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Another Relation Between π, e, γ and $\zeta(n)$

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Abstract. We obtain a relation between π, e, γ, $\zeta(n)$ and the triangular numbers based on a recursion formula for $n!$.

Otra relación entre π, e, γ y $\zeta(n)$

Resumen. Se obtiene una relación entre π, e, γ, $\zeta(n)$ y los números triangulares basada en una fórmula de recursión para $n!$.

1 The formula

The relation is

$$2\pi = \mathrm{e}^{\gamma} \prod_{n=2}^{\infty} \exp\left(\frac{\zeta(n)}{\binom{n+1}{2}}\right), \tag{1}$$

where γ is Euler's constant, $\binom{m}{2} := m(m-1)/2$ are the triangular numbers, and

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s} \qquad (\mathrm{Re}\, s > 1) \tag{2}$$

is Riemann's Zeta function.

The derivation of (1) is rather simple. Recall first Stirling's result, which we shall use for $n \gg 1$ [1].

$$\log n! = n(\log n - 1) + \log \sqrt{2\pi n} + \mathrm{o}(1). \tag{3}$$

We shall describe a recursion formula for log $n!$. Notice

$$\begin{aligned} \log n! &= \log n + \log (n-1)! \\ &= n \log n + \log (n-1)! - (n-1) \log n \\ &\equiv n \log n + R(n), \end{aligned} \tag{4}$$

with

$$R(m) := \log (m-1)! - (m-1) \log m = (m-1) \log((m-1)/m) + R(m-1). \tag{5}$$

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Expanding $\log(1+x) = x - x^2/2 + x^3/3 - \cdots, \quad (|x| < 1, \text{ and } x = 1)$

$$\begin{aligned} R(m) &= (m-1)(-1/m - 1/2m^2 - 1/3m^3 - \ldots) + R(m-1) \\ &= -1 + (1/m - 1/2m) + (1/2m^2 - 1/3m^2) + \cdots + R(m-1). \end{aligned} \tag{6}$$

The process continues until we reach $R(1) = 0$. Hence, as there are precisely n steps,

$$\begin{aligned} \log n! &= n \log n - n + \frac{1}{1 \cdot 2} \sum_{m=1}^{n} 1/m + \frac{1}{2 \cdot 3} \sum_{m=1}^{n} 1/m^2 + \frac{1}{3 \cdot 4} \cdots \\ &= n(\log n \ - 1) + \sum_{k=1}^{\infty} \zeta_n(k)/k(k+1), \end{aligned} \tag{7}$$

where the truncated Zeta functions are

$$\zeta_N(k) := \sum_{j=1}^{N} j^{-k}. \tag{8}$$

For *any* n we obtain therefore from (7)

$$\log n! = n(\log n - 1) + \sum_{k=1}^{\infty} \zeta_n(k)/k(k+1). \tag{7'}$$

For $n \gg 1,\ \zeta_n(1) = \sum_{k=1}^{n} 1/k = \log n + \gamma + o(1)$ and we obtain

$$\log n! \to n(\log n - 1) + (1/2)(\log n + \gamma) + \sum_{k=2}^{\infty} \zeta(k)/k(k+1). \tag{9}$$

Comparing with Stirling's result (3) we have

$$\log(\sqrt{2\pi}) = \gamma/2 + \sum_{n=2}^{\infty} \zeta(n)/n(n+1), \tag{10}$$

which is our *final result*. Exponentiation gives (1) at once.

2 Remarks

1. We find formula (1) rather pretty: it relates 2π, e, γ and $\zeta(n)$ with the triangular numbers and *nothing else*.
2. The asymptotic behaviour of $\log n!$ in (3)) is reproduced: from (9)

$$\log n! = n(\log n \ - 1) + \log \sqrt{n} + O(1).$$

3. $\gamma/2$ in (10) reflects the *residual* (finite part) of $\zeta(1)/2$, of course.
4. Numerical evaluation of (10) gives 0.9189385332; the convergence, although quadratic, is rather slow, as $\zeta(n) \to 1$ for $n \gg 1$. Still is faster than the conventional, linear form.
5. Both (10) and (1) can be obtained using *Mathematica* for the righthand side.

3 Functional method

J. Sondow ([2] and private communication) has suggested another derivation, which we explain now in some detail.

Start from Weierstraß' infinite product formula for the Gamma *function* $\Gamma(z)$

$$1/\Gamma(z) = ze^{\gamma z} \prod_{n=1}^{\infty} (1 + z/n) e^{-z/n}, \tag{11}$$

where $\gamma = 0.577...$ is Euler's constant. Taking logarithms,

$$\log \Gamma(z) + \log z = -\gamma z + \sum_{n=1}^{\infty} \{z/n - \log(1 + z/n)\} \tag{12}$$

or, using the functional equation $z\Gamma(z) = \Gamma(z+1)$ for $z \neq 0$ and (7),

$$\log \Gamma(1+z) = -\gamma z + \sum_{n=2}^{\infty} (-1)^n \zeta(n) z^n / n \qquad (|z| < 1,\ z = 1) \tag{13}$$

This formula is very useful, allowing e.g. several forms for Euler's γ. As an example, for $z = +1/2$ we obtain

$$\log\ 4/\pi = \gamma + 2 \sum_{k=2}^{\infty} (-1)^{k+1} \zeta(k)/k \cdot 2^k \tag{14}$$

which is formula (4) in [2].

Sondow suggested to integrate (13) in z from -1 to 0; it gives

$$\int_{-1}^{0} \log\ \Gamma(1+z)\ dz = \gamma/2 + \sum_{k=2}^{\infty} \zeta(k)/k(k+1) \tag{15}$$

which is our formula (10), provided

$$\int_{-1}^{0} \log \Gamma(1+z)\ dz = \log \sqrt{2\pi} \tag{16}$$

But obviously

$$\int_{-1}^{0} \log\ \Gamma(1+z)\ dz = \int_{0}^{1} \log\ \Gamma(z)\ dz \tag{17}$$

and exercise 21 (p. 261) of Ch. XII of [3] can be applied to get the result (cf. exercise 50, p. 264, of the same chapter).

Formula (16) is given also in *Mathematica*.

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