}$ in $\mathcal{L}^q(\Omega)$ such that

$$\langle \tilde{\varphi}, (T_j)_\alpha \rangle = \int_\Omega \varphi \cdot g_{\alpha,j} \, \mathrm{d}x, \qquad j \in J, \quad \varphi \in \tilde{\varphi} \in \mathcal{L}^p(\Omega).$$

Given $h > 0$, we take r in $\mathbb{N}$ such that $r > h$. From Proposition 3, we obtain that

$$\sup_{\substack{\alpha\in\mathbb{N}_0^k \\ j\in J}} h^{|\alpha|} M_{|\alpha|} \|g_{\alpha,j}\|_q < \infty.$$

Having in mind Proposition 2, it follows that, for each $(\tilde{f}_\alpha)$ in Y and $j \in J$,

$$\langle (\tilde{f}_\alpha), T_j \rangle = \sum_{\alpha\in\mathbb{N}_0^k} \int_\Omega f_\alpha \cdot g_{\alpha,j} \, \mathrm{d}x, \qquad f_\alpha \in \tilde{f}_\alpha, \quad \alpha \in \mathbb{N}_0^k,$$

and, in particular, if φ is in $\mathcal{D}_{(L^p)}^{\{M_n\}}(\Omega)$, then

$$\langle (\tilde{D}^\alpha \varphi), T_j \rangle = \sum_{\alpha\in\mathbb{N}_0^k} \int_\Omega D^\alpha \varphi \cdot g_{\alpha,j} \, \mathrm{d}x$$

and besides

$$\langle (\tilde{D}^\alpha \varphi), T_j \rangle = \langle \lambda(\varphi), T_j \rangle = \langle \varphi, {}^t\lambda(T_j) \rangle = \langle \varphi, S_j \rangle.$$

The result now follows without difficulty. ■

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Manuel Valdivia
Departamento de Análisis Matemático,
Facultad de Matemáticas
Universidad de Valencia
Calle Doctor Moliner, 50
46100 Burjassot (Valencia, Spain).