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Holomorphically Dependent Generalised Inverses

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Abstract. In this article we investigate when the pointwise existence of a generalised inverse for holomorphic operator-valued mappings defined on domains in a Banach space implies the existence of a holomorphic generalised inverse.

Inversas Generalizadas Holomórficamente Dependientes

Resumen. En este artículo investigamos cuándo la existencia puntual de una inversa generalizada de una aplicación holomorfa operador-valuada definida en un dominio de un espacio de Banach implica la existencia de una inversa generalizada holomorfa.

1 Introduction

Let f denote a holomorphic mapping from a domain Ω in a Banach space into $\mathcal{L}(X, Y)$, the space of continuous linear mappings from the Banach space X into the Banach space Y. Over many years different authors, e.g. [1, 2, 4, 5, 7, 12], have considered when pointwise invertibility properties, of various kinds, imply the existence of a globally smooth inverse of the same kind. For example, if $f(z)$ has a right inverse for each $z \in \Omega$ does there exist g, holomorphic on Ω with values in $\mathcal{L}(Y, X)$, such that $g(z)$ is a right inverse for a $f(z)$ for all $z \in \Omega$? In this paper we continue our investigations of such problems. Many results are known when Ω is a domain in a finite dimensional space and our interest is focused on the problems that arise when Ω is a domain in an infinite dimensional space.

We refer to [6, 10] for background information on operators between Banach spaces, to [3, 9] for the theory of holomorphic mappings on Banach spaces and to [6, 7, 12] for classical results on holomorphic dependence of operator-valued functions over finite dimensional complex manifolds.

2 Linear Preliminaries

If X and Y are Banach spaces over $\mathbb{C}$, $\mathcal{L}(X, Y)$ will denote the space of all continuous linear operators from X to Y and $GL(X, Y)$ will denote the set of all invertible linear operators from X to Y. If X and Y are subspaces of the Banach space Z we use the notation $Z = X \oplus Y$ to indicate that X and Y are closed complemented subspaces of Z and that Z is the direct sum of X and Y. We let $\mathcal{H}(\Omega, X)$ denote the set of all X-valued holomorphic mappings defined on an open subset Ω of a Banach space. We also use the standard notation $\mathcal{L}(X) := \mathcal{L}(X, X)$ and $GL(X) := GL(X, X)$.

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Definition 1. *Let $T \in \mathcal{L}(X,Y)$. If $S \in \mathcal{L}(Y,X)$ and $TST = T$ we call S a* pseudo-inverse *for T. If, in addition, $STS = S$ we call S a generalised inverse for T. If $TS = \mathbf{1}_Y$ we call S a right inverse for T. The operator T is called splitting if $\ker(T)$ and $\operatorname{im}(T)$ are complemented in X and Y respectively.*

The following proposition contains some important known results about generalised inverses ([2, 12]).

Proposition 1. *If X and Y are Banach spaces and $T \in \mathcal{L}(X,Y)$ then the following are equivalent:*

(a) *T has a pseudo-inverse,*

(b) *T has a generalised inverse,*

(c) *T is a splitting operator.*

Right inverses are generalised inverses and generalised inverses are pseudo-inverses. If S is a pseudo-inverse for T then STS is a generalised inverse for T.

We require the following construction of a generalised inverse. Let $T \in \mathcal{L}(X,Y)$ and suppose $X = \ker(T) \oplus X_1$ and $Y = Y_1 \oplus \operatorname{im}(T)$ are direct sum decompositions. The restriction of T to X_1, T_R, is a continuous bijective linear mapping from X_1 onto $\operatorname{im}(T)$ and has, by the open mapping theorem, a continuous inverse, T_R^{-1}. We define $S\colon Y \to X$ by letting $S(y_1 + y_2) = T_R^{-1}(y_2)$ for $y_1 \in Y_1$ and $y_2 \in \operatorname{im}(T)$. If $x_1 \in \ker(T)$ and $x_2 \in X_1$ then

$$TST(x_1 + x_2) = TST(x_2) = T(T_R^{-1}T(x_2)) = T(x_2) = T(x_1 + x_2)$$

and $TST = T$. Moreover, if $y_1 \in Y_1$ and $y_2 \in \operatorname{im}(T)$, then

$$STS(y_1 + y_2) = S(TT_R^{-1}(y_2)) = S(y_2) = S(y_1 + y_2),$$

and S is a generalised inverse for T.

Lemma 1. *If P and Q are projections in $\mathcal{L}(X)$ and $\|P - Q\| < 1$ then $(\mathbf{1}_X - P + Q) \in GL(X)$ and $(\mathbf{1}_X - P + Q)(P(X)) = Q(X)$. In particular, $P(X) \simeq Q(X)$.*

PROOF. Let $R := \mathbf{1}_X - P + Q$. Since $(\mathbf{1}_X - P + Q)P = QP$ we have

$$R(P(X)) = (\mathbf{1}_X - P + Q)(P(X)) \subseteq Q(X). \tag{1}$$

Since $\|P - Q\| < 1$, $R := \mathbf{1}_X - P + Q \in GL(X)$ and

$$R^{-1} = (\mathbf{1}_X - P + Q)^{-1} = \sum_{n=0}^{\infty} (P - Q)^n = \left[\sum_{n=0}^{\infty} (P - Q)^{2n}\right] (\mathbf{1}_X + P - Q).$$

Interchanging P and Q in (1) we obtain $(\mathbf{1}_X - Q + P)(Q(X)) \subseteq P(X)$ and as $(P - Q)^2 P = P(\mathbf{1}_X - QP)$ we see that $(P - Q)^2 P(X) \subseteq P(X)$. Hence $R^{-1}(Q(X)) \subseteq P(X)$ and $Q(X) \subseteq R(P(X))$. Combining this with (1) completes the proof. ∎

3 Vector Bundles

In this section we recall the definition of holomorphic Banach vector bundles and generalise to Banach spaces a result of Shubin [11] (see also [12, Theorem 3.11]).

Let $\pi\colon \mathcal{E} \to \Omega$ be a surjective holomorphic map of complex Banach manifolds. We assume that the fibre above $z \in \Omega, \mathcal{E}_z := \pi^{-1}(z)$, has been given a Banach space structure whose topology coincides with the topology induced from $\mathcal{E}$. A collection $(U_\alpha, \tau_\alpha)_{\alpha \in \Lambda}$ is called a *trivialising cover* for π if $(U_\alpha)_{\alpha \in \Lambda}$ is an open cover of Ω and for each $\alpha \in \Lambda$ there is a Banach space X_α such that $\tau_\alpha\colon \pi^{-1}(U_\alpha) \to U_\alpha \times X_\alpha$ is a biholomorphic mapping and conditions (i), (ii) and (iii) below are satisfied.

(i) $\tau_{\alpha,z} := \tau_\alpha|_{\mathcal{E}_z}$ is a linear isomorphism[1], from $\mathcal{E}_z$ onto X_α for each $z \in U_\alpha$.

(ii) $\pi|_{\pi^{-1}(U_\alpha)} = \pi_\alpha \circ \tau_\alpha$, where π_α is the canonical projection from $U_\alpha \times X_\alpha$ onto U_α.

Conditions (i) and (ii) imply that $\rho_{\alpha\beta} := \tau_\alpha \circ \tau_\beta^{-1}|_{U_{\alpha\beta}\times X_\beta}$ has the form $\rho_{\alpha\beta}(z,x) = (z, g_{\alpha\beta}(z)x)$, where $g_{\alpha\beta}(z) \in \mathcal{L}(X_\beta, X_\alpha)$ and $x \in X_\beta$ whenever $\alpha, \beta \in \Lambda$ and $z \in U_{\alpha\beta} := U_\alpha \cap U_\beta \neq \emptyset$.

(iii) If $\alpha, \beta \in \Lambda$ and $U_\alpha \cap U_\beta \neq \emptyset$ then the map $z \mapsto g_{\alpha\beta}(z)$ from $U_{\alpha\beta}$ into $\mathcal{L}(X_\beta, X_\alpha)$ is holomorphic.

Two trivialising covers are said to be *equivalent* if their union is also a trivialising cover.

Definition 2. *A holomorphic vector bundle is a triple $(\mathcal{E}, \pi, \Omega)$, where $\pi\colon \mathcal{E} \to \Omega$ is a surjective holomorphic map of complex Banach manifolds, together with a class of equivalent trivialising covers for π.*

We call $\mathcal{E}$ the *bundle space*, π the *projection* of the bundle, Ω the *base* of the bundle, $\{\tau_\alpha\colon \pi^{-1}(U_\alpha) \to U_\alpha \times X_\alpha\}$, $(U_\alpha, \tau_\alpha, X_\alpha)$, (U_α, τ_α) or just τ_α a *trivialization* of $\pi^{-1}(U_\alpha)$ and $g_{\alpha\beta}$ a *transition map*. Note that $g_{\alpha\alpha}(z) = \mathbf{1}_{X_\alpha}$ for all $z \in U_\alpha$, $g_{\alpha\beta}g_{\beta\gamma} = g_{\alpha\gamma}$ on $U_{\alpha\beta\gamma} := U_\alpha \cap U_\beta \cap U_\gamma \neq \emptyset$, and $g_{\alpha\beta}(z)^{-1} = g_{\beta\alpha}(z)$ for all $z \in U_{\alpha\beta}$. For convenience, we often write $\mathcal{E}$ in place of $(\mathcal{E}, \pi, \Omega)$.

If X is a Banach space and Ω is a complex manifold, the triple $(\Omega \times X, \pi, \Omega)$, where π is the canonical projection from $\Omega \times X$ onto Ω, together with the covering trivialisation $(\mathbf{1}_{\Omega\times X}\colon \Omega \times X \to \Omega \times X)$ is called the *trivial bundle*. If $\mathcal{E}$ is a holomorphic vector bundle and (U, τ, X) is a trivialisation of $\pi^{-1}(U)$ then $\mathcal{E}_U := (\pi^{-1}(U), \pi|_{\pi^{-1}(U)}, U)$ is a trivial bundle with covering trivialisation (U, τ, X).

A *holomorphic section* of the holomorphic vector bundle $(\mathcal{E}, \pi, \Omega)$ is a holomorphic mapping $f\colon \Omega \to \mathcal{E}$ such that $\pi \circ f = \mathbf{1}_\Omega$. We let $\Gamma(\mathcal{E})$ denote the set of all holomorphic sections of $\mathcal{E}$. For any complex manifold Ω and any Banach space X, $\Gamma(\Omega \times X) \simeq \mathcal{H}(\Omega, X)$.

In proving the main result in this section we require the following important theorem of Lempert [8].

Theorem 1. *Let Z be a Banach space with an unconditional basis, $\Omega \subset Z$ pseudo-convex open, $\mathcal{E} \to \Omega$ a holomorphic Banach vector bundle, then the sheaf coholomogy groups $H^q(\Omega, \mathcal{E})$ vanish for all $q \geq 1$.*

Let $(U_\alpha)_{\alpha\in\Gamma}$ be an open covering of Ω. A *Cousin data* for $(U_\alpha)_{\alpha\in\Gamma}$ is a collection of functions $f_{\alpha\beta} \in \mathcal{H}(U_{\alpha\beta}, \mathcal{E})$ satisfying $f_{\alpha\beta} + f_{\beta\alpha} = 0$ on $U_{\alpha\beta} := U_\alpha \cap U_\beta \neq \emptyset$, and $f_{\alpha\beta} + f_{\beta\gamma} + f_{\gamma\alpha} = 0$ on $U_{\alpha\beta\gamma} := U_\alpha \cap U_\beta \cap U_\gamma$ whenever $U_{\alpha\beta\gamma} \neq \emptyset$. The *additive Cousin problem* consists in finding $f_\alpha \in \mathcal{H}(U_\alpha, \mathcal{E})$, for all α, such that

$$f_\alpha|_{U_{\alpha\beta}} - f_\beta|_{U_{\alpha\beta}} = f_{\alpha\beta}$$

whenever $U_{\alpha\beta} \neq \emptyset$. Since the Cousin data form a 1-cocycle, when $q = 1$ Theorem 1 implies the following result.

Corollary 1. *Let Z be a Banach space with an unconditional basis, Ω be a pseudo-convex open subset of Z, and $(\mathcal{E}, \pi, \Omega)$ a holomorphic Banach vector bundle. If $(U_\alpha)_{\alpha\in\Gamma}$ is an open cover of Ω and $f_{\alpha\beta} \in \mathcal{H}(U_{\alpha\beta}, \mathcal{E})$ is a Cousin data then the corresponding Cousin problem is solvable.*

Example 1. If $(\mathcal{E}, \pi, \Omega)$ is a holomorphic vector bundle we let $\mathcal{L}(\mathcal{E}) = \bigcup_{z\in\Omega} \mathcal{L}(\mathcal{E}_z)$ and let $\theta(T_z) = z$ for all $T_z \in \mathcal{L}(\mathcal{E}_z)$. Then $\theta\colon \mathcal{L}(\mathcal{E}) \to \Omega$ is surjective and $\theta^{-1}(\{z\}) = \mathcal{L}(\mathcal{E})_z = \mathcal{L}(\mathcal{E}_z)$. We endow $\mathcal{L}(\mathcal{E})_z$ with the Banach space structure from $\mathcal{L}(\mathcal{E}_z)$. Let $\{\,\tau_\alpha\colon \pi^{-1}(U_\alpha) \to U_\alpha \times X_\alpha\,\}_{\alpha\in\Lambda}$ be a trivialising cover for $\mathcal{E}$. For $z \in U_\alpha$ and $T_z \in \mathcal{L}(\mathcal{E}_z)$ let

$$\overset{\wedge}{\tau}_\alpha(T_z) = (z, \tau_{\alpha,z} \circ T_z \circ \tau_{\alpha,z}^{-1}).$$

Then $\overset{\wedge}{\tau}_\alpha\colon \theta^{-1}(U_\alpha) \to U_\alpha \times \mathcal{L}(X_\alpha)$ is a bijective mapping and $\overset{\wedge}{\tau}_{\alpha,z}\colon \mathcal{L}(\mathcal{E})_z \to \mathcal{L}(X_\alpha)$ is a continuous linear mapping for all $z \in U_\alpha$. If

[1]Here and elsewhere we identify, when necessary, $\{z\} \times X_\alpha$ and X_α.

$$\overset{\wedge}{\tau}_{\alpha\beta} := \overset{\wedge}{\tau}_{\alpha} \circ \overset{\wedge}{\tau}_{\beta}^{\;-1} : U_{\alpha\beta} \times \mathcal{L}(X_\beta) \longrightarrow U_{\alpha\beta} \times \mathcal{L}(X_\alpha) \tag{2}$$

then, for $z \in U_{\alpha\beta}$ and $T \in \mathcal{L}(X_\beta)$, we have

$$\overset{\wedge}{\tau}_{\alpha\beta}(z, T) = (z, g_{\alpha\beta}(z) \circ T \circ g_{\beta\alpha}(z))$$

where, as previously, $\rho_{\alpha\beta}$, and the transition mappings $g_{\alpha\beta}$ are defined by

$$\rho_{\alpha\beta}(z, x) := \tau_\alpha \circ \tau_\beta^{-1}(z, x) =: (z, g_{\alpha\beta}(z)x)$$

for $z \in U_{\alpha\beta}$ and $x \in X_\beta$. This implies that $\overset{\wedge}{\tau}_{\alpha\beta}$ is biholomorphic for all $\alpha, \beta \in \Lambda$ whenever $U_{\alpha\beta} \neq \emptyset$. The bijective mappings $(\overset{\wedge}{\tau}_\alpha)_{\alpha\in\Lambda}$ can now be used with (2) to define a unique complex manifold structure on $\mathcal{L}(\mathcal{E})$ such that $\overset{\wedge}{\tau}_\alpha \colon \theta^{-1}(U_\alpha) \to U_\alpha \times \mathcal{L}(X_\alpha)$ is biholomorphic for all α and such that $(\mathcal{L}(\mathcal{E}), \theta, \Omega)$ is a holomorphic vector bundle with trivialising cover $(U_\alpha, \overset{\wedge}{\tau}_\alpha)_{\alpha\in\Lambda}$. This bundle has transition maps $\hat{g}_{\alpha\beta} \in \mathcal{H}\big(U_{\alpha\beta}, \mathcal{L}(\mathcal{L}(X_\beta), \mathcal{L}(X_\alpha))\big)$ where

$$\left[\hat{g}_{\alpha\beta}(z)\right](T) = g_{\beta\alpha}(z) \circ T \circ g_{\alpha\beta}(z)$$

for $z \in U_{\alpha\beta}$ and $T \in \mathcal{L}(X_\beta)$.

A *sub-bundle* of $(\mathcal{E}, \pi, \Omega)$ is a bundle $(\mathcal{F}, \eta, \Omega)$ where $\mathcal{F}$ is a subset of $\mathcal{E}$, $\eta = \pi|_{\mathcal{F}}$, $\mathcal{F}_z$ is a closed subspace of $\mathcal{E}_z$ for all $z \in \Omega$ and the following condition holds:

There exists a trivialising cover $\{\, \tau_\alpha \colon \pi^{-1}(U_\alpha) \to U_\alpha \times X_\alpha \,\}_{\alpha\in\Lambda}$ *for* $\mathcal{E}$*, and a collection of Banach spaces* $(Y_\alpha)_{\alpha\in\Lambda}$*,* $Y_\alpha \subset X_\alpha$*, such that* $\{\, \tau_\alpha|_{\eta^{-1}(U_\alpha)} \colon \eta^{-1}(U_\alpha) \to U_\alpha \times Y_\alpha \,\}_{\alpha\in\Lambda}$ *is a trivialising cover for* $\mathcal{F}$*.*

Note that a sub-bundle is defined locally, that is given a bundle $(\mathcal{E}, \pi, \Omega)$ and an open cover of Ω, $(U_\alpha)_\alpha$, and for each α a sub-bundle $\mathcal{F}_\alpha$ of $\mathcal{E}_{U_\alpha}$, then there exists a unique sub-bundle $\mathcal{F}$ of $\mathcal{E}$ such that $\mathcal{F}_{U_\alpha} = \mathcal{F}_\alpha$.

This means that we may and do identify Y_α with a subspace of X_α and, moreover, that $[g_{\alpha\beta}(z)]Y_\beta = Y_\alpha$ for the transition functions $g_{\alpha\beta}$ where $z \in U_{\alpha\beta}$ and $\alpha, \beta \in \Lambda$. If each Y_α is a complemented subspace of X_α the sub-bundle is called a *direct sub-bundle*.

Sub-bundles can also be characterised by using transition functions. Suppose we are given a trivialising cover $\{\, \tau_\alpha \colon \pi^{-1}(U_\alpha) \to U_\alpha \times X_\alpha \,\}_{\alpha\in\Lambda}$ for $\mathcal{E}$ with transition functions $(g_{\alpha\beta})_{\alpha,\beta\in\Lambda}$, and a collection of Banach spaces $(Y_\alpha)_{\alpha\in\Lambda}$, $Y_\alpha \subset X_\alpha$, such that $[g_{\alpha\beta}(z)]Y_\beta \subset Y_\alpha$ for all $\alpha, \beta \in \Lambda$ and all $z \in U_\alpha \cap U_\beta$. Since $g_{\alpha\beta}(z)^{-1} = g_{\beta\alpha}(z)$ this implies

$$[g_{\alpha\beta}(z)]Y_\beta = Y_\alpha \tag{3}$$

for all $z \in U_{\alpha\beta}$. Let $\mathcal{F} = \cup_{\alpha\in\Lambda}\pi^{-1}(U_\alpha \times Y_\alpha)$, $\eta = \pi|_{\mathcal{F}}$ and $\varphi_\alpha = \tau_\alpha|_{\eta^{-1}(U_\alpha)}$ for all $\alpha \in \Lambda$. Then $\varphi_{\alpha,z} \colon \eta^{-1}(\{z\}) = \mathcal{F}_z \to \{z\} \times Y_\alpha$ is bijective and the Banach space $\mathcal{E}_z$ induces on $\mathcal{F}_z$ a Banach space structure. Since each φ_α is the restriction of a bijective mapping it also is bijective onto its image and as $\varphi_{\alpha\beta} := \varphi_\alpha \circ \varphi_\beta^{-1}(z, y) = (z, g_{\alpha\beta}(z)y)$ for all $(z, y) \in U_{\alpha\beta} \times Y_\beta$ we see, by (3), that $(\mathcal{F}, \eta, \Omega)$ is a holomorphic vector bundle with trivialising cover $\{\, \varphi_\alpha \colon \eta^{-1}(U_\alpha) \to U_\alpha \times Y_\alpha \,\}_{\alpha\in\Lambda}$. Since $\varphi_\alpha = \tau_\alpha|_{\eta^{-1}(U_\alpha)}$, $\mathcal{F}$ is a sub-bundle of $\mathcal{E}$.

Example 2. Let $(\mathcal{F}, \eta, \Omega)$ be a sub-bundle of the holomorphic vector bundle $(\mathcal{E}, \pi, \Omega)$. By definition we can find a trivialising cover for π, $\{\, \tau_\alpha \colon \pi^{-1}(U_\alpha) \to U_\alpha \times X_\alpha \,\}_{\alpha\in\Lambda}$ and a collection of Banach spaces $(Y_\alpha)_{\alpha\in\Lambda}$, $Y_\alpha \subset X_\alpha$, such that $\{\, \tau_\alpha|_{\eta^{-1}(U_\alpha)} \colon \eta^{-1}(U_\alpha) \to U_\alpha \times Y_\alpha \,\}$ is a trivialising cover for η. Let $(\mathcal{L}(\mathcal{E}), \theta, \Omega)$ denote the holomorphic vector bundle with trivialising cover $\{\, \overset{\wedge}{\tau}_\alpha \colon \theta^{-1}(U_\alpha) \to U_\alpha \times \mathcal{L}(X_\alpha) \,\}_{\alpha\in\Lambda}$ constructed in Example 1.

For each $\alpha \in \Lambda$ let

$$Z_\alpha := \{\, T \in \mathcal{L}(X_\alpha) : T(X_\alpha) \subset Y_\alpha, T(Y_\alpha) = 0 \,\}.$$

For α, $\beta \in \Lambda$, $z \in U_{\alpha\beta}$ and $T \in Z_\beta$ we have

$$\begin{aligned} [\hat{g}_{\alpha\beta}(z)(T)](X_\alpha) &\subset g_{\alpha\beta}(z) \circ T(g_{\beta\alpha}(z)X_\alpha) \\ &\subset g_{\alpha\beta}(z) \circ T(X_\beta) \\ &\subset g_{\alpha\beta}(z)(Y_\beta) \\ &\subset Y_\alpha \end{aligned}$$

and

$$[\hat{g}_{\alpha\beta}(z)(T)](Y_\alpha) \subset g_{\alpha\beta}(z)(T(Y_\beta)) = \{0\}.$$

Hence $\hat{g}_{\alpha\beta}(z)(Z_\beta) \subset Z_\alpha$ for all $z \in U_{\alpha\beta}$. This implies, following our discussion above, that $\mathcal{L}(\mathcal{E} \circledcirc \mathcal{F}) := \cup_{\alpha\in\Lambda} \overset{\wedge}{\tau}_\alpha^{\,-1}(U_\alpha \times Z_\alpha)$ can be endowed with the structure of a sub-bundle of $\mathcal{L}(\mathcal{E})$.

An *endomorphism* of the holomorphic vector bundle $(\mathcal{E}, \pi, \Omega)$ is a holomorphic mapping $f\colon \mathcal{E} \to \mathcal{E}$ such that $f \circ \pi = \pi$, $f_z := f|_{\mathcal{E}_z}$ is a continuous linear mapping for all $z \in \Omega$, and the mapping

$$z \in U \longrightarrow \tau_z \circ f_z \circ \tau_z^{-1} \in \mathcal{L}(X) \tag{4}$$

is holomorphic for any trivialising map $\tau\colon \pi^{-1}(U) \to U \times X$. We denote by $\mathcal{M}(\mathcal{E})$ the set of all endomorphisms of $\mathcal{E}$. If $f_z^2 = f_z$ for all $z \in \Omega$ we call f a *projection*.

Using the notation of Examples 1 and 2 we see that the mapping

$$\theta\colon \mathcal{M}(\mathcal{E}) \longrightarrow \Gamma(\mathcal{L}(\mathcal{E})), \qquad [\theta(A)](z) := A|_{\mathcal{E}_z} \tag{5}$$

is bijective and, moreover, if $\mathcal{F}$ is a sub-bundle of $\mathcal{E}$ then

$$A(\mathcal{E}) \subset \mathcal{F} \Longleftrightarrow [\theta(A)(z)]\mathcal{E}_z \subset \mathcal{F}_z \text{ for all } z \in \Omega \tag{6}$$

and

$$A(\mathcal{F}) = \{0\} \Longleftrightarrow [\theta(A)(z)]\mathcal{F}_z = \{0\} \text{ for all } z \in \Omega. \tag{7}$$

Clearly $A \in \mathcal{M}(\mathcal{E})$ is a projection if and only if $[\theta(A)](z)$ is a (linear) projection for all $z \in \Omega$. For the trivial bundle, $\mathcal{M}(\Omega \times X) \simeq \mathcal{H}(\Omega, \mathcal{L}(X))$.

Proposition 2. *Let Ω be a pseudo-convex open subset of a Banach space with an unconditional basis. If $\mathcal{F} := (\mathcal{F}, \eta, \Omega)$ is a sub-bundle of the holomorphic vector bundle $(\mathcal{E}, \pi, \Omega)$ then $\mathcal{F}$ is a direct sub-bundle if and only if there exists a projection $p \in \mathcal{M}(\mathcal{E})$ such that $p(\mathcal{E}) = \mathcal{F}$.*

PROOF. We first suppose that $\mathcal{F}$ is a direct sub-bundle of $\mathcal{E}$. Let $\{\tau_\alpha\colon \pi^{-1}(U_\alpha) \to U_\alpha \times X_\alpha\}_{\alpha\in\Lambda}$ denote a trivialising cover for $\mathcal{E}$ such that $\{\tau_\alpha|_{\eta^{-1}(U_\alpha)}\colon \eta^{-1}(U_\alpha) \to U_\alpha \times Y_\alpha\}$ is a trivialising cover for $\mathcal{F}$. By our hypothesis Y_α is a complemented subspace of X_α and we let $P_\alpha \in \mathcal{L}(X_\alpha)$ denote a continuous projection onto Y_α for each $\alpha \in \Lambda$. For each α let $\mathcal{E}_\alpha$ denote the holomorphic vector bundle $(\pi^{-1}(U_\alpha), \pi|_{\pi^{-1}(U_\alpha)}, U_\alpha)$ with trivialising cover $(U_\alpha, \tau_\alpha, X_\alpha)$. Then $\mathcal{F}_\alpha := (\eta^{-1}(U_\alpha), \eta|_{\eta^{-1}(U_\alpha)}, U_\alpha)$ with trivialising cover $(U_\alpha, \tau_\alpha|_{\eta^{-1}(U_\alpha)}, Y_\alpha)$ is a direct sub-bundle of $\mathcal{E}_\alpha$. We define $f_\alpha\colon \mathcal{E}_\alpha \to \mathcal{E}_\alpha$ as follows: if $z \in U_\alpha$ let $f_\alpha|_{\mathcal{E}_z} =: f_{\alpha,z}$ where

$$f_{\alpha,z}(\xi) = \tau_{\alpha,z}^{-1} \circ P_\alpha \circ \tau_{\alpha,z}(\xi)$$

for all $\xi \in \mathcal{E}_z$. Then $f_{\alpha,z} \in \mathcal{L}(\mathcal{E}_z)$ is a projection with $f_{\alpha,z}(\mathcal{E}_z) = \mathcal{F}_z$ for all $z \in U_\alpha$. Since $\tau_{\alpha,z} \circ f_\alpha \circ \tau_{\alpha,z}^{-1} = P_\alpha$, $f_\alpha \in \mathcal{M}(\mathcal{E}_\alpha)$ and $f_\alpha(\mathcal{E}_\alpha) = \mathcal{F}_\alpha$.

If $\alpha, \beta \in \Lambda$ and $U_{\alpha\beta} \neq \emptyset$ let $f_{\alpha\beta} = f_\alpha|_{\mathcal{E}_{\alpha\beta}} - f_\beta|_{\mathcal{E}_{\alpha\beta}}$. Then $f_{\alpha\beta} \in \mathcal{M}(\mathcal{E}_{\alpha\beta})$ and $f_{\alpha\beta}(\mathcal{E}_{\alpha\beta}) \subset \mathcal{F}_{\alpha\beta}$. Since $f_\alpha(\xi) = f_\beta(\xi) = \xi$ for all $z \in U_{\alpha\beta}$ and all $\xi \in \mathcal{F}_z$, $f_{\alpha\beta}(\mathcal{F}_{\alpha\beta}) = \{0\}$. By (5) we can identify $f_{\alpha\beta}$ with $g_{\alpha\beta} \in \Gamma(\mathcal{L}(\mathcal{E}_{\alpha\beta}))$ and, by Example 2 and (6) and (7), $g_{\alpha\beta} \in \Gamma(\mathcal{L}(\mathcal{E}_{\alpha\beta} \circledcirc \mathcal{F}_{\alpha\beta}))$. Since $(g_{\alpha\beta})_{\alpha,\beta\in\Lambda}$ forms a 1-cocycle in the sheaf of $\mathcal{L}(\mathcal{E} \circledcirc \mathcal{F})$-valued holomorphic germs on Ω, Corollary 1 implies that there exist, for all $\alpha \in \Lambda$, $g_\alpha \in \Gamma(\mathcal{L}(\mathcal{E}_\alpha \circledcirc \mathcal{F}_\alpha))$ such that

$$g_\alpha|_{U_{\alpha\beta}} - g_\beta|_{U_{\alpha\beta}} = g_{\alpha\beta}. \tag{8}$$

By (5) each g_α can be identified with $h_\alpha \in \mathcal{M}(\mathcal{E}_\alpha)$, satisfying $h_\alpha(\mathcal{E}_\alpha) \subset \mathcal{F}_\alpha$ and $h_\alpha(\mathcal{F}_\alpha) = 0$ and, by (8),

$$h_\alpha|_{\mathcal{E}_{\alpha\beta}} - h_\beta|_{\mathcal{E}_{\alpha\beta}} = f_\alpha|_{\mathcal{E}_{\alpha\beta}} - f_\beta|_{\mathcal{E}_{\alpha\beta}}$$

for all α, $\beta \in \Lambda$ whenever $U_{\alpha\beta} \neq \emptyset$. Hence

$$(f_\alpha - h_\alpha)|_{\mathcal{E}_{\alpha\beta}} = (f_\beta - h_\beta)|_{\mathcal{E}_{\alpha\beta}}$$

whenever $U_{\alpha\beta} \neq \emptyset$ and the mapping

$$p(\xi) := f_\alpha(\xi) - h_\alpha(\xi)$$

for all $\xi \in \pi^{-1}(U_\alpha)$ is well defined on $\mathcal{E}$ and belongs to $\mathcal{M}(\mathcal{E})$. Since f_α and h_α both map $\mathcal{E}_\alpha$ into $\mathcal{F}_\alpha$ for all $\alpha \in \Lambda$ it follows that $p(\mathcal{E}) \subset \mathcal{F}$ and as $f_\alpha(\mathcal{F}_\alpha) = \mathcal{F}_\alpha$ and $h_\alpha(\mathcal{F}_\alpha) = \{0\}$ this implies $p(\mathcal{E}) = \mathcal{F}$. If $z \in U_\alpha$ and $\xi \in \mathcal{E}_z$ then $f_{\alpha,z}(h_{\alpha,z}(\xi)) = h_{\alpha,z}(\xi)$, $h_{\alpha,z}(f_{\alpha,z}(\xi)) = 0$, and $h_{\alpha,z}(h_{\alpha,z}(\xi)) = 0$. Hence

$$\begin{aligned}
p(p(\xi)) &= p(f_{\alpha,z}(\xi) - h_{\alpha,z}(\xi)) \\
&= f_{\alpha,z}^2(\xi) - f_{\alpha,z}(h_{\alpha,z}(\xi)) - h_{\alpha,z}(f_{\alpha,z}(\xi)) + h_{\alpha,z}(h_{\alpha,z}(\xi)) \\
&= f_{\alpha,z}(\xi) - h_{\alpha,z}(\xi) \\
&= p(\xi).
\end{aligned}$$

This completes the proof in one direction.

Since the converse is a local result we may suppose that $\mathcal{E}$ is the trivial bundle, $\Omega \times X$, that $p \in \mathcal{H}(\Omega, \mathcal{L}(X))$ and $p(z)$ is a projection for all $z \in \Omega$. We must show that $\mathcal{F} := \{\, (z,x) : x = p(z)x \,\}$ is a direct sub-bundle of $\mathcal{E}$. Fix $w \in \Omega$, and let $X_0 := p(w)X$, $X_1 := (\mathbf{1}_X - p(w))X$. For $z \in \Omega$ let

$$A(z) := p(z)p(w) + (\mathbf{1}_X - p(z))(\mathbf{1}_X - p(w)).$$

Since $A(w) = \mathbf{1}_X$ we can choose a neighbourhood of w, V_w, such that $A(z)$ is invertible on V_w. Then

$$A(z)(X_0) = p(z)p(w)X \subset p(z)X$$

and

$$\begin{aligned}
A(z)(X_1) &= (\mathbf{1}_X - p(z))(\mathbf{1}_X - p(w))X \\
&\subset (\mathbf{1}_X - p(z))X.
\end{aligned}$$

Since $A(z)$ is invertible on V_w we have $A(z)(X_0 + X_1) = X$, hence $A(z)(X_0) = p(z)X$ and $A(z)(X_1) = (\mathbf{1}_X - p(z))X$. If $B(z)$ denotes the inverse of $A(z)$ then $X_0 = B(z)(p(z)X)$ for all $z \in V_w$ and the mapping

$$V_w \times X \to V_w \times X : (z,x) \to (z, B(z)x)$$

provides the required trivialization. This completes the proof. ■

Note that we did not require pseudo-convexity or Corollary 1 for the second half of the proof.

4 Generalised Inverses

In this section we consider the following question: if $f \in \mathcal{H}(\Omega, \mathcal{L}(X,Y))$ and $f(z)$ has a generalised inverse at all points in Ω, does f have a holomorphic generalised inverse?

Definition 3. *Let $f \in \mathcal{H}(\Omega, \mathcal{L}(X,Y))$, where X and Y are Banach spaces and Ω is an open subset of a Banach space. A mapping $g \in \mathcal{H}(\Omega, \mathcal{L}(Y,X))$ is called a holomorphic generalised inverse for f if $g(z)$ is a generalised inverse for $f(z)$ for all $z \in \Omega$.*

The following example shows that a holomorphic generalised inverse need not always exist.

Example 3. If $h(z) = z\mathbf{1}_H$, where H is a one dimensional Hilbert space, then $h \in \mathcal{H}(\mathbb{C}, \mathcal{L}(H))$. If $z \neq 0$, $f(z)$ is invertible and we have a unique generalised inverse $g(z) := (f(z))^{-1} = z^{-1}\mathbf{1}_H$. Since $\lim_{z\to 0} g(z)$ does not exist f does not have a holomorphic generalised inverse.

Proposition 3. *Let $f \in \mathcal{H}(\Omega, \mathcal{L}(X,Y))$, where X and Y are Banach spaces and Ω is an open subset of a Banach space. Then f has a holomorphic generalised inverse if and only if there exist $P \in \mathcal{H}(\Omega, \mathcal{L}(X))$ and $Q \in \mathcal{H}(\Omega, \mathcal{L}(Y))$ such that $P(z)$ is a continuous projection onto $\ker(f(z))$ and $Q(z)$ is a continuous projection onto $\mathrm{im}(f(z))$ for all $z \in \Omega$.*

PROOF. If g is a holomorphic generalised inverse for f then the mappings P and Q, defined by letting $P(z) := g(z) \circ f(z)$ and $Q(z) := f(z) \circ g(z)$, are the required projection-valued holomorphic mappings.

Conversely, suppose we are given the projection-valued holomorphic mappings P and Q. For convenience let $P^*(z) = \mathbf{1}_X - P(z)$ and let I_z denote the natural injection from $P^*(z)X$ into X for all $z \in \Omega$. Let

$$g(z) := I_z \circ \big(f^*(z)\big)^{-1} \circ Q(z) \tag{9}$$

where $f^*(z) = f(z)|_{P^*(z)X}$. The linear result in the second section shows that $g(z)$ is a generalised inverse for $f(z)$ for all $z \in \Omega$.

To show that g is holomorphic we fix $w \in \Omega$ and choose $\epsilon > 0$ such that $W := \{\, z : \|z-w\| < \epsilon \,\} \subset \Omega$, $\|P(z) - P(w)\| < 1$ and $\|Q(z) - Q(w)\| < 1$ for all $z \in W$. Let $U(z) = \mathbf{1}_X + P(z) - P(w) = \mathbf{1}_X - P^*(z) + P^*(w)$ and $V(z) = \mathbf{1}_Y - Q(z) + Q(w) = \mathbf{1}_Y + Q^*(z) - Q^*(w)$ for all $z \in W$. By Lemma 1, $U \in \mathcal{H}(W, GL(X))$, $V \in \mathcal{H}(W, GL(Y))$, $U(z)(P^*(z)X) = P^*(w)X$ and $V(z)(Q(z)Y) = Q(w)Y$ for all $z \in W$. We have

$$\begin{aligned} g(z) : &= (I_w \circ U(z)^{-1}) \circ (U(z) \circ \big(f^*(z)\big)^{-1} \circ V(z)^{-1}) \circ (V(z) \circ Q(z)) \\ &= (I_w \circ U(z)^{-1}) \circ (V(z) \circ f(z) \circ U(z)^{-1})^{-1} \circ (V(z) \circ Q(z)). \end{aligned}$$

Since $V(z) \circ Q(z) = Q(w) \circ Q(z)$ for all $z \in W$ the mapping $z \to V(z) \circ Q(z)$ lies in $\mathcal{H}(W, \mathcal{L}(Y, Q(w)Y)$. By Lemma 1, the mapping $z \in W \to I_w \circ U(z)^{-1}$ belongs to $\mathcal{H}(W, \mathcal{L}(P^*(w)X, X)$. It remains to show that the mapping

$$z \longrightarrow k(z) := \big(V(z) \circ f(z) \circ U(z)^{-1}\big)^{-1}$$

lies in $\mathcal{H}(W, \mathcal{L}(Q(w)Y, P^*(w)X))$. By construction the mapping

$$z \longrightarrow k^*(z) := V(z) \circ f(z) \circ U(z)^{-1}$$

lies in $\mathcal{H}(\Omega, GL(P^*(w)X, Q(w)Y))$ and, as $k(z) = (k^*(z))^{-1}$, this proves that k is holomorphic. This completes the proof. ■

We now present the main result in this article. Note that for $z \in \Omega$, $\ker(f(z))$ is the kernel of a linear operator while $\ker(f)$ is a holomorphic vector bundle.

Theorem 2. *Let Ω be a pseudo-convex open subset of a Banach space with an unconditional basis and let X and Y be Banach spaces. If $f \in \mathcal{H}(\Omega, \mathcal{L}(X,Y))$ has a generalised inverse for each $z \in \Omega$, then the following conditions are equivalent:*

(1) *f has a holomorphic generalised inverse on Ω,*

(2) *There exist holomorphic projections $P \in \mathcal{H}(\Omega, \mathcal{L}(X)) \simeq \mathcal{M}(\Omega \times X)$ onto $\ker(f) := \{ (z,x) : z \in \Omega, x \in X, f(z)x = 0 \}$ and $Q \in \mathcal{H}(\Omega, \mathcal{L}(Y)) \simeq \mathcal{M}(\Omega \times Y)$ onto $\mathrm{im}(f) := \{ (z,y) : z \in \Omega, y \in Y, y = f(z)x$ for some $x \in X \}$,*

(3) *$\ker(f)$ and $\mathrm{im}(f)$ are direct sub-bundles of the trivial bundles $\Omega \times X$ and $\Omega \times Y$ respectively,*

(4) *For every $w \in \Omega$ there exist a neighbourhood V_w of w and closed subspaces $X_w \subset X$ and $Y_w \subset Y$ such that for all $z \in V_w$, $\ker(f(z)) \oplus X_w = X$ and $\mathrm{im}(f(z)) \oplus Y_w = Y$.*

PROOF. By Proposition 3, (1) and (2) are equivalent. By Proposition 2, (2) and (3) are equivalent. By the definition of sub-bundle, (3) implies (4), and it remains to show that (4) implies (3).

Since the result is local we fix $w \in \Omega$ and show that (3) holds on a neighbourhood V_w of w. If $z \in V_w$, $x \in X$ and $y \in Y_w$ let $g(z)(x + y) = f(z)x + y$. Then $g \in \mathcal{H}(V_w, \mathcal{L}(X + Y_w, Y))$,

$$\ker(g(z)) = \ker(f(z)) + \{0\} \quad \text{and} \quad \mathrm{im}(g(z)) = \mathrm{im}(f(z)) + Y_w = Y$$

for all $z \in V_w$. Hence g is surjective with complemented kernel for all $z \in V_w$. By the proof of Proposition 1 (see also Theorem 4 in [4]), $\ker(g) = \{ (z, x, y) \in V_w \times (X + Y_w) : f(z)x = 0, y = 0 \}$ is a direct holomorphic sub-bundle of the trivial bundle $V_w \times (X + Y_w)$. Since $\ker(f|_{V_w}) \simeq \ker(g) \subset V_w \times (X + \{0\}) \simeq V_w \times X$ this implies $\ker(f|_{V_w})$ is a direct sub-bundle of the trivial bundle $V_w \times X$.

By Proposition 2 there exist a holomorphic projection $p \in \mathcal{H}(V_w, \mathcal{L}(X))$ such that $\ker(f(z)) = p(z)(X)$ for all $z \in V_w$. By Lemma 1 and, if necessary, by restricting ourselves to a smaller neighbourhood of w we have $p(z)(X) = p(w)(X) =: Z_w$ for all $z \in V_w$. Hence $X = Z_w \oplus X_w$ and $f(z)(x+y) = f(z)(y)$ for all $z \in V_w$, all $x \in Z_w = \ker(f(z))$, and all $y \in X_w$. If $h(z) := f(z)|_{X_w}$ then $h \in \mathcal{H}(V_w, \mathcal{L}(X_w, Y))$, $h(z)$ is injective and $\mathrm{im}(f(z)) = \mathrm{im}(h(z))$ is a complemented subspace of Y for all $z \in V_w$. By adapting the proof of Proposition 1 in [4] we see that $\mathrm{im}(h) = \mathrm{im}(f|_{V_w})$ is a complemented sub-bundle of the trivial bundle $V_w \times Y$. Hence (4) implies (3) and this completes the proof. ■

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