**RAC**$\mathbb{SAM}$
*Rev. R. Acad. Cien. Serie A. Mat.*
VOL. **102** (2), 2008, pp. 205–209
Análisis Matemático / Mathematical Analysis

# On Completions of LB-spaces of Moscatelli Type

## (Dedicated to the Memory of Walter Roelcke, doctoral promotor of the first author)

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**Abstract.** We first present a class of LF-spaces, extending the class of LF-spaces of Moscatelli type, for which regularity implies completeness. Then we utilize the obtained results to describe the completions of LB-spaces of Moscatelli type. In particular, we prove that the completions of LB-spaces of that type are again LB-spaces.

### Sobre las completaciones de espacios LB de tipo Moscatelli

**Resumen.** Presentamos primero una clase de espacios LF, que extiende los espacios LF de tipo Moscatelli, para la cual la regularidad del límite inductivo implica la completitud. A continuación utilizamos los resultados obtenidos para describir la completación de los espacios LB de tipo Moscatelli usual. En particular, demostramos que la completación de un espacio LB de ese tipo es también un espacio LB.

LB- and LF-spaces of Moscatelli type have been thoroughly investigated since the late eighties, and have acted as a rich source of examples and counterexamples in various contexts (see e.g. [1, 2, 3]). In particular it is known for almost twenty years that LF-spaces of that class are complete, if they are regular. On the other hand, no information about the complete hull of LB-space of that type had been obtained. In view of the following well-known implication: "If it is true that every regular LB-space is complete, then the completion of every LB-space is again an LB-space" a description of the complete hulls of certain LB-spaces is desirable. We will prove that completions of LB-spaces of Moscatelli type are always LB-spaces and present a description.

In our proof we will utilize a result which we prove in the first part of the present article, which also answers a question from the early nineties. In fact, in [2] J. Bonet, C. Fernández and S. D. had presented a class of LF-spaces, extending LF-spaces of Moscatelli type, and could only show that for these spaces regularity implies completeness under certain additional assumptions. We will show now that in that class of generalized LF-spaces of Moscatelli type regularity always implies completeness.

1. Let $(\lambda, \|\cdot\|_\lambda) = \lambda$ be a normal Banach sequence space and let $X$ be a Fréchet space with zero basis $(U_n)_{n\in\mathbb{N}}$ of absolutely convex $U_n$ and the corresponding fundamental sequence of continuous seminorms $(p_{U_n})_{n\in\mathbb{N}}$, where $p_{U_n}$ denotes the Minkowski functional for $U_n (n \in \mathbb{N})$. Then the vector space

$$\lambda(X) := \{x = (x_k)_k \in X^{\mathbb{N}} : (p_{U_n}(x_k))_k \in \lambda \text{ for all } n \in \mathbb{N}\}$$

Presentado por / Submitted by José Bonet.
*Recibido / Received*: 29 de abril de 2008. *Aceptado / Accepted*: 4 de junio de 2008.
*Palabras clave / Keywords*: LB-spaces, LF-spaces, spaces of Moscatelli type, regular, complete
*Mathematics Subject Classifications*: 46A13

endowed with the locally convex topology generated by the seminorms

$$\hat{p}_{U_n} : x \mapsto \|(p_{U_n}(x_k))_k\|_\lambda$$

is again a Fréchet space. If $X$ is a Banach space, then $\lambda(X)$ is a Banach space as well.

Now let $X, Y$ be Fréchet spaces with continuous inclusion $Y \hookrightarrow X$. Then the continuous inclusions

$$\begin{aligned} &F_n := X^n \times \lambda((Y)_{k>n}) := \\ &\{((x_1,\dots,x_n),(y_k)_{k>n}) \in X^n \times \prod_{k>n} Y : ((0,\dots,0),(y_k)_{k>n}) \in \lambda(Y)\} \\ &\hookrightarrow F_{n+1} \quad (n \in \mathbb{N}) \end{aligned}$$

define an inductive sequence of Fréchet spaces and hence lead to the LF-space

$$F := \operatorname*{ind}_{n\to} F_n$$

which can also be represented as the range of the addition map

$$\bigoplus_{\mathbb{N}} X \times \lambda(Y) \longrightarrow X^{\mathbb{N}}, (x,y) \mapsto x+y,$$

where the corresponding quotient (= final) topology coincides with the inductive limit topology. From this representation one obviously obtains a useful description of a 0-basis in $F$. We will frequently write $F = \bigoplus X + \lambda(Y)$. LF-spaces of that type had been studied previously in [2] for the special case $\lambda = \ell^\infty$ and in [3] for the general case.

In [2] also the following extension of LF-spaces of Moscatelli type had been investigated: Let $X$, $Y$, $Z$ be Fréchet spaces with continuous inclusions $Z \hookrightarrow Y \hookrightarrow X$, let again $\lambda$ be a normal Banach sequence space and let $\mu$ denote the closure of $\varphi := \bigoplus_{\mathbb{N}} \mathbb{K}$ (with $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$) in $(\lambda, \|\cdot\|_\lambda)$. Then $\mu$ is again a normal Banach sequence space w.r.t. $\|\cdot\|_\mu := \|\cdot\|_\lambda|\mu$, with the special property of "Abschnittskonvergenz", i.e. every element in $\mu$ is the limit of its finite sections. In this situation the range $E$ of the additon map

$$\bigoplus_{\mathbb{N}} X \times \mu(Y) \times \lambda(Z) \to X^{\mathbb{N}}, (x,y,z) \mapsto x+y+z$$

is an LF-space for the corresponding quotient topology, which we will write $E = \bigoplus X + \mu(Y) + \lambda(Z)$. Moreover, with $E_n := X^n \times (\mu((Y)_{k>n}) + \lambda((Z)_{k>n})$ one has $E = \operatorname*{ind}_{n\to} E_n$ also topologically.

Notation: Given a locally convex space $X$, then $\mathfrak{U}_0(X)$ will denote the filter basis of absolutely convex 0-nbhds.

**Remark 1** *Let $X$, $Y$, $Z$, $\lambda$, $\mu$ be as before. Then $G := \bigoplus X + \mu(Y)$ is a closed topological subspace of $E := \bigoplus X + \mu(Y) + \lambda(Z)$ and $E$ is continuously embedded into $F := \bigoplus X + \lambda(Y)$.*

PROOF. The continuity of the inclusions $G \hookrightarrow E \hookrightarrow F$ is obvious.

Now let $(U_n)_{n\in\mathbb{N}} \in \mathfrak{U}_0(X)^{\mathbb{N}}$ and $V \in \mathfrak{U}_0(Y)$ be given. Put $\mathcal{U} := \bigoplus U_n$, $\mathcal{V} := \{y \in \mu(Y) : \hat{p}_V(y) \leq 1\}$, $\mathcal{W} := \{z \in \lambda(Z) : \hat{p}_{V\cap Z}(z) \leq 1\}$; then $\mathcal{U}+\mathcal{V}+\mathcal{W} \in \mathfrak{U}_0(E)$. Let $x \in \bigoplus X$, $y \in \mu(Y)$, $u \in \mathcal{U}$, $v \in \mathcal{V}$, $w \in \mathcal{W}$ such that $x+y = u+v+w \in G \cap (\mathcal{U}+\mathcal{V}+\mathcal{W})$. By the special property of $\mu$ there is $\tilde{k} \in \mathbb{N}$ such that $\tilde{y} := ((0,\dots,0),(y_k)_{k>\tilde{k}}) \in \mathcal{V}$ and such that $x_k = u_k = 0$ for all $k > \tilde{k}$. Then $x + y - \tilde{y} = u + ((v_k)_{k\leq\tilde{k}}, (0)_{k>\tilde{k}}) + ((w_k)_{k\leq\tilde{k}}, (0)_{k>\tilde{k}}) \in \mathcal{U} + 2\mathcal{V})$ and $x + y \in U + 3\mathcal{V}$, which shows that the inclusion $G \hookrightarrow E$ is open onto its range.

Finally, let $x \in \bigoplus X$, $y \in \mu(Y)$, $z \in \lambda(Z)$ such that $x+y+z \notin \bigoplus X + \mu(Y)$, hence $z \notin \mu(Y)$. Thus there is $V \in \mathfrak{U}_0(Y)$ and $\varepsilon > 0$ such that - with $z^{(k)} := ((0,\dots,0),(z_\ell)_{\ell>k}) - \hat{p}_V(z^{(k)}) > \varepsilon$ for all $k \in \mathbb{N}$. Put $\mathcal{V} := \{v \in \lambda(Y) : \hat{p}_V(v) < \frac{\varepsilon}{z}\}$; then $\bigoplus X + \mathcal{V} \in \mathfrak{U}_0(F)$, hence $\mathcal{U} := (\bigoplus X + \mathcal{V}) \cap E \in \mathfrak{U}_0(E)$ and $(x+y+z+U) \cap G = \emptyset$ (note that $y$ is eventually in $\mathcal{V}$). ■

**Proposition 1** *Let $X$, $Y$, $Z$ be Fréchet spaces with continuous inclusions $Z \hookrightarrow Y \hookrightarrow X$, let $\lambda = (\lambda, \|\cdot\|_\lambda)$ be a normal Banach sequence space and $\mu := \overline{\varphi}^\lambda$. Then the following statements are equivalent:*

(a) $F := \bigoplus X + \lambda(Y)$ *is regular.*

(b) $E := \bigoplus X + \mu(Y) + \lambda(Z)$ *is regular.*

(c) $G := \bigoplus X + \mu(Y)$ *is regular.*

(d) $Y$ *has a* $0$*-basis consisting of* $X$*-closed sets.*

PROOF. a) $\Longleftrightarrow$ d) $\Longleftrightarrow$ c) is true by [3, Chap. III, Prop. 1].
a) $\Longrightarrow$ b). Let $B \subset E$ be bounded. Then $B$ is a bounded subset of $F$, hence there is $n \in \mathbb{N}$ such that $B$ is a bounded subset of $F_n = X^n \times \lambda((Y)_{k>n})$. As $F_n \cap E = E_n := X^n \times (\mu((Y)_{k>n}) + \lambda((Z)_{k>n}),$ $B$ is contained in $E_n$. Moreover, $E_n$ carries the initial topology w.r. to the inclusions $E_n \hookrightarrow F_n$ and $E_n \hookrightarrow E$. In fact, let $U \in \mathfrak{U}_0(X)$, $V \in \mathfrak{U}_0(Y)$, $W \in \mathfrak{U}_0(Z)$, $W \subset V$, $\mathcal{V} := \{y \in \mu(Y) : \hat{p}_V(y) \leq 1\}$, $\mathcal{W} := \{z \in \lambda(Z) : \hat{p}_W(z) \leq 1\}$. Then

$$\begin{aligned}&(\bigoplus X + \mathcal{V} + \mathcal{W}) \cap (U^n \times \{(y_k)_{k>n} : ((0,\ldots,0),(y_k)_{k>n}) \in \mathcal{V}\})\\ &\subset U^n \times (\{(v_k)_{k>n} : ((0,\ldots,0),(v_k)_{k>n}) \in 5\mathcal{V}\} +\\ &\{(w_k)_{k>n} : ((0,\ldots,0),(w_k)_{k>n}) \in \mathcal{W}\},\end{aligned}$$

as can easily be verified.

Consequently $B$ is bounded in $E_n$.

b) $\Longrightarrow$ c). Let $B \subset G$ be bounded. Then $B$ is a bounded subset of $E$, hence there is $n \in \mathbb{N}$ such that $B$ is a bounded subset of $E_n$. Consequently, $B \subset (\bigoplus X + \mu(Y)) \cap E_n = G_n$. Now let $U \in \mathfrak{U}_0(X)$, $V \in \mathfrak{U}_0(Y)$, put $W := V \cap Z$ and put $\mathcal{V} := \{y \in \mu(Y) : \hat{p}_V(y) \leq 1\}$, $\mathcal{W} := \{z \in \lambda(Z) : \hat{p}_W(z) \leq 1\}$. Then $(\bigoplus X + \mathcal{V}) \cap (U^n \times \{(y_k)_{k>n} + (z_k)_{k>n} : ((0,\ldots,0),(y_k)_{k>n}) \in \mathcal{V}$, $((0,\ldots,0),(z_k)_{k>n}) \in \mathcal{W}\}) \subset U^n \times \{(v_k)_{k>n} : ((0,\ldots,0),(v_k)_{k>n}) \in 2\mathcal{V}\}$, which proves that $B$ is bounded in $G_n$. ■

We would like to remark that Prop. 1 remains also valid for "regular" replaced by "$\alpha$-regular" (see the proof and [3, loc. cit.]).

**Remark 2** *Let $X$, $Y$, $Z$, $\lambda$, $\mu$, $E$ and $G$ be as in Proposition 1. Then by Remark 1, $G = \bigoplus X + \mu(Y)$ is a closed topological linear subspace of $E = \bigoplus X + \mu(Y) + \lambda(Z)$. Let $q : E \to E/G$ denote the quotient map. Then*

$$\hat{q} \cdot \lambda(Z) \longrightarrow E/G, z \mapsto q(z)$$

*is linear and continuous and also surjective, as for all $x \in \bigoplus X$, $y \in \mu(Y)$, $z \in \lambda(Z)$, $q(x + y + z) = q(z) = \hat{q}(z)$. $\lambda(Z)$ being a Fréchet space and $E/G$ being barrelled, $\hat{q}$ is open. Since clearly $\ker \hat{q} = \lambda(Z) \cap \mu(Y)$, the quotient $E/G$ is topologically isomorphic to the Fréchet space $\lambda(Z)/(\lambda/Z) \cap \mu(Y))$.*

With the help of these facts we can prove now that for LF-spaces $E$ of generalized Moscatelli type regularity implies completeness.

**Theorem 1** *Let $X$, $Y$, $Z$ be Fŕechet spaces with continuous inclusions $Z \hookrightarrow Y \hookrightarrow X$, let $\lambda$ be a normal Banach sequence space and $\mu := \overline{\varphi}^\lambda$. If $E := \bigoplus X + \mu(Y) + \lambda(Z)$ is regular, then $E$ is complete.*

PROOF. By Remark 1, $G := \bigoplus X + \mu(Y)$ is a closed topological subspace of $E$; by Remark 2 the quotient $E/G$ is complete. If $E$ is regular, then $G$ is regular by Proposition 1, hence complete by [3, Chap. III, Prop. 11]. Now Pasynkov [4] yields the completeness of $E$, as $E$ contains a complete linear subspace such that the corresponding quotient is also complete. ■

2. We will utilize now the above result about generalized inductive limits of Moscatelli type in order to show that the completion of any LB-space of usual Moscatelli type is again an LB-space and to give a description of such a completion.

**Remark 3** *Let $X$ and $Z$ be Banach spaces with unit balls $A$ and $C$ respectively such that $Z \subset X$ and $C \subset A$. Let $\lambda$ be a normal Banach sequence space and $\mu := \overline{\varphi}^{\lambda}$. Put $B := \overline{C}^{X}$ and let $Y := [B]$ be provided with the Minkowski functional $p_B$ such that $Y$ becomes a Banach space admitting the continuous inclusions*

$$Z \hookrightarrow Y \hookrightarrow X.$$

*$F := \bigoplus X + \lambda(Z)$ is then an LB-space, not necessarily regular, whose completion is denoted by $\tilde{F}$. $E := \bigoplus X + \mu(Y) + \lambda(Z)$ is a regular LB-space by Proposition 1, hence complete according to Theorem 1. Furthermore $G := \bigoplus X + \mu(Z)$ is an LB-space which is a topological dense linear subspace of the complete LB-space $\bigoplus X + \mu(Y)$ by* [1, Theorem 3.2 and 2.2] *such that $\tilde{G} := \bigoplus X + \mu(Y)$ is a completion of $G$. Consequently, the inclusion $G \hookrightarrow F$ which is a topological isomorphism onto its range by Remark 1 (case $Y = Z$) has a continuous linear extension $\eta : \tilde{G} \to \tilde{F}$ (which is also a topological isomorphim onto its range). Therefore,*

$$\begin{aligned} g \quad : \quad & \lambda(Z) \times \tilde{G} \longrightarrow \tilde{F} \\ & (z, y) \mapsto z + \eta(y) \end{aligned}$$

*is continuous and linear; its range will be denoted by $H$; $H$ contains $\lambda(Z) + \bigoplus X = F$ and is consequently dense in $\tilde{F}$.*

(1) We will show now that $g$ is open onto its range.

Let $(\varepsilon_k)_k \in (0,1)^{\mathbb{N}}$ and $\varepsilon > 0$ be given. Then putting $\mathcal{W} := \{z \in \lambda(Z) : \|(\|z_k\|_Z)_k\|_\lambda \le \varepsilon\}$, the set $\bigoplus_k \frac{\varepsilon_k}{z} A + \mathcal{W}$ belongs to $\mathfrak{U}_0(F)$, hence there is an absolutely convex $\tilde{F}$-open $U \in \mathfrak{U}_0(\tilde{F})$ such that $U \cap F \subset \bigoplus \frac{\varepsilon_k}{2} A + \mathcal{W}$. We show that $U \cap H \subset g(\mathcal{W} \times (\bigoplus_k \varepsilon_k A + \mathcal{V})$, where $\mathcal{V} := \{y \in \mu(Y) : \|(\|y_k\|_Y)_k\|\mu \le \varepsilon\} \in \mathfrak{U}_0(\mu(Y))$. Let $z \in \lambda(Z), y \in \tilde{G}$ such that $g((z, y)) \in U$. There is a net $(y^{(\iota)})_{\iota \in I}$ in $G = \bigoplus X + \mu(Z)$ such that $y^{(\iota)} \longrightarrow y$ in $\tilde{G} = \bigoplus X + \mu(Y)$, hence $z + y^{(\iota)} = q(z, y^{(\iota)}) \underset{i \in I}{\longrightarrow} g((z, y)) \in U$. $U$ being open in $\tilde{F}$, there is $\iota_0 \in I$ such that $z + y^{(\iota)} \in U \cap F \subset \bigoplus_k \frac{\varepsilon_k}{2} A + \mathcal{W}$. As $y^{(\iota)} \longrightarrow y$ in $\bigoplus X + \mu(Y)$ we may assume in addition that $y^{(\iota)} - y \in \bigoplus \frac{\varepsilon_k}{2} A + \mathcal{V}$ for all $\iota \ge \iota_0$. Taking into account that $\eta | \bigoplus X + \mu(Z)$ is the inclusion map, we obtain

$$\begin{aligned} g((z, y)) &= z + \eta(y) \\ &= z + y^{(\iota_0)} + \eta(y - y^{(\iota_0)}) \in \bigoplus \frac{\varepsilon_k}{2} A + \mathcal{W} + \eta(\bigoplus \frac{\varepsilon_k}{2} A + \mathcal{V}) \\ &= \mathcal{W} + \eta(\bigoplus \varepsilon_k A + \mathcal{V}) = g(\mathcal{W} \times (\bigoplus \varepsilon_k A + \mathcal{V})). \end{aligned}$$

Thus $H$ is topologically isomorphic to the LB-space $(\lambda(Z) \times (\bigoplus X + \mu(Y)))/\ker g$. Moreover, we show that $\ker g = \{(z, -z) : z \in \mu(Z)\}$, hence $\ker g$ is topologically isomorphic to the Banach space $\mu(Z)$. In fact, since "$\supset$" is obvious let $(z, y) \in \lambda(z) \times (\bigoplus X + \mu(Y))$ such that $g((z, g)) = 0$. Again there is a net $(y^{(\iota)})_{\iota \in I}$ in $G = \bigoplus X + \mu(Z)$ such that $y^{(\iota)} \longrightarrow y$ in $\tilde{G}$, whence $y^{(\iota)} = \eta(y^{(\iota)}) \longrightarrow \eta(y) = -z \in \lambda(Z) \subset F$. Since $y^{(\iota)} \in \bigoplus X + \mu(Z)$, which is a closed subspace of $F$, we obtain $\eta(y) = -z \in (\bigoplus X + \mu(Z)) \cap \lambda(Z) = \mu(Z)$. $\eta$ being injective, we obtain $y \in \mu(Z)$, which completes the argument.

(2) As the inclusion $F \hookrightarrow E$ is continuous and $E$ is complete by Theorem 1, it has a continuous linear extension $\varrho : H \longrightarrow E$. We show that $\varrho$ is onto. In fact, we first show that $\varrho \circ \eta : \bigoplus X + \mu(Y) \longrightarrow E$ is the inclusion map. Let $y \in \bigoplus X + \mu(Y) = \tilde{G}$ and let $(y^{(\iota)})_{\iota \in I}$ be a net in $G$ converging to $y$ in $\tilde{G}$.

As the inclusion $\tilde{G} \hookrightarrow E$ is continuous, $y^{(\iota)} \longrightarrow y$ in $E$. On the other hand, $y^{(\iota)} = (\varrho \circ \eta)(y^{(\iota)}) \longrightarrow (\varrho \circ \eta)(y)$ in $E$. Uniqueness of limits implies $y = (\varrho \circ \eta)(y)$.

Now let $x \in \bigoplus X$, $y \in \mu(Y)$, $z \in \lambda(Z)$ be given. Then $g((z, x+y)) \in H$ and $\varrho(g((z, x+y))) = \varrho(z + \eta(x+y)) = z + x + y$.

$H$ and $E$ being LB-spaces, $\varrho : H \longrightarrow E$ is open, consequently $E$ is topologically isomorphic to $H/\ker\varrho$.

(3) Finally we will prove that $\ker\varrho$ is complete. (Utilizing Pasynkov's result [4] again, $\ker\varrho$ and $H/\ker\varrho$ being complete, we obtain the completeness of $H$ and hence the equality $\tilde{F} = H$, whence $\tilde{F}$ is an LB-space). For the proof of the completeness of $\ker\varrho$ we first have

$$g^{-1}(\ker\varrho) = \{(z, -z) : z \in \lambda(Z) \cap \mu(Y)\}$$

In fact, given $z \in \lambda(Z) \cap \mu(Y)$, $(\varrho \circ g)((z, -z)) = z + (\varrho \circ \eta)(-z) = z - z = 0$. For the converse inclusion, let $(z, y) \in \lambda(z) \times (\bigoplus X + \mu(Y))$ satisfying $0 = (\varrho \circ g)((z, y)) = z + (\varrho \circ \eta) y) = z + y$; then $y = -z \in \lambda(Z) \cap (\bigoplus X + \mu(Y)) = \lambda(Z) \cap \mu(Y)$.

As the inclusion $\lambda(Z) \hookrightarrow \lambda(Y)$ is continuous and $\mu(Y)$ is closed in $\lambda(Y)$, $\lambda(Z) \cap \mu(Y)$ is a Banach subspace of $\lambda(Z)$. Moreover $j : \lambda(Z) \cap \mu(Y) \longrightarrow g^{-1}(\ker\varrho), z \mapsto (z, -z)$ is well defined, linear, bijective and obviously continuous. $j$ is also open. In fact, let $\mathcal{W}$ denote the unit ball in $\lambda(Z)$. Then $j(\mathcal{W} \cap \mu(Y)) \supset j(\lambda(Z) \cap \mu(Y)) \cap (\mathcal{W} \times \mu(Y)) = j(\lambda(Z) \cap \mu(Y)) \cap (\mathcal{W} \times (\bigoplus X + \mu(Y))$.

Finally, $g$ being a continuous linear open onto its range, $\ker\varrho$ is topologically isomorphic to $g^{-1}(\ker\varrho)/\ker g$, hence topologically isomorphic to the Banach space $(\lambda(Z) \cap \mu(Y))/\mu(Z)$, thus complete.

We have proved the following

**Theorem 2** *Let $X$, $Z$ be Banach spaces with continuous inclusion $Z \hookrightarrow X$, let $\lambda$ be a normal Banach sequence space and $\mu := \overline{\varphi}^{\lambda}$. Then the completion of the LB-space of Moscatelli type $\bigotimes X + \lambda(Z)$ is topologically isomorphic to a quotient of the LB-space $\lambda(Z) \times \widetilde{\bigoplus X + \mu(Z)}$ by a Banach subspace, hence an LB-space itself.*

*Moreover, $\bigoplus X + \lambda(Z)$ is Mackey sequentially dense in its completion.*

PROOF. Because of Remark 3 it remains to prove the last statement. Because of [1, 3.2 Theorem], $\bigoplus X + \mu(Z)$ is Mackey sequentially dense in its completion, which implies the assertion. ■

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