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On the defining polynomials of maximal real cyclotomic extensions

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Abstract. The aim of this paper is to show that the simplest techniques of linear algebra allow us to make explicit the defining equations of the maximal real cyclotomic extensions $\mathbb{Q}(\zeta+\zeta^{-1})$ of $\mathbb{Q}(\zeta)$, where ζ stands for a primitive p^ν-th rooot of unity with p a rational prime and ν any positive integer.

Sobre los polinomios que definen las extensiones ciclotómicas reales maximales

Resumen. El objetivo de este artículo es dar de forma explícita, utilizando técnicas sencillas de álgebra lineal, las ecuaciones correspondientes a las extensiones ciclotómicas reales maximales $\mathbb{Q}(\zeta+\zeta^{-1})$ de $\mathbb{Q}(\zeta)$, en donde ζ denota una p^ν-ésima raíz primitiva de la unidad, siendo p un número primo y ν un entero positivo cualquiera.

1 Introduction

The explicit computation of the monic irreducible polynomials, over $\mathbb{Q}$, of the generators $\zeta+\zeta^{-1}$ of the maximal real subextensions of the cyclotomic fields $\mathbb{Q}(\zeta)$, where ζ stands for a primitive n-th root of unity is an interesting problem coming back to C. F. Gauss (see [1, article 337]), at least when n is a prime, and has recently been solved in [3], closely following the ideas appearing in [2], which essentially rely on expanding the Taylor series of the logarithms of the reciprocals of the irreducible polynomials associated with the Gaussian 2-periods of the roots of unity and then working in the classical framework of power series, Chebyshev polynomials (see [6]), and generating functions and recursive formulae connecting coefficients. This machinery, though quite efficient, seems, to some extent, to be a bit sophisticated compared to the simple nature of the problem posed. So we have wondered whether a simpler, elementary treatment, namely that of just working with systems of linear equations arising from polynomials, could be found. In the case $n=p^r$, a prime power, we have succeeded in doing so, and this will be described in what follows.

Thus, let us begin by recalling (see, for instance, [5, p. 7], or [4, Ch. 1 §10]), that the irreducible polynomial of a p^ν-th root of unity, p prime, $\nu\geq 1$, is just

$$\Phi_{p^\nu}(X)=\Phi(X)=X^{p^{\nu-1}(p-1)}+X^{p^{\nu-1}(p-2)}+\cdots+X^{p^{\nu-1}}+1. \tag{1}$$

If $\Psi(X)=\Psi_{p^\nu}(X)$ denotes the $\mathbb{Q}$-irreducible polynomial of $\zeta+\zeta^{-1}$ then it is obvious that $\deg\Psi(X)=\frac{1}{2}(\varphi(p^\nu))=p^{\nu-1}(p-1)/2$, since $\mathbb{Q}(\zeta+\zeta^{-1})\subset\mathbb{Q}(\zeta)$ is a quadratic extension. As $X^{p^{\nu-1}(p-1)/2}\Psi(X+$

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X^{-1}) is obviously a monic polynomial in X with rational coefficients, vanishing when $X = \zeta$ and of degree $p^{\nu-1}(p-1)$, we see that

$$X^{p^{\nu-1}(p-1)/2}\Psi(X+X^{-1}) = \Phi(X) \tag{2}$$

We write

$$\Psi_{p^\nu}(X) = \Psi(X) = a_m X^m + a_{m-1}X^{m-1} + \cdots + a_1 X + a_0 \tag{3}$$

where, of course $a_m = 1$, and $m = \frac{1}{2}\varphi(p^\nu) = p^{\nu-1}(p-1)/2$.

This paper is devoted to the evaluation of the coefficients $a_{m-1}, \ldots, a_1, a_0$.

2 Preliminaries

In order to achieve our aim we first proceed to expand the relation (2) using the notation of (3):

$$\Phi(X) = X^m \sum_{i=0}^{m} a_i \left(X + \frac{1}{X}\right)^i = \sum_{i=0}^{m} a_i\, X^{m-i} \left(\sum_{j=0}^{i} \binom{i}{j} X^{2(i-j)}\right) = \sum_{t=0}^{2m} \left(\sum_{\substack{i-2j=t-m \\ 0\le j\le i\le m}} a_i \binom{i}{j}\right) X^t.$$

This shows that $a_0, \ldots, a_m$ is a solution of the system of linear equations

$$\sum_{\substack{i-2j=t-m \\ 0\le j\le i\le m}} a_i \binom{i}{j} = b_t\,, \tag{4}$$

where by (1), for $0 \le t \le \varphi(p^\nu) = 2m$,

$$b_t = \begin{cases} 1, & \text{if } t = p^{\nu-1}\, l, \quad 0 \le l < p \\ 0, & \text{otherwise,} \end{cases}$$

For the sake of clarity we rewrite the preceding system of equations (4) more explicitly as follows:

$$b_{2m-k} = \begin{cases} \displaystyle\sum_{j=0}^{k/2} \binom{m-k+2j}{j} a_{m-k+2j}, & \text{for } k = 2, 4, \ldots, r. \\ \displaystyle\sum_{j=0}^{(k-1)/2} \binom{m-k+2j}{j} a_{m-k+2j}, & \text{for } k = 1, 3, \ldots, s. \end{cases} \tag{5}$$

where $r = m$, $s = m-1$, if $p \equiv 1 \pmod 4$ or $p = 2$; and $r = m-1$, $s = m$ in the case $p \equiv 3 \pmod 4$.

In order to get the explicit solution of (5) we are led to compute the following determinants of order k, for strictly positive integers r, k:

$$A_r(k) := \begin{vmatrix} \binom{r+2(k-1)}{1} & \binom{r+2(k-2)}{0} & 0 & 0 & \cdots & \cdots & 0 \\ \binom{r+2(k-1)}{2} & \binom{r+2(k-2)}{1} & \binom{r+2(k-3)}{0} & 0 & \cdots & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & & \vdots \\ \vdots & \vdots & \vdots & \vdots & & \ddots & \vdots \\ \vdots & \vdots & \vdots & \vdots & & & 0 \\ \binom{r+2(k-1)}{k-1} & \binom{r+2(k-2)}{k-2} & \binom{r+2(k-3)}{k-3} & \binom{r+2(k-4)}{k-4} & \cdots & \cdots & \binom{r}{0} \\ \binom{r+2(k-1)}{k} & \binom{r+2(k-2)}{k-1} & \binom{r+2(k-3)}{k-2} & \binom{r+2(k-4)}{k-4} & \cdots & \cdots & \binom{r}{1} \end{vmatrix}$$

for $k \geq 2$, and $A_r(1) := \binom{r}{1} = r$.

Obviously $A_1(k) = 0$, for all $k \geq 2$ (since there appear two equal rows in such a case), and the remaining cases are covered by the following

Lemma 1 *For $r \geq 2$, and $k \geq 2$, we have*

$$A_r(k) = A_{r-1}(k) + A_r(k-1).$$

PROOF. This follows from the equality

$$A_r(k) = \begin{vmatrix} \binom{r+2k-3}{1} & \binom{r+2k-5}{0} & 0 & 0 & \cdots & \cdots & 0 \\ \binom{r+2(k-1)}{2} & \binom{r+2(k-2)}{1} & \binom{r+2(k-3)}{0} & 0 & \cdots & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & & \vdots \\ \vdots & \vdots & \vdots & \vdots & & \ddots & \vdots \\ \vdots & \vdots & \vdots & \vdots & & & 0 \\ \binom{r+2(k-1)}{k-1} & \binom{r+2(k-2)}{k-2} & \binom{r+2(k-3)}{k-3} & \binom{r+2(k-4)}{k-4} & \cdots & \cdots & \binom{r}{0} \\ \binom{r+2(k-1)}{k} & \binom{r+2(k-2)}{k-1} & \binom{r+2(k-3)}{k-2} & \binom{r+2(k-4)}{k-3} & \cdots & \cdots & \binom{r}{1} \end{vmatrix}$$

$$+ \begin{vmatrix} 1 & 0 & 0 & \cdots & \cdots & 0 \\ \binom{r+2(k-1)}{1} & \binom{r+2(k-2)}{0} & 0 & \cdots & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & & \vdots \\ \vdots & \vdots & \vdots & & \ddots & 0 \\ \binom{r+2(k-1)}{k-1} & \binom{r+2(k-2)}{k-2} & \binom{r+2(k-3)}{k-3} & \cdots & \cdots & \binom{r}{0} \\ \binom{r+2(k-1)}{k} & \binom{r+2(k-2)}{k-1} & \binom{r+2(k-3)}{k-2} & \cdots & \cdots & \binom{r}{1} \end{vmatrix},$$

obtained by performing in the first determinant successive subtractions of each row to the following one, starting from above. ■

We will define $A_r(0) = 1$, for $r \geq 2$, and by convention we will consider $\binom{i}{j} = 0$ either if $i < j$ or if $i < 0$. With these notations it is easy to establish the following

Theorem 1 *For any strictly positive integers r, k, we have*

$$A_r(k) = \binom{r+k-2}{k} + \binom{r+k-3}{k-1}$$

PROOF. From Lemma 1 we deduce that the $A_r(k)$ are easily computed by adding two Pascal triangles, namely the usual one with that obtained by shifting the usual Pascal triangle by means of adding an edge of zeros on the left sloping side. This immediately yields the result. ■

3 Evaluation of the coefficients

Next we give explicit expressions for the coefficients a_i in terms of what we have introduced in the preceding section.

Theorem 2 *The coefficients a_j of the polynomial $\Psi_{p^\nu}(X)$ are given by the following formulae:*

If p is odd,

$$a_j = \begin{cases} 0, & \text{if } j > m - p^{\nu-1}. \\ \displaystyle\sum_{\substack{k=1 \\ k\equiv 1(\bmod 2)}}^{\left[\frac{m-j}{p^{\nu-1}}\right]} (-1)^{(m-j-kp^{\nu-1})/2} A_{j+2}\left(\frac{m-j-kp^{\nu-1}}{2}\right), & \text{if } m+j \equiv 1 (\bmod 2). \\ (-1)^{\frac{m-j}{2}} \displaystyle\sum_{k=0}^{\left[\frac{m-j}{2p^{\nu-1}}\right]} (-1)^k A_{j+2}\left(\frac{m-j}{2} - kp^{\nu-1}\right), & \text{if } m+j \equiv 0 (\bmod 2). \end{cases}$$

and in the case $p = 2$, $\nu \geq 3$

$$a_j = \begin{cases} (-1)^{\frac{m-j}{2}} A_{j+2}\left(\frac{m-j}{2}\right) & \text{if } j \text{ is even,} \\ 0 & \text{otherwise.} \end{cases}$$

PROOF. Let us first examine the case $p \equiv 1 \pmod 4$. According to (5), for $j = 1, 3, 5, \ldots, m-1$, the coefficients a_j satisfy the system of linear equations

$$\begin{aligned} b_{2m-1} &= \binom{m-1}{0} a_{m-1} \\ b_{2m-3} &= \binom{m-1}{1} a_{m-1} + \binom{m-3}{0} a_{m-3} \\ &\vdots \\ b_{m+1} &= \binom{m-1}{\frac{m-2}{2}} a_{m-1} + \binom{m-3}{\frac{m-4}{2}} a_{m-3} + \cdots + \binom{1}{0} a_1 \end{aligned}$$

By Cramer's rule, we have

$$a_j = \begin{vmatrix} \binom{m-1}{0} & 0 & \cdots & 0 & b_{2m-1} & 0 & \cdots & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots & \vdots & \vdots & \ddots & & \vdots \\ \vdots & & \ddots & 0 & \vdots & \vdots & & \ddots & \vdots \\ \binom{m-1}{\frac{m-3-j}{2}} & \cdots & \cdots & \binom{j+2}{0} & b_{m+j+2} & 0 & \cdots & \cdots & 0 \\ \binom{m-1}{\frac{m-1-j}{2}} & \cdots & \cdots & \binom{j+2}{1} & b_{m+j} & 0 & \cdots & \cdots & 0 \\ \binom{m-1}{\frac{m+1-j}{2}} & \cdots & \cdots & \binom{j+2}{2} & b_{m+j-2} & \binom{j-2}{0} & 0 & \cdots & 0 \\ \vdots & \ddots & & \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \vdots & \vdots & \vdots & & \ddots & 0 \\ \binom{m-1}{\frac{m-2}{2}} & \cdots & \cdots & \binom{j+2}{\frac{j+1}{2}} & b_{m+1} & \binom{j-2}{\frac{j-3}{2}} & \cdots & \cdots & \binom{1}{0} \end{vmatrix}$$

or, more simply,

$$a_j = \begin{vmatrix} \binom{m-1}{0} & 0 & \cdots & 0 & b_{2m-1} \\ \vdots & \ddots & \ddots & \vdots & \vdots \\ \vdots & & \ddots & 0 & \vdots \\ \binom{m-1}{\frac{m-3-j}{2}} & \cdots & \cdots & \binom{j+2}{0} & b_{m+j+2} \\ \binom{m-1}{\frac{m-1-j}{2}} & \cdots & \cdots & \binom{j+2}{1} & b_{m+j} \end{vmatrix}.$$

Since $b_i = 1$ for $i = p^{\nu-1}(p-2),\ p^{\nu-1}(p-4),\ \ldots,\ p^{\nu-1}$, and $b_i = 0$ otherwise, with our preceding notations, by expanding the determinant according to its last column, we obtain

$$\begin{aligned} a_j &= (-1)^{(m-j-p^{\nu-1})/2} A_{j+2}\left(\frac{m-j-p^{\nu-1}}{2}\right) + (-1)^{(m-j-3p^{\nu-1})/2} A_{j+2}\left(\frac{m-j-3p^{\nu-1}}{2}\right) \\ &\quad + \cdots + (-1)^{(m-j-kp^{\nu-1})/2} A_{j+2}\left(\frac{m-j-kp^{\nu-1}}{2}\right), \end{aligned}$$

for j satisfying $m - kp^{\nu-1} \geq j \geq m-(k+2)p^{\nu-1}$, with k odd.

Again from (5), we know that for $j = 0,\ 2, \ldots,\ m$, the a_j are the solution of the system of $m/2+1$ linear equations

$$\begin{aligned} b_{2m} &= \binom{m}{0} a_m \\ b_{2m-2} &= \binom{m}{1} a_m + \binom{m-2}{0} a_{m-2} \\ &\vdots \\ b_m &= \binom{m}{\frac{m}{2}} a_m + \binom{m-2}{\frac{m-2}{2}} a_{m-2} + \cdots + \binom{0}{0} a_0 \end{aligned}$$

Here Cramer's rule yields

$$a_j = \begin{vmatrix} \binom{m}{0} & 0 & \cdots & 0 & b_{2m} \\ \vdots & \ddots & \ddots & \vdots & \vdots \\ \vdots & & \ddots & 0 & \vdots \\ \binom{m}{\frac{m-2-j}{2}} & \cdots & \cdots & \binom{j+2}{0} & b_{m+j+2} \\ \binom{m}{\frac{m-j}{2}} & \cdots & \cdots & \binom{j+2}{1} & b_{m+j} \end{vmatrix},$$

where $b_i = 1$ for $i = p^{\nu-1}(p-1),\ p^{\nu-1}(p-3), \ldots,\ p^{\nu-1}2,\ 0$, and $b_i = 0$ otherwise. It follows that

$$\begin{aligned} a_j &= (-1)^{(m-j)/2} A_{j+2}\left(\frac{m-j}{2}\right) + (-1)^{(m-j-2p^{\nu-1})/2} A_{j+2}\left(\frac{m-j-2p^{\nu-1}}{2}\right) \\ &\quad + \cdots + (-1)^{(m-j-kp^{\nu-1})/2} A_{j+2}\left(\frac{m-j-kp^{\nu-1}}{2}\right), \end{aligned}$$

for $m - kp^{\nu-1} \geq j > m-(k+2)p^{\nu-1}$, with k even.

The same reasoning applies to the case $p \equiv 3 \pmod 4$. From (5), it follows that, for $j = 1,\ 3,\ \ldots,\ m$, the a_j satisfy the system of linear equations

$$\begin{aligned} b_{2m} &= \binom{m}{0} a_m \\ b_{2m-2} &= \binom{m}{1} a_m + \binom{m-2}{0} a_{m-2} \\ &\vdots \\ b_{m+1} &= \binom{m}{\frac{m-1}{2}} a_m + \binom{m-2}{\frac{m-3}{2}} a_{m-2} + \cdots + \binom{1}{0} a_1 \end{aligned}$$

whose discussion is just that of the case for j even, when $p \equiv 1 \pmod 4$ and for $j = 0, 2, \ldots, m-1$, according to (5), the coefficients a_j satisfy the following system of $(m+1)/2$ equations:

$$\begin{aligned} b_{2m-1} &= \binom{m-1}{0} a_{m-1} \\ b_{2m-3} &= \binom{m-1}{1} a_{m-1} + \binom{m-3}{0} a_{m-3} \\ &\vdots \\ b_m &= \binom{m-1}{\frac{m-1}{2}} a_{m-1} + \binom{m-3}{\frac{m-3}{2}} a_{m-3} + \cdots + \binom{0}{0} a_0 \end{aligned}$$

and here again the solution follows the lines for j odd in the case $p \equiv 1 \pmod 4$.

The preceding calculations carry over without any changes when $p = 2$ (and $\nu \geq 3$). Here the cyclotomic polynomial corresponding to a primitive 2^ν-th root of unity is $X^{2^{\nu-1}} + 1$. Therefore, in this case $b_i = 1$ for $i = 2^{\nu-1}$ and 0, and $b_i = 0$ otherwise. As now we have $m = 2^{\nu-2}$, the system of linear equations for the coefficients a_j, with j odd, is homogeneous, so that all these a_j vanish. For the coefficients a_j, with j even, however, the system is just that of the case $p \equiv 1 \pmod 4$ for j even, the only difference being that now only b_{2m} equals 1 while the other b_i vanish, and hence as a result we get, for j even,

$$a_j = (-1)^{\frac{m-j}{2}} A_{j+2} \left(\frac{m-j}{2} \right).$$

■

Remark 1 *Observe that a_0 may also be easily obtained by considering a simple alternate sum. Namely, if $p \equiv 1 \pmod 4$ as a_0 occurs in the system of equations*

$$b_{2m-k} = \sum_{j=0}^{k/2} \binom{m-k+2j}{j} a_{m-k+2j}, \qquad k = 0, 2, 4, \ldots 2m,$$

by considering the alternate sum of these equations we get $1 = (-1)^{\frac{m}{2}} a_0$, i.e., $a_0 = (-1)^{\frac{m}{2}} = (-1)^{\frac{p-1}{4}}$. The case $p \equiv 3 \pmod 4$ is dealt with in exactly the same way and the result is $a_0 = (-1)^{\frac{p-3}{4}}$. Finally, when $p = 2$ and $\nu \geq 3$, we also apply the same method but now the alternate sum of the b_i is 2 and then we get $a_0 = -2$ for $\nu = 3$, and $a_0 = 2$ for $\nu > 3$. In particular, if ζ is a primitive p^νth root of unity then $N(\zeta + \zeta^{-1}) = \pm 1$, if p is odd, and $N(\zeta + \zeta^{-1}) = \pm 2$ if $p = 2$, $\nu \geq 3$.

4 Numerical exemples

Here we list a few polynomials $\Psi_{p^\nu}(X)$ of degree less than 100. The polynomials have been computed with *Mathematica*, solving the system of equations given in (5).

$\Psi_{2^3}(X) = X^2 - 2$

$\Psi_{2^4}(X) = X^4 - 4X^2 + 2$

$\Psi_{2^5}(X) = X^8 - 8X^6 + 20X^4 - 16X^2 + 2$

$$\Psi_{2^6}(X) = X^{16} - 16X^{14} + 104X^{12} - 352X^{10} + 660X^8 - 672X^6 + 336X^4 - 64X^2 + 2$$

$$\begin{aligned}\Psi_{2^7}(X) = {} & X^{32} - 32X^{30} + 464X^{28} - 4032X^{26} + 23400X^{24} - 95680X^{22} + 283360X^{20} \\ & - 615296X^{18} + 980628X^{16} - 1136960X^{14} + 940576X^{12} - 537472X^{10} + 201552X^8 \\ & - 45696X^6 + 5440X^4 - 256X^2 + 2\end{aligned}$$

$$\begin{aligned}\Psi_{2^8}(X) = {} & X^{64} - 64X^{62} + 1952X^{60} - 37760X^{58} + 520144X^{56} - 5430656X^{54} + 44662464X^{52} \\ & - 296854272X^{50} + 1623421800X^{48} - 7398867840X^{46} + 28362326720X^{44} \\ & - 92043777280X^{42} + 254005423840X^{40} - 597659820800X^{38} + 1200442440064X^{36} \\ & - 2057901325824X^{34} + 3006465218196X^{32} - 3732682723968X^{30} \\ & + 3922021702720X^{28} - 3467892873984X^{26} + 2561511781920X^{24} \\ & - 1565841089280X^{22} + 782920544640X^{20} - 315492902400X^{18} + 100563362640X^{16} \\ & - 24754058496X^{14} + 4559958144X^{12} - 602516992X^{10} + 53796160X^8 \\ & - 2968064X^6 + 87296X^4 - 1024X^2 + 2\end{aligned}$$

$$\Psi_3(X) = X + 1$$

$$\Psi_{3^2}(X) = X^3 - 3X + 1$$

$$\Psi_{3^3}(X) = X^9 - 9X^7 + 27X^5 - 30X^3 + 9X + 1$$

$$\begin{aligned}\Psi_{3^4}(X) = {} & X^{27} - 27X^{25} + 324X^{23} - 2277X^{21} + 10395X^{19} - 32319X^{17} + 69768X^{15} - 104652X^{13} \\ & + 107406X^{11} - 72930X^9 + 30888X^7 - 7371X^5 + 819X^3 - 27X + 1\end{aligned}$$

$$\begin{aligned}\Psi_{3^5}(X) = {} & X^{81} - 81X^{79} + 3159X^{77} - 79002X^{75} + 1423575X^{73} - 19690290X^{71} + 217468314X^{69} \\ & - 1969809516X^{67} + 14915235753X^{65} - 95752130760X^{63} + 526771581336X^{61} \\ & - 2503875308688X^{59} + 10348141674312X^{57} - 37365688036656X^{55} \\ & + 118311400286640X^{53} - 329359817363616X^{51} + 807564936805020X^{49} \\ & - 1745765378093205X^{47} + 3328346338128315X^{45} - 5594334252541650X^{43} \\ & + 8281448901713295X^{41} - 10779028729214130X^{39} + 12307026638440170X^{37} \\ & - 12288575324139660X^{35} + 10689623271729675X^{33} - 8063030124961812X^{31} \\ & + 5243788822527612X^{29} - 2920409138472168X^{27} + 1381487341784004X^{25} \\ & - 549663398587800X^{23} + 181784104369560X^{21} - 49257628280784X^{19} \\ & + 10743691882671X^{17} - 1844876383893X^{15} + 242443079235X^{13} - 23491379106X^{11} \\ & + 1595093643X^9 - 70544682X^7 + 1813266X^5 - 22140X^3 + 81X + 1\end{aligned}$$

$$\Psi_5(X) = X^2 + X - 1$$

$$\Psi_{5^2}(X) = X^{10} - 10X^8 + 35X^6 + X^5 - 50X^4 - 5X^3 + 25X^2 + 5X - 1$$

$$\begin{aligned}\Psi_{5^3}(X) = {} & X^{50} - 50X^{48} + 1175X^{46} - 17250X^{44} + 177375X^{42} - 1357510X^{40} + 8021650X^{38} \\ & - 37469900X^{36} + 140512125X^{34} - 427248250X^{32} + 1059575660X^{30} - 2148789800X^{28} \\ & + 3562467300X^{26} + X^{25} - 4814145000X^{24} - 25X^{23} + 5272635000X^{22} + 275X^{21}\end{aligned}$$

$$
\begin{aligned}
&- 4639918800X^{20} - 1750X^{19} + 3241119750X^{18} + 7125X^{17} - 1767883500X^{16}\\
&- 19380X^{15} + 736618125X^{14} + 35700X^{13} - 227613750X^{12} - 44200X^{11}\\
&+ 50075025X^{10} + 35750X^{9} - 7400250X^{8} - 17875X^{7} + 672750X^{6} + 5005X^{5}\\
&- 32500X^{4} - 650X^{3} + 625X^{2} + 25X - 1
\end{aligned}
$$

$$
\Psi_7(X) = X^3 + X^2 - 2X - 1
$$

$$
\begin{aligned}
\Psi_{7^2}(X) = {}& X^{21} - 21X^{19} + 189X^{17} - 952X^{15} + X^{14} + 2940X^{13} - 14X^{12} - 5733X^{11} + 77X^{10}\\
&+ 7007X^{9} - 210X^{8} - 5147X^{7} + 294X^{6} + 2072X^{5} - 196X^{4} - 371X^{3} + 49X^{2}\\
&+ 14X - 1
\end{aligned}
$$

$$
\Psi_{11}(X) = X^5 + X^4 - 4X^3 - 3X^2 + 3X + 1
$$

$$
\begin{aligned}
\Psi_{11^2}(X) = {}& X^{55} - 55X^{53} + 1430X^{51} - 23375X^{49} + 269500X^{47} - 2330636X^{45} + X^{44}\\
&+ 15696120X^{43} - 44X^{42} - 84366645X^{41} + 902X^{40} + 367982175X^{39} - 11440X^{38}\\
&- 1317269525X^{37} + 100529X^{36} + 3899117794X^{35} - 649572X^{34} - 9586673914X^{33}\\
&+ 3196578X^{32} + 19619239607X^{31} - 12243264X^{30} - 33417385705X^{29} + 36984860X^{28}\\
&+ 47273371134X^{27} - 88763664X^{26} - 55309822425X^{25} + 169695240X^{24}\\
&+ 53182431720X^{23} - 258048959X^{22} - 41656556260X^{21} + 310465133X^{20}\\
&+ 26269293600X^{19} - 292746091X^{18} - 13133636571X^{17} + 213285468X^{16}\\
&+ 5102112862X^{15} - 117671620X^{14} - 1499508714X^{13} + 47797607X^{12}\\
&+ 321503053X^{11} - 13737009X^{10} - 47746996X^{9} + 2634412X^{8} + 4537511X^{7}\\
&- 306977X^{6} - 241142X^{5} + 18271X^{4} + 5489X^{3} - 363X^{2} - 33X + 1
\end{aligned}
$$

$$
\Psi_{13}(X) = X^6 + X^5 - 5X^4 - 4X^3 + 6X^2 + 3X - 1
$$

$$
\begin{aligned}
\Psi_{13^2}(X) = {}& X^{78} - 78X^{76} + 2925X^{74} - 70226X^{72} + 1212822X^{70} - 16049124X^{68} + 169258817X^{66}\\
&+ X^{65} - 1461006690X^{64} - 65X^{63} + 10519248168X^{62} + 2015X^{61} - 64064084656X^{60}\\
&- 39650X^{59} + 333510087768X^{58} + 556075X^{57} - 1496043894384X^{56} - 5916638X^{55}\\
&+ 5817948478160X^{54} + 49639590X^{53} - 19705288218335X^{52} - 336962340X^{51}\\
&+ 58324134324755X^{50} + 1884328875X^{49} - 151210718618596X^{48} - 8793534750X^{47}\\
&+ 343882440712733X^{46} + 34566585690X^{45} - 686438430180555X^{44}\\
&- 115221952300X^{43} + 1202538433766241X^{42} + 327186864550X^{41}\\
&- 1847253500444352X^{40} - 793766949499X^{39} + 2484237427915958X^{38}\\
&+ 1647566693461X^{37} - 2917993021435999X^{36} - 2926078446954X^{35}\\
&+ 2984310576915395X^{34} + 4441369064600X^{33} - 2646952512165572X^{32}\\
&- 5747654035266X^{31} + 2026059804050724X^{30} + 6318342833374X^{29}\\
&- 1330313881030628X^{28} - 5870129857956X^{27} + 743866969521565X^{26}\\
&+ 4578698208045X^{25} - 351127341416826X^{24} - 2973173393775X^{23}\\
&+ 138436551316115X^{22} + 1590288757005X^{21} - 45002650983004X^{20}\\
&- 691411044090X^{19} + 11871743606359X^{18} + 240286948785X^{17} - 2491355718306X^{16}
\end{aligned}
$$

$$- 65341075696X^{15} + 405458684553X^{14} + 13521774841X^{13}$$
$$- 49483859945X^{12} - 2051360909X^{11} + 4324257587X^{10} + 216422271X^{9}$$
$$- 252930808X^{8} - 14656655X^{7} + 8903427X^{6} + 556387X^{5} - 157001X^{4}$$
$$- 9061X^{3} + 1014X^{2} + 39X - 1$$

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