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Coarse dimensions and partitions of unity

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Abstract Gromov [11] and Dranishnikov [2] introduced asymptotic and coarse dimensions of proper metric spaces via quite different ways. We define coarse and asymptotic dimension of all metric spaces in a unified manner and we investigate relationships between them generalizing results of Dranishnikov [2] and Dranishnikov-Keesling-Uspienskij [5].

Dimensiones a gran escala y particiones de la unidad

Resumen. Gromov [11] y Dranishnikov [2] han introducido dimensiones asintóticas y a gran escala para espacios métricos propios de varias formas diferentes. Nosotros definimos dimensiones a gran escala y asintóticas para todos los espacios métricos de modo unificado e investigamos las relaciones entre ellas, generalizando resultados de Dranishnikov [2] y Dranishnikov-Keesling-Uspienskij [5].

1 Introduction

There are three concepts of dimension associated with variants of the coarse category of proper metric spaces. The original one, the asymptotic dimension of Gromov [11], and dimensions $\mathrm{asdim}^*(X)$ and $\dim^c(X)$ introduced by Dranishnikov [2]. All three dimensions are defined in seemingly different ways:

1. The asymptotic dimension of Gromov (see [11] or [2, Definitions 1–2 on p. 1103]) is the smallest integer n such that for every $M > 0$ there is a uniformly bounded family $\mathcal{U}$ of Lebesque number at least M and multiplicity (or order) at most $n + 1$.

2. The asymptotic dimension $\mathrm{asdim}^*(X)$ of Dranishnikov (see [2, Def. 3 on p. 1104]) is the smallest integer n such that for every proper function $f\colon X \to \mathbb{R}_+$ there is a contracting map $\phi\colon X \to K$ to an n-dimensional asymptotic polyhedron such that for each $M > 0$ there is a compact subset C of X with the property that $\phi^{-1}(B(\phi(x), M)) \subset B(x, f(x))$ for all $x \in X \setminus C$.

3. The coarse dimension $\dim^c(X)$ of Dranishnikov (see [2, Def. 4 on p. 1105]) is the smallest integer n such that $\mathbb{R}^{n+1}$ is an absolute extensor of X in the category of proper asymptotically Lipschitz functions. That dimension coincides with the dimension of the Higson corona $\nu(X)$ of X (see in [2, Theorem 6.6 on p. 1111]).

One of the main motivations behind the research in asymptotic dimension is the result of Yu (see [16] and [17]) that the Novikov Conjecture holds for groups of finite asymptotic dimension.

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In this paper we work in the coarse category of all metric spaces and we devise a unified way of defining five dimensions: *coarse dimension* $\dim{}^{\mathrm{coa}}_{\mathrm{rse}}(X)$, *major coarse dimension* $\dim{}^{\mathrm{COA}}_{\mathrm{RSE}}(X)$, *asymptotic dimension* $\mathrm{asdim}(X)$, *minor asymptotic dimension* $\mathrm{ad}(X)$, and *large scale dimension* $\dim{}^{\mathrm{large}}_{\mathrm{scale}}(X)$.

In case of proper metric spaces, three of them coincide with the above dimensions. Namely, $\dim{}^{\mathrm{COA}}_{\mathrm{RSE}}(X) = \dim^*(X)$, $\dim{}^{\mathrm{coa}}_{\mathrm{rse}}(X) = \dim^c(X)$, and $\mathrm{asdim}(X)$ coincides with Gromov's asymptotic dimension. The fourth one, the minor asymptotic dimension, is a variant of Gromov's dimension. The large scale dimension is always equal to the coarse dimension and the reason we are introducing it is to simplify proofs of the relations between the three basic dimensions which we do in a much simpler way than as described in Dranishnikov's paper [2]. The main relations between dimensions are as follows:

1. There are two strands of inequalities:

$$\mathrm{asdim}(X) \geq \dim{}^{\mathrm{COA}}_{\mathrm{RSE}}(X) \geq \dim{}^{\mathrm{coa}}_{\mathrm{rse}}(X) \qquad \text{and} \qquad \mathrm{asdim}(X) \geq \mathrm{ad}(X) \geq \dim{}^{\mathrm{coa}}_{\mathrm{rse}}(X),$$

2. In each strand (for unbounded spaces X), finiteness of a larger dimension implies its equality with all smaller dimensions in the strand.

We do not know of any unbounded space X such that a larger dimension in a strand is infinite and a smaller dimension is finite.

Our fundamental concept is that of a coarse family and we follow the well-established route of defining the covering dimension by refining covers with covers of a prescribed multiplicity. In classical dimension theory one deals with two cases: finite covers and infinite covers. There, for paracompact spaces, the two concepts coincide. In the case of coarse covers we get two concepts of coarse dimension whose equality remains unresolved.

A finite family $\mathcal{U}$ of subsets of X is coarse if and only if there is a slowly oscillating partition of unity f on $X \setminus B$ for some bounded subset B of X whose carriers $\mathrm{Carr}(f)$ refine $\mathcal{U}$. That explains why, in the case of a proper metric space X, its coarse dimension equals the covering dimension of the Higson corona of X.

Our basic strategy is to associate natural functions with objects and declare those objects to be coarse, asymptotic, or large scale if the function is coarsely proper. A function f is *coarsely proper* if $f(E_n) \to \infty$ whenever $E_n \to \infty$. Elements E_n related to objects could be points in a metric space, bounded subsets in a metric space, or covers of a metric space (in which case divergence to infinity is measured by the size of the Lebesque number). In [2, p. 1089] coarsely proper functions were defined as those $f \colon X \to Y$ such that $f^{-1}(A)$ is bounded whenever A is bounded in Y. Notice that our definition generalizes the one from [2].

2 Preliminaries

Given a subset $A \neq \emptyset$ of a metric space X the most basic function is *the distance function* $d_A \colon X \to \mathbb{R}_+$: $d_A(x) = \mathrm{dist}(x, A)$.

Definition 1 *Given a subset A of a metric space (X, d_X) the ball $B(A, M)$ is defined to be the set $\{ x \in X \mid \mathrm{dist}(x, A) < M \}$ if $M > 0$, it is defined to be the set $\{ x \in X \mid \mathrm{dist}(x, X \setminus A) > -M \}$ if $M < 0$, and it is simply A if $M = 0$.*

The distance function leads to the first concept of divergence to infinity: $x_n \to \infty$ if $d_X(x_n, x_0) \to \infty$ for some (and hence for all) $x_0 \in X$. However, $\mathrm{dist}(x, A)$ is a function of two arguments and we can use the second one to define divergence to infinity for bounded subsets of X. Here is a more general concept.

Definition 2 *A family $\mathcal{U}$ of bounded subsets of X is called* coarsely proper *if the function $U \to d_U(x_0)$ is coarsely proper for some (and hence for all) $x_0 \in X$. Here $\mathcal{U}$ is considered as a subspace of all bounded subsets of X with the Hausdorff metric.*

Notice that a sequence $\{A_n\}$ of bounded subsets of X containing points $x_n \in A_n$ so that $x_n \to \infty$ is coarsely proper if and only if every bounded subset of X intersects at most finitely many elements of the sequence. In that case we write $A_n \to \infty$ and that form of divergence to infinity is of most interest to us.

Lemma 1 *If $\mathcal{U}$ is a coarsely proper cover of X, then every selection function $\phi\colon X \to \mathcal{U}$ (that means $x \in \phi(x)$) is coarsely proper.*

PROOF. Suppose $x_n \to \infty$ and $x_n \in U_n \in \mathcal{U}$. Clearly, $U_n \to \infty$ in the Hausdorff metric. Pick $x_0 \in X$. Since $d_{U_n}(x_0) \to \infty$, every bounded subset of X intersects at most finitely many elements of the sequence $\{U_n\}$ and any selection function ϕ is coarsely proper. ∎

Definition 3 *Given a family $\mathcal{U}$ in X,* the local Lebesque number $L_{\mathcal{U}}(x) \in \mathbb{R}_+ \cup \infty$ *is defined as the supremum of* $\mathrm{dist}(x, X \setminus U)$*, $U \in \mathcal{U}$. If $U = X$ for some $U \in \mathcal{U}$ it is defined to be infinity.*

Notice that either $L_{\mathcal{U}} \equiv \infty$ at all points or it is a natural Lipschitz function associated with $\mathcal{U}$. More precisely $|L_{\mathcal{U}}(x) - L_{\mathcal{U}}(y)| \le d_X(x, y)$.

Definition 4 *The* Lebesque number $L(\mathcal{U}, A)$ *is* $\inf\{\, L_{\mathcal{U}}(x) \mid x \in A\,\}$.

Definition 5 *A family of subsets $\mathcal{U}$ of a metric space X is called* coarse *if $L_{\mathcal{U}}$ is coarsely proper (as a function from X to $\mathbb{R} \cup \infty$).*

An alternative way to define coarse families is to require $L(\mathcal{U}, A) \to \infty$ as $A \to \infty$. Yet another way is to state that $L(\mathcal{U}, X \setminus B(x_0, t)) \to \infty$ as $t \to \infty$.

Proposition 1

1. *A family $\mathcal{U} = \{A\}$ consisting of one subset A of X is coarse if and only if $X \setminus A$ is bounded.*
2. *A family $\mathcal{U} = \{X_1, X_2\}$ consisting of two subsets of X is coarse if and only if d_X restricted to $(X \setminus X_1) \times (X \setminus X_2)$ is coarsely proper.*
3. *A family $\mathcal{U} = \{X_1, X_2, \ldots, X_n\}$ consisting of finitely many subsets of X is coarse if and only if the function $d_{\mathcal{U}}(x) := \sum_{i=1}^{n} \mathrm{dist}(x, X \setminus X_i)$ is coarsely proper.*

PROOF. 1. If $X \setminus A$ is bounded, then $L_{\mathcal{U}}(x) \ge \mathrm{dist}(x, X \setminus A)$ and $L_{\mathcal{U}}$ is coarsely proper. If $X \setminus A$ is unbounded, then $L_{\mathcal{U}}(x) = 0$ at all $x \in X \setminus A$ and $L_{\mathcal{U}}$ is not coarsely proper.

2. Suppose $\mathcal{U} = \{X_1, X_2\}$ is coarse and $x_n \to \infty$, $y_n \to \infty$, for some $x_n \in X \setminus X_1$, $y_n \in X \setminus X_2$. Notice $L_{\mathcal{U}}(x_n) \le d_X(x_n, y_n)$, so $d_X(x_n, y_n) \to \infty$.

If $\mathcal{U} = \{X_1, X_2\}$ is not coarse, then there is a sequence $z_n \to \infty$ with $L_{\mathcal{U}}(z_n)$ bounded by M. We can produce $x_n \in X \setminus X_1$ and $y_n \in X \setminus X_2$ so that $d_X(z_n, x_n) < M + 1$ and $d_X(z_n, y_n) < M + 1$ for all n. Now, $d_X(x_n, y_n) < 2M + 2$, a contradiction.

3. Notice $d_{\mathcal{U}}(x) \ge L_{\mathcal{U}}(x)$ and $m \cdot L_{\mathcal{U}}(x) \ge d_{\mathcal{U}}(x)$. ∎

Definition 6 *Given a function $f\colon X \to Y$ of metric spaces, its* Lebesque number transfer $L^f\colon \mathbb{R}_+ \to \mathbb{R}_+ \cup \infty$ *is the supremum of all functions $\alpha\colon \mathbb{R}_+ \to \mathbb{R}_+ \cup \infty$ such that $L(\mathcal{U}, Y) \ge t$ implies $L(f^{-1}(\mathcal{U}), X) \ge \alpha(t)$ for all families $\mathcal{U}$ of subsets of Y.*

Definition 7 *A function $f\colon X \to Y$ of metric spaces is* coarse *if the Lebesque number transfer L^f is coarsely proper.*

An alternative definition of coarse functions is to require the function $\mathcal{U} \to L(f^{-1}(\mathcal{U}), X)$ to be coarsely proper on the set of covers of Y.

Let us show that our definition of coarse functions coincides with that of Roe [14].

Proposition 2 *A function $f\colon X \to Y$ is coarse if and only if for every $R > 0$ there is $M > 0$ such that $d_X(x,y) \le R$ implies $d_Y(f(x), f(y)) \le M$ for all $x, y \in X$.*

PROOF. Notice that if $M > 0$ and $N > 0$ are numbers such that $d_X(x,y) < M$ implies $d_Y(f(x), f(y)) < N$, then $L^f(N) \ge M$. Therefore f being coarse in the sense of Roe implies L^f being coarsely proper.

Conversely, if $L^f(N) \ge M$, then consider the cover $\mathcal{U} = \{B(z,N)\}_{z\in Y}$ whose Lebesque number is clearly at least N. If $d_X(x,y) < M$, then there is z so that $x, y \in f^{-1}(B(z,N))$. Hence $d_Y(f(x), f(y)) < 2 \cdot N$ and f is coarse. ■

Dranishnikov [2, p. 1088] defined *asymptotically Lipschitz functions* $f\colon X \to Y$ as those for which there are constants $M > 0$ and A such that $d_Y(f(x), f(y)) \le M \cdot d_X(x,y) + A$ for all $x, y \in X$. Let us relate this concept to the Lebesque number transfer.

Proposition 3 *A function $f\colon X \to Y$ is asymptotically Lipschitz if and only if there is a linear function $t \to m \cdot t + b$ so that $m > 0$ and $L^f(t) \ge m \cdot t + b$ for all t.*

PROOF. Suppose there are constants $M > 0$ and A such that $d_Y(f(x), f(y)) \le M \cdot d_X(x,y) + A$ for all x, $y \in X$. Given a cover $\mathcal{U}$ of Y with $L(\mathcal{U}, Y) \ge t$ and given $x \in X$, the ball $B(x, (t - A - \delta)/M)$ is mapped by f into the ball $B(f(x), t - \delta)$ which is contained in an element of $\mathcal{U}$ for all $\delta > 0$. That shows the Lebesque number of $f^{-1}(\mathcal{U})$ to be at least $(t - A)/M$. Conversely, if $L^f(t) \ge m \cdot t + b$ for all t and $m > 0$, then we claim $d_Y(f(x), f(y)) < 2 \cdot d_X(x,y)/m + 2(1-b)/m$. Indeed, put $d_X(x,y) = s$ and consider the cover $\mathcal{U} = \{B(z, (s+1-b)/m)\}_{z\in Y}$ whose Lebesque number is clearly at least $(s+1-b)/m$. There is z so that x, $y \in f^{-1}(B(z, (s+1-b)/m))$. Hence $d_Y(f(x), f(y)) < 2 \cdot (s+1-b)/m$ and f is asymptotically Lipschitz. ■

Proposition 4 *Given a function $f\colon X \to Y$ of metric spaces the following conditions are equivalent:*

1. *f sends bounded subsets of X to bounded subsets of Y and $f^{-1}(\mathcal{U})$ is coarse for every coarse family $\mathcal{U}$ in Y.*

2. *f is coarse and coarsely proper.*

PROOF. $1 \Longrightarrow 2$. Given a bounded subset A of Y the family $\{Y \setminus A\}$ is coarse (see Proposition 1). Since $\{f^{-1}(Y \setminus A)\}$ is coarse and $f^{-1}(Y \setminus A) = X \setminus f^{-1}(A)$, $f^{-1}(A)$ must be bounded and f is coarsely proper.

If f is not coarse, we find sequences x_n, $y_n \in X$ so that $d_Y(f(x_n), f(y_n)) > n$ for each n but $d_X(x_n, y_n) < M$ for all n. Since f sends bounded subsets of X to bounded subsets of Y, we may assume $x_n \to \infty$, hence $y_n \to \infty$. Put $A = \{x_n\}$ and $B = \{y_n\}$. Using Proposition 1 we see that $\mathcal{U} = \{Y \setminus f(A), Y \setminus f(B)\}$ is a coarse family in Y. Since $f^{-1}(\mathcal{U})$ is coarse, the family $\mathcal{V} = \{X \setminus A, X \setminus B\}$, to which $\mathcal{U}$ is a shrinking, is coarse as well. That however contradicts Proposition 1.

$2 \Longrightarrow 1$. Obviously, coarse functions $f\colon X \to Y$ send bounded subsets of X to bounded subsets of Y. Put $\mathcal{V} = f^{-1}(\mathcal{U})$ for some coarse family $\mathcal{U}$ in Y. To find points $x \in X$ such that $L_{\mathcal{V}}(x) > t$ we find $s > 0$ so that $L^f(s) > t$ and we find $u > 0$ such that $L_{\mathcal{U}}(y) > s$ for $y \in Y \setminus B(y_0, u)$. Put $\mathcal{W} = \mathcal{U} \cup \{B(y_0, u+s)\}$. Note $L(\mathcal{W}, Y) > s$. Since $L(f^{-1}(\mathcal{W}), X) > t$, points x lying outside of the bounded set $f^{-1}(B(y_0, u+s))$ satisfy $L_{\mathcal{V}}(x) > t$. ■

In the end of this section let us demonstrate the usefulness of the concept of a coarse family by rewording notions from [6].

In [6, section 5.2] the concept of *asymptotic neighborhood* W of a subset A of X is introduced by requiring $\lim_{r\to\infty} \operatorname{dist}(A \setminus B(x_0, r), X \setminus W) = \infty$ for some (and hence for all) $x_0 \in X$.

Proposition 5 *W is an asymptotic neighborhood of A if and only if the pair $\{X \setminus A, W\}$ is coarse.*

PROOF. According to part 2 of Proposition 1 the pair $\{X \setminus A, W\}$ is coarse if and only if d_X restricted to $A \times (X \setminus W)$ is coarsely proper. That can be easily seen as equivalent to

$$\lim_{r\to\infty} \operatorname{dist}(A \setminus B(x_0, r), X \setminus W) = \infty$$

for some (and hence for all) $x_0 \in X$. ■

In [6, section 5.2] (see also [3]) the concept of *asymptotically disjoint subsets* A and B of X is introduced by requiring $\lim_{r\to\infty} \operatorname{dist}(A \setminus B(x_0, r), B \setminus B(x_0, r)) = \infty$ for some (and hence for all) $x_0 \in X$.

Proposition 6 *A and B are asymptotically disjoint if and only if the pair $\{X \setminus A, X \setminus B\}$ is coarse.*

PROOF. Apply part 2 of Proposition 1. ■

Also notice that the concept of an asymptotic separator of [6] (see section 5.2) can be introduced without referring to the Higson corona.

Definition 8 *A subset C of X is an* asymptotic separator *between asymptotically disjoint subsets A and B if there are asymptotic neighborhoods W_A of A and W_B of B such that $C = X \setminus (W_A \cup W_B)$ and $W_A \cap W_B = \emptyset$.*

3 Multiplicity and higher Lebesque numbers

Definition 9 *Given a family $\mathcal{U}$ of subsets of X we define* the multiplicity function $m_{\mathcal{U}}\colon X \to \mathbb{Z}_+ \cup \infty$ *by setting $m_{\mathcal{U}}(x)$ to be equal to the number of elements of $\mathcal{U}$ containing x. The* global multiplicity $m(\mathcal{U}, A)$ *is the supremum of $m_{\mathcal{U}}(x)$, $x \in A$.*

By a *coarse refinement* $\mathcal{V}$ of a coarse family $\mathcal{U}$ we mean a coarse family such that every element V of $\mathcal{V}$ is contained in an element U of $\mathcal{U}$. $\mathcal{V}$ is called a *shrinking* of $\mathcal{U}$ if they are indexed by the same set S and $V_s \subset U_s$ for all $s \in S$. If $\mathcal{V}$ is a coarse refinement of $\mathcal{U}$ indexed by a different set T, then one can create a shrinking $\mathcal{V}'$ of $\mathcal{U}$ as follows: find a function $\phi\colon T \to S$ satisfying $V_t \subset U_{\phi(t)}$ for all $t \in T$. Define V'_s as $\bigcup\{ V_t \mid s = \phi(t) \}$. Notice that $\mathcal{V}'$ has multiplicity at most that of $\mathcal{V}$ and is a coarse shrinking of $\mathcal{U}$.

Given a family $\phi = \{\phi_s : X \to \mathbb{R}_+\}_{s\in S}$ of functions its *carrier family* $\operatorname{Carr}(\phi)$ is the family $\{\phi_s^{-1}(0,\infty)\}_{s\in S}$. The *multiplicity* $m(\phi)$ of ϕ is defined as the multiplicity of its carrier family and its *Lebesque number* $L(\phi)$ is defined as the Lebesque number of its carrier family.

Lemma 2 *If $\mathcal{U} = \{U_s\}_{s\in S}$ is a family in X such that $L_{\mathcal{U}}(x_0) = \infty$ for some $x_0 \in X$, then it has a coarse refinement $\mathcal{V}$ of multiplicity at most 2.*

PROOF. Put $V_n = \{ x \in X \mid (n-1)^2 \le d(x, x_0) < (n+1)^2 \}$ for $n \ge 1$. ■

Lemma 3 *If $\mathcal{U} = \{U_s\}_{s\in S}$ is a family in X of multiplicity at most $n + 1$, then it can be refined by $\mathcal{V} = \bigcup_{i=1}^{n+1} \mathcal{V}^i$ such that $L_{\mathcal{V}}(x) \ge L_{\mathcal{U}}(x)/(2n + 2)$ for each $x \in X$ and each $\mathcal{V}^i$ consists of disjoint sets.*

PROOF. Define $f_s(x) = \operatorname{dist}(x, X \setminus V_s)$. For each finite set T of S define

$$W_T = \left\{ x \in X \mid \min\{ f_t(x) \mid t \in T \} > \sup\{ f_s(x) \mid s \in S \setminus T \} \right\}.$$

Notice $W_T = \emptyset$ if T contains at least $n + 2$ elements. Also, notice that $W_T \cap W_F = \emptyset$ if both T and F are different but contain the same number of elements. Let us estimate the Lebesque number of $\mathcal{W} = \{W_T\}_{T\subset S}$. Given $x \in X$ arrange all non-zero values $f_s(x)$ from the largest to the smallest. Add 0 at the end and look at gaps between those values. The largest number is at least $L_{\mathcal{U}}(x)$, there are at most $n+1$ gaps, so one of them is at least $L_{\mathcal{U}}(x)/(n+1)$. That implies the ball $B(x, L_{\mathcal{U}}(x)/(2n+2))$ is contained in one W_T (T consists of all t to the left of the gap). Define $\mathcal{V}_i$ as $\{W_T\}$, all T containing exactly i elements. ■

Lemma 4 *If $\mathcal{U} = \{U_s\}_{s\in S}$ is a coarse family in X, then it has a coarse refinement $\mathcal{V}$ that is coarsely proper. Moreover, if $\mathcal{U}$ is of finite multiplicity, then we may require $\mathcal{V}$ to be of finite multiplicity as well.*

PROOF. Let $\mathcal{V} = \{V_{s,m}\}_{(s,m)\in S\times N}$, where $V_{s,m} = \{x \in U_s \mid 2^m < d(x,x_0) \le 2^{m+2}\}$. Notice $\mathcal{V}$ is coarse of multiplicity at most $2\cdot m(\mathcal{U})$. Also, it consists of bounded sets so that for any sequence $x_k \to \infty$ the conditions $x_k \in V_{s(k),m(k)}$ imply $V_{s(k),m(k)} \to \infty$. ■

Proposition 7 *If $\mathcal{U} = \{U_s\}_{s\in S}$ is a coarse family in X, then it has a coarse shrinking $\mathcal{V} = \{V_s\}_{s\in S}$ such that for any $M > 0$ there is a bounded subset A_M of X with the property that $B(x,M)\cap V_s \neq \emptyset$ implies $B(x,M) \subset U_s$ provided $x \in X \setminus A_M$.*

PROOF. Pick $x_0 \in X$ and define $f(x) = \min(d(x,x_0)/2, L_{\mathcal{U}}(x)/2)$. Notice f is a coarsely proper function of Lipschitz constant $1/2$. For each $x \in X$ pick $s(x) \in S$ so that $B(x,f(x)) \subset U_{s(x)}$. Define V_s as the union of those balls $B(x,f(x)/2)$ so that $s = s(x)$. It suffices to observe that $B(x,M)\cap V_s \neq \emptyset$ and $M < f(x)/3$ implies $B(x,M) \subset U_s$. Indeed, $B(y,f(y)) \subset U_s$ for some $y \in B(x,M)$. Since $f(x) - f(y) \le d(x,y)/2 < M/2$, one has $f(y) > f(x) - M/2 > 3M - M/2 > 2M$ and $B(x,M) \subset B(y,f(y)) \subset U_s$. ■

Lemma 5 *If $\mathcal{U}$ is a coarse family in X that is coarsely proper, then there is a coarsely proper function $f\colon \mathcal{U} \to \mathbb{R}_+$ such that the family $\{B(U,-f(U))\}_{U\in\mathcal{U}}$ is coarse.*

PROOF. Define $f(U) = \inf\{\, L_{\mathcal{U}}(x)/4 \mid x \in U\}$. Notice f is a coarsely proper function. Pick $s(x) \in S$ so that $B(x, L_{\mathcal{U}}(x)/2) \subset U_{s(x)}$. $f(U_{s(x)}) \le L_{\mathcal{U}}(x)/4$ which implies $B(x, L_{\mathcal{U}}(x)/4) \subset B(U_{s(x)}, -f(U_{s(x)}))$. Thus $\{B(U,-f(U))\}_{U\in\mathcal{U}}$ is coarse. ■

In the large scale geometry one should think of bounded subsets of X as points. Here is a generalization of the Lebesque number.

Definition 10 *Let $n \ge 0$. Suppose $\mathcal{U}$ is a family in X and A is a bounded subset of X. The n-*th Lebesque number *$L^n(\mathcal{U},A)$ is the supremum of $t \in [0,\infty]$ such that $\mathcal{U}|_A$ has a refinement of multiplicity at most $n+1$ and Lebesque number at least t.*

Notice such supremum exists as the cover of A consisting of points is of Lebesque number 0 and multiplicity 1.

Observe that $L^n(\mathcal{U},A),\ n \ge 0$, form an increasing sequence of numbers bounded by $L(\mathcal{U},A)$. If $\mathcal{U}|_A$ is of finite order, then they eventually stabilize and are equal to $L(\mathcal{U}|_A, A)$.

Let us point out that Sperner's Lemma can be used to estimate higher Lebesque numbers as follows: Consider a 2-simplex Δ with vertices labeled 0, 1, and 2. Let $\mathcal{U}$ be the cover of Δ by stars U_i, $i = 0$, 1, 2, of its vertices. Consider a subdivision L of Δ with mesh M (in this case it coincides with the longest edge in the subdivision). Let $X = A$ be the set of vertices of L. Suppose $\mathcal{V} = \{V_0, V_1, V_2\}$ is a shrinking of $\mathcal{U}|_A$. Obviously, there is a shrinking of multiplicity 1. However, if we request $\mathcal{V}$ to be of large Lebesque number, we run into problems. Namely, $L^1(\mathcal{U},A) \le M$. Indeed, if $L(\mathcal{V}) > M$, we assign to each vertex v of L number i such that $B(v,M) \subset V_i$. We are in the situation of the classical Sperner's Lemma: vertices on the edges of Δ must be labeled with a number of one of the vertices of that edge. Therefore one has a simplex in L whose vertices were assigned all three numbers 0, 1, 2. Since $L(\mathcal{V}) > M$, the three vertices belong to $V_0 \cap V_1 \cap V_2$ and multiplicity of $\mathcal{V}$ is 3. Thus $L^1(\mathcal{V},A) \le M$.

We will use the observation above in the case of M-scale connected spaces.

Definition 11 *Suppose M is a positive number. A metric space X is called M-*scale connected *if for every two points $x, y \in X$ there is a chain of points $x = x_1, x_2, \ldots, x_k = y$ such that $d_X(x_i, x_{i+1}) < M$ for all $i < k$.*

Here is an application of Sperner's Lemma for 1-simplices.

Lemma 6 *Let M be a positive number and X be an M-scale connected metric space. If $L^0(\mathcal{U},X) > M$ for some cover $\mathcal{U}$ of X, then $\mathcal{U}$ contains X as an element.*

PROOF. Suppose $\mathcal{V}$ is a refinement of $\mathcal{U}$ of multiplicity at most 1 and Lebesque number bigger than M. If X is not an element of $\mathcal{V}$, then there are disjoint non-empty elements V_1, $V_2 \in \mathcal{V}$. Pick a chain of points $x = x_1, x_2, \ldots, x_k = y$ such that $d_X(x_i, x_{i+1}) < M$ for all $i < k$ and $x \in V_1$, $y \in V_2$. There is an index $j < k$ such that $x_j \in V_1$ and $x_{j+1} \notin V_1$. The ball $B(x_{j+1}, M)$ is contained in an element W of $\mathcal{V}$ and intersects V_1. Therefore $W = V_1$, a contradiction. ■

4 The coarse category

Let us introduce the coarse category in a way that explains why two coarse functions are considered equivalent if their distance is bounded.

Definition 12 *Given a metric space (X, d_X) and its two subsets X_1 and X_2 the notation $X_1 \leq X_2$ means there is a positive number R such that X_1 is contained in the ball $B(X_2, R) = \{x \in X \mid \operatorname{dist}(x, X_2) < R\}$.*

Proposition 8 *A function $f \colon X \to Y$ of metric spaces is coarse if and only if it preserves the relation $\leq$ of sets. Thus, $X_1 \leq X_2$ implies $f(X_1) \leq f(X_2)$.*

PROOF. Suppose $f \colon X \to Y$ preserves the relation $\leq$ of sets but not in the sense of Roe. Therefore, for some $M > 0$ there is a sequence of points x_n, y_n so that $d_X(x_n, y_n) < M$ for each n but $d_Y(f(x_n), f(y_n)) \to \infty$ as $n \to \infty$. If $f(A)$ is bounded for some subsequence A of x_n, then $f(B)$ is bounded for the corresponding subsequence of y_n (in view of $f(B) \leq f(A)$) contradicting

$$d_Y(f(x_n), f(y_n)) \to \infty \qquad \text{as } n \to \infty.$$

Thus $f(x_n) \to \infty$ and $f(y_n) \to \infty$ as $n \to \infty$. By induction define a subsequence a_n of $\{x_n\}_{n\geq 1}$ and the corresponding subsequence b_n of $\{y_n\}_{n\geq 1}$ with the property that $d_Y\big(f(a_k), f(b_i)\big) > k$ and $d_Y\big(f(b_k), f(a_i)\big) > k$ for all $k \geq i$. Since $A = \{a_n\}_{n\geq 1} \leq B = \{b_n\}_{n\geq 1}$ one has $f(A) \leq f(B)$, a contradiction.

Suppose $f \colon X \to Y$ is coarse in the sense of Roe and $X_1 \leq X_2$ in X. Pick $R > 0$ so that $X_1 \subset B(X_2, R)$ and choose $M > 0$ satisfying $d_Y\big(f(x), f(y)\big) < M$ if $d_X(x, y) < R$ for all x, $y \in X$. Given $x \in X_1$ pick $y \in X_2$ so that $d_X(x, y) < R$ since $d_Y\big(f(x), f(y)\big) < M$ one gets $f(X_1) \subset B(f(X_2), M)$. Thus $f(X_1) \leq f(X_2)$. ■

Notice that $X_1 \leq X_2$ for every bounded subset X_1 of X provided $X_2 \neq \emptyset$. Also, $X_1 \leq X_2$ implies X_1 is bounded provided X_2 is bounded. Therefore $f(A)$ is bounded for every bounded subset A of X and every coarse function $f \colon X \to Y$.

Given a function $f \colon X \to Y$ of metric spaces one can identify it with its graph $\Gamma(f) \subset X \times Y$. Therefore it makes sense to ponder the meaning of $\Gamma(f) \leq \Gamma(g)$ for $f, g \colon X \to Y$.

Proposition 9 *Suppose $f, g \colon X \to Y$ are functions of metric spaces.*

1. *If g is coarse, then $\Gamma(f) \leq \Gamma(g)$ implies that the distance $\operatorname{dist}(f, g)$ between f and g is finite. In particular, f is coarse.*

2. *If $\operatorname{dist}(f, g)$ is finite, then $\Gamma(f) \leq \Gamma(g)$.*

PROOF.

1. Suppose the distance $\operatorname{dist}(f, g)$ is not finite, so there are points $x_n \in X$ with $d_Y\big(f(x_n), g(x_n)\big) > n$ for all $n \geq 1$. Let $R > 0$ be a number such that $B(\Gamma(g), R)$ contains $\Gamma(f)$. For each n pick $y_n \in X$ satisfying $d_X(x_n, y_n) + d_Y\big(f(x_n), g(y_n)\big) < R$. There is $M > 0$ so that $d_Y(g(x_n), g(y_n)) < M$ for all $n \geq 1$ as g is coarse. Now, $d_Y\big(f(x_n), g(x_n)\big) \leq d_Y\big(f(x_n), g(y_n)\big) + d_Y\big(g(y_n), g(x_n)\big) < R + M$ for all $n \geq 1$, a contradiction.

2. Notice $\Gamma(f) \subset B(\Gamma(g), \operatorname{dist}(f, g))$. ■

Definition 13 *Given a function $f\colon X \to Y$ of metric spaces we define the* forward distance transfer *function $d_f\colon \mathbb{R}_+ \to \mathbb{R}_+ \cup \infty$ as the infimum of all functions $\alpha\colon \mathbb{R}_+ \to \mathbb{R}_+ \cup \infty$ with the property that $d_X(x,y) \le t$ implies $\alpha(t) \ge d_Y\big(f(x), f(y)\big)$ for all $x, y \in X$.*

The reverse distance transfer *function $d^f\colon \mathbb{R}_+ \to \mathbb{R}_+ \cup \infty$ as the infimum of all functions $\alpha\colon \mathbb{R}_+ \to \mathbb{R}_+ \cup \infty$ with the property that $d_Y\big(f(x), f(y)\big) \le t$ implies $d_X(x,y) \le \alpha(t)$ for all $x, y \in X$.*

Notice that f is coarse if and only if d_f maps $\mathbb{R}_+$ to $\mathbb{R}_+$, i.e. the values of d_f are finite. Also, f is asymptotically Lipschitz if and only if d_f is bounded by a linear function.

Proposition 10 *If $f, g\colon X \to Y$ are two coarsely proper coarse functions, then the following conditions are equivalent:*

1. $\mathrm{dist}(f,g)$ *is finite.*

2. *For every coarse family $\mathcal{U} = \{U_s\}_{s\in S}$ in Y the family $\{f^{-1}(U_s) \cap g^{-1}(U_s)\}_{s\in S}$ is coarse.*

PROOF.

$1 \Longrightarrow 2$. Let $\mathrm{dist}(f,g) < M$. Consider $\mathcal{V} = \{B(U_s, -M)\}_{s\in S}$. It is a coarse family, so $f^{-1}(\mathcal{V})$ is coarse by Proposition 4. Notice $f^{-1}\big(B(U_s,-M)\big) \subset f^{-1}(U_s) \cap g^{-1}(U_s)$ for all $s \in S$ which is sufficient to establish coarseness of $\{f^{-1}(U_s) \cap g^{-1}(U_s)\}_{s\in S}$.

$2 \Longrightarrow 1$. If $\mathrm{dist}(f,g)$ is not finite, there is a sequence $x_n \to \infty$ such that $d_Y\big(f(x_n), g(x_n)\big) > n$ for all n. Put $A = \{x_n\}_{n\ge 1}$. By Proposition 1, the family $\mathcal{U} = \{Y \setminus f(A), Y \setminus g(A)\}$ is coarse. However, $\{f^{-1}(U_s) \cap g^{-1}(U_s)\}_{s\in S}$ is not coarse as it refines $\{X \setminus A\}$ which is not coarse. ∎

Our category is that of metric spaces and equivalence classes of coarse functions.

$$f \sim g \quad \text{if } d_Y(f(x), g(x)) \text{ is a bounded function of } x.$$

Generalizing the concept of $A \le B$ for subsets of a given metric space X, we say Y *coarsely dominates* X (notation: $X \le_{\text{rse}}^{\text{coa}} Y$) if there are coarse functions $f\colon X \to Y$ and $g\colon Y \to X$ such that $g \circ f$ is at a finite distance from id_X.

Proposition 11 *Suppose $f\colon X \to Y$ and $g\colon Y \to X$ are coarse functions. If $g \circ f$ is at a finite distance from id_X, then both $f\colon X \to f(X)$ and $g\colon f(X) \to X$ are coarsely proper and $f \circ g$ is at finite distance from $\mathrm{id}_{f(X)}$.*

PROOF. Suppose $x_n \to \infty$. None of the subsequences of $\{f(x_n)\}$ can be bounded as g would send it to a bounded subset of X. Thus $f(x_n) \to \infty$. If $f(x_n) \to \infty$, then none of subsequences of $\{x_n\}$ is bounded. Therefore none of the subsequences of $\{g\big(f(x_n)\big)\}$ is bounded and $g\colon f(X) \to X$ is coarsely proper. If $d_X\big(g(f(x)), x\big) < M$ for all $x \in X$, then $d_Y\big(f(g(f(x))), f(x)\big) \le d_f(M)$ and $f \circ g$ is at finite distance from $\mathrm{id}_{f(X)}$. ∎

Proposition 12 *A surjective coarse function $f\colon X \to Y$ of metric spaces is a coarse isomorphism if and only if the reverse distance transfer function d^f is finite.*

PROOF. If there is a coarse function $g\colon Y \to X$ such that $g \circ f$ is at finite distance M to id_X, then $d^f(a) \le d_g(a) + 2M$ is finite.

Assume d^f is finite and pick a right inverse $g\colon Y \to X$. Notice $d_X\big(g(x), g(y)\big) \le d^f\big(d_Y(x,y)\big)$, so g is coarse. ∎

5 Coarse dimensions

Definition 14 *The* coarse dimension $\dim_{\text{rse}}^{\text{coa}}(X)$ *(respectively,* the major coarse dimension $\dim_{\text{RSE}}^{\text{COA}}(X)$*) is the smallest integer n such that any finite coarse family in X (respectively, any coarse family in X) has a coarse refinement with multiplicity at most $n+1$.*

Remark 1 *Using* [2, Proposition 4.4 on p. 1104] *(notice that the words 'uniformly bounded' are erroneously inserted there) one can show that, for proper metric spaces X, the major coarse dimension of X coincides with the asymptotic dimension of Dranishnikov. In view of Corollary* 6*, our coarse dimension and Dranishnikov coarse dimension are identical.*

Given a coarse family $\mathcal{U} = \{U_s\}_{s\in S}$ in a subset A of X one can extend it to a coarse family $\mathcal{U}' = \{U_s \cup (X \setminus A)\}_{s\in S}$ in X. Notice that $\mathcal{V} \cap A$ is a coarse refinement of $\mathcal{U}$ for any coarse refinement $\mathcal{V}$ of $\mathcal{U}'$. Therefore the following holds.

Corollary 1 *If A is a subset of a metric space X, then*

$$\dim_{\text{rse}}^{\text{coa}}(A) \leq \dim_{\text{rse}}^{\text{coa}}(X) \qquad \textit{and} \qquad \dim_{\text{RSE}}^{\text{COA}}(A) \leq \dim_{\text{RSE}}^{\text{COA}}(X).$$

Proposition 13 *If Y coarsely dominates X, then*

$$\dim_{\text{rse}}^{\text{coa}}(X) \leq \dim_{\text{rse}}^{\text{coa}}(Y) \qquad \textit{and} \qquad \dim_{\text{RSE}}^{\text{COA}}(X) \leq \dim_{\text{RSE}}^{\text{COA}}(Y).$$

PROOF. The proof is almost the same for both dimensions. Suppose $\mathcal{U}$ is a coarse family in X and $f\colon X \to Y$, $g\colon Y \to X$ are coarse functions such that there is $M > 0$ with $d_X\big(x, g(f(x))\big) < M$ for all $x \in X$. Replacing Y by $f(X)$ we may assume f is onto and both f and g are coarsely proper (see Proposition 11). The idea of the proof is to refine $g^{-1}(\mathcal{U})$ by $\mathcal{V}$ and then refine $f^{-1}(\mathcal{V})$ to obtain a desired refinement $\mathcal{W}$ of $\mathcal{U}$ of multiplicity at most $n+1$, where n is the dimension of Y. Consider $\mathcal{U}' = \{B(U_s, -M)\}_{s\in S}$. It is a coarse family in X, so $\{g^{-1}\big(B(U_s, -M)\big)\}_{s\in S}$ is coarse and it has a coarse shrinking $\mathcal{V} = \{V_s\}_{s\in S}$ of multiplicity at most $n+1$. Suppose $x \in f^{-1}(V_s) \setminus U_s$. Since $d_X\big(x, g(f(x))\big) < M$, $g(f(x)) \notin B(U_s, -M)$. However, $f(x) \in V_s \subset g^{-1}\big(B(U_s, -M)\big)$, a contradiction. ■

Definition 15 *The* minor asymptotic dimension $\mathrm{ad}(X)$ *(resp., the* asymptotic dimension $\mathrm{asdim}(X)$*) is the smallest integer n such that the function $\mathcal{U} \to L^n(\mathcal{U}, X)$ is coarsely proper on the space of finite covers (resp., arbitrary covers) $\mathcal{U}$ of X.*

Let us show that our definition of asymptotic dimension is equivalent to that of Gromov.

Proposition 14 $\mathrm{asdim}(X) \leq n$ *if and only if for each $M > 0$ there is a uniformly bounded family $\mathcal{U}$ in X of Lebesque number at least M and multiplicity at most $n+1$.*

PROOF. If $\mathrm{asdim}(X) \leq n$ as in Definition 15 and $M > 0$, then there is $N > 0$ such that every cover $\mathcal{V}$ of X satisfying $L(\mathcal{V}, X) \geq N$ has a refinement $\mathcal{U}$ of multiplicity at most $n+1$ and Lebesque number at least M. Pick $\mathcal{V}$ to be the cover of X by balls of radius N. The resulting $\mathcal{U}$ is uniformly bounded.

Suppose for each $M > 0$ there is a uniformly bounded family $\mathcal{U}^M$ of multiplicity at most $n+1$ and Lebesque number at least M. Let $\alpha(M)$ be the supremum of diameters of elements of $\mathcal{U}^M$. Given any family $\mathcal{V}$ of Lebesque number at least $\alpha(M)+1$, $\mathcal{U}^M$ is a refinement of of $\mathcal{V}$ which proves that the function $\mathcal{V} \to L^n(\mathcal{V}, X)$ is coarsely proper on the space of all covers $\mathcal{V}$ of X. ■

Quite often it is useful to have even stronger conditions imposed on covers appearing in Proposition 14.

Proposition 15 (Gromov) *If Gromov asymptotic dimension $\mathrm{asdim}(X)$ does not exceed n, then for any $M, N > 0$ there exist uniformly bounded families $\mathcal{U}^i$, $1 \leq i \leq n+1$, such that each $\mathcal{U}^i$ is N-disjoint and $\mathcal{U} = \bigcup_{i=1}^{n+1} \mathcal{U}^i$ is of Lebesque number at least M.*

PROOF. Consider a uniformly bounded family $\mathcal{V} = \{V_s\}_{s\in s}$ of multiplicity at most $n+1$ and Lebesque number at least $2(n+1)\cdot(M+N)$. Lemma 3 says it can be refined by $\mathcal{V}' = \bigcup_{i=1}^{n+1} \mathcal{V}^i$ such that $L_{\mathcal{V}}(x) \geq L_{\mathcal{U}}(x)/(2n+2) \geq M+N$ for each $x \in X$ and each $\mathcal{V}^i$ consists of disjoint sets. Define $\mathcal{U}_i$ as $\{B(W,-N)\}$, $W \in \mathcal{V}^i$. ■

Let us characterize spaces of asymptotic dimension 0.

Proposition 16 $\operatorname{asdim}(X) > 0$ *if and only if there exist a number $M > 0$ and a coarsely proper sequence $\{(x_n, y_n)\}_{n=1}^{\infty}$ of pairs of points in X such that $\operatorname{dist}(x_n, y_n) \to \infty$ and the points x_n and y_n can be M-scale connected in $X \setminus B(x_0, n)$.*

PROOF. If $\operatorname{asdim}(X) = 0$, then for any $M > 0$ there exists an M-disjoint cover of X by uniformly bounded sets. Therefore, the distance between two points x and y which can be M-scale connected in X is uniformly bounded.

Suppose $\operatorname{asdim}(X) > 0$. Let n be a positive integer and x_0 be the base point in X. There is $L > 0$ such that X does not have a uniformly bounded cover of Lebesque number bigger than L and multiplicity 1. Define an equivalence relation on $X \setminus B(x_0, n)$ by saying $x \sim y$ if and only if x and y can be $2L$-scale connected in $X \setminus B(x_0, n)$. The cover of X by the equivalence classes has Lebesque number at least $2L$, therefore these classes are not uniformly bounded by the choice of L. Thus, there exist points x_n and y_n which can be $2L$-scale connected in $X \setminus B(x_0, n)$ such that $\operatorname{dist}(x_n, y_n)$ is arbitrarily large. ■

Proposition 17 *If Y coarsely dominates X, then* $\operatorname{asdim}(X) \leq \operatorname{asdim}(Y)$ *and* $\operatorname{ad}(X) \leq \operatorname{ad}(Y)$.

PROOF. The proof is almost the same for both dimensions. Suppose $\mathcal{U}$ is a coarse family in X and $f\colon X \to Y$, $g\colon Y \to X$ are coarse functions such that there is $M > 0$ with $d_X(x, g(f(x))) < M$ for all $x \in X$. By replacing Y with $f(X)$ we may assume f is onto and both f and g are coarsely proper (see Proposition 11). The idea of the proof is to refine $g^{-1}(\mathcal{U})$ by $\mathcal{V}$ and then refine $f^{-1}(\mathcal{V})$ to obtain a desired refinement $\mathcal{W}$ of $\mathcal{U}$ of multiplicity at most $n+1$, where n is the dimension of X. Take a coarsely proper function $\alpha\colon \mathbb{R}_+ \to \mathbb{R}_+$ with the property that any finite cover (respectively, arbitrary cover) $\mathcal{U}$ of Y satisfying $L(\mathcal{U}, Y) \geq \alpha(t)$ has a refinement $\mathcal{V}$ of multiplicity at most $n+1$ so that $L(\mathcal{V}, Y) \geq t$.

Given $t > 0$ pick $\beta(t)$ so that $L^g(\beta(t)) > \alpha(t)$ (see Definition 7). Assume $L(\mathcal{U}) > M + \beta(t)$. Consider $\mathcal{U}' = \{B(U_s, -M)\}_{s\in S}$. $L(\mathcal{U}') > \beta(t)$, so $\{g^{-1}\big(B(U_s, -M)\big)\}_{s\in S}$ is of Lebesque number at least $\alpha(t)$ and it has a shrinking $\mathcal{V} = \{V_s\}_{s\in S}$ of multiplicity at most $n+1$ and $L(\mathcal{V}) \geq t$. Suppose $x \in f^{-1}(V_s) \setminus U_s$. Since $d_X\big(x, g(f(x))\big) < M$, $g(f(x)) \notin B(U_s, -M)$. However, $f(x) \in V_s \subset g^{-1}\big(B(U_s, -M)\big)$, a contradiction. ■

Theorem 1 *The major coarse dimension of X does not exceed the asymptotic dimension of X.*

PROOF. Suppose $\operatorname{asdim}(X) = n < \infty$ and $\mathcal{U} = \{U_s\}_{s\in S}$ is a coarse family in X. By Lemma 4 we may assume U is coarsely proper. By induction on k find a sequence of numbers $M_0 = 1$, M_1, M_2, …, and covers $\mathcal{V}^k = \{V_t\}_{t\in T(k)}$, $k \geq 1$, of multiplicity at most $n+1$ and satisfying the following conditions:

a. $L(\mathcal{V}^k, X) \geq M_{k-1}$ for $k \geq 1$.

b. The diameter of each element of $\mathcal{V}^k$ is smaller than M_k.

c. The family $\{\, B(x, M_{k-1}) \mid d(x, x_0) \geq M_k \,\}$ refines $\mathcal{U}$ for each $k \geq 1$.

d. $M_{k+1} > 2M_k$ for all $k \geq 1$.

Find functions $j(k)\colon T(k) \to T(k+1)$ so that $V_t \subset V_{j(k)(t)}$. Denote $\{\, x : M_k \leq d(x, x_0) < M_{k+1} \,\}$ by A_k. Given $t \in T(k)$ so that V_t is contained in some element of $\mathcal{U}$ define $\alpha(t) \in S$ by looking at the sequence $V_t \subset V_{j(k)(t)} \subset \cdots$, picking the latest element contained in some U_s and setting $\alpha(t) = s$ (it is possible each element of the sequence is contained in some U_s in which case all of them are contained in some U_s and that s is picked as $\alpha(t)$). Define W_s as follows: it is the union of non-empty sets of the form $V_t \cap A_k$ so that $V_t \in \mathcal{V}^{k-1}$ and $\alpha(t) = s$. Notice that $m(\mathcal{W}) \le n+1$ as in the annulus A_k the family $\mathcal{W}$ is obtained from $\mathcal{V}^{k-1}$ by assembling some of its elements together.

We plan to show $\mathcal{W}$ is coarse by proving that if $M_k \le d(x, x_0) < M_{k+1}$, then $B(x, M_{k-3})$ is contained in some W_s. Indeed, there is $t \in T(k-2)$ so that $B(x, M_{k-3}) \subset V_t$. Put $r = j(k-2)(t)$ and $u = j(k-1)(r)$. Points of $B(x, M_{k-3})$ can belong to only two of the following three annuli: A_{k-1}, A_k, and A_{k+1}. If $z \in B(x, M_{k-3}) \cap A_{k+1}$, then $V_u \subset B(z, M_k) \subset U_s$ for some $s \in S$. We might as well put $s = \alpha(t) = \alpha(u) = \alpha(r)$. In this case $B(x, M_{k-3}) \subset W_s$. If $B(x, M_{k-3})$ misses the last annulus, then only $\alpha(r)$ is definitely defined ($\alpha(u)$ may not exist) and $\alpha(t) = \alpha(r)$. Now, $B(x, M_{k-3}) \subset W_s$, where $s = \alpha(r)$. ■

Remark 2 *Theorem* 1 *generalizes* [2, Proposition 4.5 on p. 1105].

6 The large scale dimension

In this section we prove that any dimension of X (asymptotic, major coarse, or minor asymptotic), if finite, equals the coarse dimension of X. That corresponds to results of Dranishnikov [2] that $\mathrm{asdim}(X)$ or $\mathrm{asdim}^*(X)$, if finite, are equal to the dimension of the Higson corona of any proper metric space X. Our proofs are direct and become simpler by introducing a new dimension, the *large scale dimension* of X. That dimension turns out to be identical with the coarse dimension.

Definition 16 *The* large scale dimension $\dim^{\text{large}}_{\text{scale}}(X)$ *of X is the smallest integer n such that $A \to L^n(\mathcal{U}, A)$ is a coarsely proper function on the set of bounded subsets of X for all finite coarse families $\mathcal{U}$ in X.*

Notice $\dim^{\text{large}}_{\text{scale}}(X) = -1$ if X is bounded.

Obviously, $\dim^{\text{large}}_{\text{scale}}(X) \ge \dim^{\text{large}}_{\text{scale}}(A)$ for any subset A of X.

Proposition 18 $\mathrm{ad}(X) \ge \dim^{\text{large}}_{\text{scale}}(X)$ *and* $\dim^{\text{coa}}_{\text{rse}}(X) \ge \dim^{\text{large}}_{\text{scale}}(X)$.

PROOF. The inequality $\dim^{\text{coa}}_{\text{rse}}(X) \ge \dim^{\text{large}}_{\text{scale}}(X)$ is almost obvious. Indeed, given $n = \dim^{\text{coa}}_{\text{rse}}(X)$ and given a coarse family $\mathcal{U}$ in X consisting of m elements one has a coarse refinement $\mathcal{V}$ of $\mathcal{U}$ such that the multiplicity $m(\mathcal{V})$ is at most $n+1$. In that case

$$L^n(\mathcal{U}, A) \ge L(\mathcal{V}, A) \ge \inf_{a \in A} L_{\mathcal{V}}(a)$$

and is a coarsely proper function of A.

Suppose $\mathrm{ad}(X) = n$ and $\mathcal{U}$ is a coarse cover of X consisting of m elements. Given $t > 0$ find a bounded subset U of X such that $\mathcal{U}|_{(X \setminus U)}$ has a refinement $\mathcal{V}$ of multiplicity at most $n+1$ and Lebesque number at least t. For any bounded subset A of $X \setminus U$, $L^n(\mathcal{U}, A) \ge L(\mathcal{V}, A) \ge t$ which proves $\dim^{\text{large}}_{\text{scale}}(X) \le n$. ■

As shown in [5], the asymptotic dimension of $\mathbb{R}^n$ is at most n (see p. 793). For the convenience of the reader let us reword the argument from [5] as follows: Given $M > 0$ consider the triangulation on the unit n-cube I^n obtained by starring at the center of each face. It is invariant under symmetries of I^n and the cover of I^n by stars of vertices has a positive Lebesque number k and is of multiplicity at most $n+1$. Rescale I^n by the factor of M/k and extend its triangulation over the whole $\mathbb{R}^n$ by reflections. The cover of $\mathbb{R}^n$ by stars of vertices has Lebesque number at least M and is of multiplicity at most $n+1$.

Let us show how to use the large scale dimension to estimate asymptotic dimension from below.

Proposition 19 $\dim^{\text{large}}_{\text{scale}}(\mathbb{R}^n) \ge n$.

PROOF. Since $\dim(I^n) = n$, there is a finite open cover $\mathcal{U}$ of I^n with no open refinement of multiplicity at most n. Let $I^n_k \subset \mathbb{R}^n$ be a copy of I^n enlarged k times with the corresponding cover $\mathcal{U}^k$. We request $I^n_k \to \infty$ so that $\mathcal{V}$ obtained by adding the corresponding elements of $\mathcal{U}^k$ is a finite coarse family on $A = \bigcup_{k=1}^{\infty} I^n_k$. Notice $L^{n-1}(\mathcal{V}, I^n_k) = 0$ for all k. Thus $\dim^{\text{large}}_{\text{scale}}(\mathbb{R}^n) \geq n$. ■

Proposition 20 *If* $\dim^{\text{large}}_{\text{scale}}(X) = 0$, *then* $\operatorname{asdim}(X) = 0$ *and* $\dim^{\text{COA}}_{\text{RSE}}(X) = 0$.

PROOF. It suffices to show $\operatorname{asdim}(X) = 0$ (see Theorem 1). Suppose $\operatorname{asdim}(X) > 0$. By Proposition 16 there exist a number $M > 0$ and a coarsely proper sequence $\{(x_n, y_n)\}_{n=1}^{\infty}$ of pairs of points in X such that $\operatorname{dist}(x_n, y_n) \to \infty$ and the points x_n and y_n can be M-scale connected in $X \setminus B(x_0, n)$ by a chain P_n. Consider a coarse family $\mathcal{U}$ consisting of two sets: $X \setminus \bigcup_{n=1}^{\infty}\{x_n\}$ and $X \setminus \bigcup_{n=1}^{\infty}\{y_n\}$.

Since $C \to L^0(\mathcal{U}, C)$ is a coarsely proper function, there is a chain P_n such that $L^0(\mathcal{U}, P_n) > M$. This contradicts Lemma 6 since P_n is M-scale connected and the cover $\mathcal{U}$ is non-trivial on P_n. ■

Definition 17 *Given a point-finite family* $\mathcal{U} = \{U_s\}_{s\in S}$ *in* X *(that means each point of* X *belongs to at most finitely many elements of* $\mathcal{U}$*) by* the canonical partition of unity *of* $\mathcal{U}$ *we mean the family of functions* $\{f_s/f\}_{s\in S}$, *where* $f_s(x) = \operatorname{dist}(x, X \setminus U_s)$ *and* $f(x) = \sum_{s\in S} f_s(x)$. *If* T *is a subset of* S, *then* X_T *is defined to be* $\{\, x \in X \mid \sum_{s\in T} f_s(x)/f(x) = 1 \,\}$ *and by* ∂X_T *we mean the set of all* $x \in X_T$ *such that* $f_s(x) = 0$ *for some* $s \in T$.

Notice that $f(x) > 0$ for all $x \in X$ such that $L_{\mathcal{U}}(x) > 0$ and f is a Lipschitz function if $\mathcal{U}$ is of finite multiplicity.

Lemma 7 *If the large scale dimension of* X *is at most* n, *then any coarse family* $\mathcal{U}$ *in* X *of finite multiplicity* m *has a coarse refinement* $\mathcal{V}$ *of multiplicity at most* $n+1$.

PROOF. Suppose $\mathcal{U}$ exists with no coarse refinement of multiplicity at most $n+1$. Using Lemma 4 we reduce the general case to that of $\mathcal{U} = \{U_s\}_{s\in S}$ consisting of bounded sets so that for any sequence $x_k \to \infty$ the conditions $x_k \in U_{s(k)} \in \mathcal{U}$ imply $U_{s(k)} \to \infty$. For induction on $m - n$ it suffices to assume the multiplicity of $\mathcal{U}$ is $n+2$.

Pick a coarse shrinking $\mathcal{W} = \{W_s\}_{s\in S}$ (see Proposition 7) so that given $M > 0$ there is a bounded subset A of X with the property that, for $x \in X \setminus A$, $B(x, M) \cap W_s \neq \emptyset$ implies $B(x, M) \subset U_s$. Consider the canonical partition of unity f of $\mathcal{W}$. Given a set T in S consisting of $n+2$ elements pick a shrinking $\mathcal{W}^T$ of $\mathcal{W}|_{X_T}$ of order at most $n+1$ and the Lebesque number at least half the maximum $L^n(\mathcal{W}^T, X_T)$ possible (if the maximum is infinity we pick a shrinking of Lebesque number twice the size of X_T). We can add $W_s \cap \partial X_T$ to W^T_s without increasing the order of W^T beyond $n+1$ (obviously, the Lebesque number does not decrease). By pasting those shrinkings for all T one gets a refinement $\mathcal{V}$ of $\mathcal{W}$ on $X \setminus A$ for some bounded subset A of X of multiplicity at most $n+1$. Therefore $\mathcal{V}$ cannot be coarse and there is $M > 0$ and a sequence of points $x_k \to \infty$ such that none of $B(x_k, M)$ is contained in an element of $\mathcal{V}$. In particular $B(x_k, M)$ is not contained in the n-skeleton of X (the points where the order of f is at most $n+1$) for large k.

Pick sets $T(k)$ so that $X_{T(k)} \setminus \partial X_{T(k)}$ contains an element $y_k \in B(x_k, M)$. For large k, $B(x_k, M)$ intersecting W_s implies $B(x_k, M) \subset U_s$. Therefore the set T of $s \in S$ so that $B(x_k, M)$ intersects W_s is of cardinality at most $n+2$ and $B(x_k, M) \subset X_{T(k)}$. For large k the cover $\mathcal{W}|_{X_{T(k)}}$ has a refinement of order at most $n+1$ and Lebesque number at least $3M$. Therefore, $B(x_k, M)$ is contained in a single element of $\mathcal{V}$, a contradiction. ■

Corollary 2 *The coarse dimension of* X *equals the large scale dimension of* X.

Corollary 3 *If the major coarse dimension of* X *is finite, then it equals the large scale dimension of* X.

Theorem 2 *If the asymptotic dimension (respectively, the minor asymptotic dimension) of unbounded* X *is finite, then it equals the large scale dimension of* X.

PROOF. Suppose $\mathrm{asdim}(X) = n$ (respectively, $\mathrm{ad}(X) = n$) and $\dim^{\text{large}}_{\text{scale}}(X) < n$. Notice $n > 0$ as $\dim^{\text{large}}_{\text{scale}}(X) < 0$ is possible only for bounded X. Therefore there is $M > 0$ and a sequence of covers (respectively, finite covers) $\mathcal{U}^k$ indexed by sets $S(k)$ of Lebesque number at least $k + 3M$ and multiplicity at most $n + 1$ so that no refinement of $\mathcal{U}^k$ of multiplicity n has Lebesque number bigger than M. Augment each $\mathcal{U}^k$ by shrinking it to the family $B(U, -M)$, $U \in \mathcal{U}^k$. Let f^k be the canonical partition of unity of that augmentation.

Notice that for any k and any $x \in X$ there is a subset T of $S(k)$ consisting of at most $(n + 1)$ elements so that $B(x, M) \subset X_T$. We are going to show that for every k there is $N > 0$ such that for any $R > N$ there is $T(k) \subset S(k)$ consisting of at most $(n+1)$ elements with $X_{T(k)} \subset X \setminus B(x_0, R)$, x_0 a fixed point in X, so that $\mathrm{Carr}(f^k|_{X_{T(k)}})$ does not admit a refinement of multiplicity at most n and Lebesque number bigger than M.

Suppose that, for some k and $R > 0$, all $\mathrm{Carr}(f^k|_{X_T})$ so that $X_T \subset X \setminus B(x_0, R)$ do admit a refinement $\mathcal{V}(T)$ of multiplicity at most n and Lebesque number bigger than M. By converting those refinements to shrinkings and pasting one gets a refinement $\mathcal{V}$ of $\mathcal{U}^k$ on $X \setminus U$ for some bounded subset U of X of multiplicity at most n and Lebesque number bigger than M. More precisely, for each $T \subset T(k)$ so that $X_T \subset X \setminus B(x_0, R)$, we pick a shrinking $\{V^T_t\}_{t \in T}$ of $\mathrm{Carr}(f^k|_{X_T})$ of multiplicity at most n and Lebesque number bigger than M. If T contains at most n elements, that shrinking is picked to be exactly $\mathrm{Carr}(f^k|_{X_T})$ as the multiplicity is at most n in such case. $\mathcal{V}$ is a shrinking of $\mathcal{U}^k|_{(X \setminus U)}$, U being the union of X_T that are not contained in $X \setminus B(x_0, R)$, and V_s, $s \in S(k)$, is defined as the union of all V^T_s with $s \in T$. The reason $\mathcal{V}$ has Lebesque number at least M is that for any $x \in X$ there is a subset T of $S(k)$ consisting of at most $(n + 1)$ elements so that $B(x, M) \subset X_T$.

Now, the cover consisting of the union of $B(U, 2M)$ and all the elements of $\mathcal{V}$ intersecting $B(U, 2M)$ and of all elements of $\mathcal{V}$ that do not intersect $B(U, 2M)$ is uniformly bounded, of multiplicity at most n (recall $n > 0$), and of Lebesque number bigger than M, a contradiction.

Construct by induction a sequence of sets $T(i) \subset S(i)$ with $X_{T(i)}$ being mutually disjoint and tending to infinity so that $\mathrm{Carr}(f^i|_{X_{T(i)}})$ does not have a refinement of multiplicity at most n and Lebesque number bigger than M. Paste all those carriers according to their index within each set $T(i)$ and get a coarse cover on a subset A of X that does not admit a refinement of multiplicity at most n and Lebesque number bigger than M on infinitely many $X_{T(i)}$, a contradiction. ■

7 Slowly oscillating functions

Definition 18 *A function $f\colon X \to Y$ is slowly oscillating if $f^{-1}(\mathcal{U})$ is coarse for every cover $\mathcal{U}$ of Y of positive Lebesque number.*

Definition 19 *Given a function $f\colon X \to Y$ of metric spaces its* oscillation function *$\mathrm{Osc}(f, M)\colon X \to \mathbb{R}_+ \cup \infty$ for every $M > 0$ is defined by declaring $\mathrm{Osc}(f, M)(a)$ to be the supremum of $d_Y\big(f(x), f(a)\big)$ over all $x \in B(a, M)$.*

Proposition 21 *f is slowly oscillating if and only if $\mathrm{Osc}(f, M)(x) \to 0$ as $x \to \infty$ for all $M > 0$.*

PROOF. Suppose $\mathrm{Osc}(f, M)(x) \to 0$ as $x \to \infty$ for all $M > 0$. Given a cover $\mathcal{U}$ of Y of positive Lebesque number and given $x_n \to \infty$ in X there is $N > 0$ such that each $f\big(B(x_n, M)\big)$ is of diameter smaller that $L(\mathcal{U}, Y)$ for $n > N$. Therefore $B(x_n, M)$ is contained in an element of $f^{-1}(\mathcal{U})$ and $f^{-1}(\mathcal{U})$ is coarse.

Suppose $f^{-1}(\mathcal{U})$ is coarse for every cover $\mathcal{U}$ of Y of positive Lebesque number. Given $x_n \to \infty$ in X and given $M > 0$ such that diameters of $f\big(B(x_n, M)\big)$ are bigger than a fixed $\delta > 0$, consider $\mathcal{U} = \{B(y, \delta/2)\}_{y \in Y}$. Since $f^{-1}(\mathcal{U})$ is coarse, there is $N > 0$ such that for all $n > N$ sets $B(x_n, M)$ are contained in an element of $f^{-1}(\mathcal{U})$. Therefore diameters of $f\big(B(x_n, M)\big)$ are smaller than a δ for $n > N$, a contradiction. ■

Our basic way of constructing slowly oscillating real-valued functions is based on the following.

Lemma 8 *Suppose $f, g\colon X \to \mathbb{R}_+$ and $\mathrm{Osc}(f, M),\ \mathrm{Osc}(g, M) < \epsilon$ for some $\epsilon > 0$. If $f(x) + g(x) > N$ for all $x \in X$, then $\mathrm{Osc}(f/(f + g), M) < 3\epsilon/N$.*

PROOF. Let $h = f/(f + g)$ and $a = 3\epsilon/N$. If $h(x) - h(y) \geq a$ for some $x,\ y \in X$ satisfying $d_X(x, y) < M$, then

$$\frac{f(x)}{f(x)+g(x)} - \frac{f(x)-\epsilon}{f(x)+g(x)+2\cdot\epsilon} \geq a$$

as well. Since

$$\frac{f(x)}{f(x)+g(x)} - \frac{f(x)-\epsilon}{f(x)+g(x)+2\cdot\epsilon} = \frac{f(x)\cdot 2\epsilon + \epsilon\cdot(f(x)+g(x))}{(f(x)+g(x))\cdot(f(x)+g(x)+2\epsilon)} \leq \frac{3\epsilon}{f(x)+g(x)+2\epsilon} < a,$$

we arrive at a contradiction. ■

Corollary 4 *If f and g are coarse functions from X to $\mathbb{R}_+$ such that $f + g$ is coarsely proper and positive, then $f/(f + g)$ is slowly oscillating.*

Here is a simple connection between oscillation and the Lebesque number.

Lemma 9 *If $\phi = \{\phi_s\colon X \to \mathbb{R}_+\}_{s\in S}$ is a family of functions with finite supremum $\sup(\phi)$ such that $\mathrm{Osc}(\phi_s, M) < \frac{1}{2}\sup(\phi)$ for each $s \in S$, then $L(\phi) \geq M$.*

PROOF. Given $a \in X$ find $s \in S$ so that $\phi_s(a) > \frac{1}{2}\sup(\phi)(a)$. If $d_X(x, a) < M$, then $|\phi_s(x)-\phi_s(a)| < \frac{1}{2}\sup(\phi)(a)$, so $\phi_s(x)$ cannot be 0 thus affirming $B(a, M) \subset \phi_s^{-1}(0, \infty)$. ■

A partition of unity $\phi = \{\phi_s\colon X \to \mathbb{R}_+\}_{s\in S}$ is called *slowly oscillating* if the corresponding function $\phi\colon X \to l^1_S$ is slowly oscillating.

ϕ is called *equi-slowly oscillating* if the oscillation of all ϕ_s is synchronized in the following way: for every $M > 0$ and every $\epsilon > 0$ there is a bounded subset U of X such that $\mathrm{Osc}(\phi_s, M)(x) < \epsilon$ for all $x \in X \setminus U$ and all $s \in S$. Obviously, every finite partition of unity into slowly oscillating functions is globally slowly oscillating and is equi-slowly oscillating. Also, every slowly oscillating partition of unity is equi-slowly oscillating.

Lemma 10 *If $\phi = \{\phi_s\colon X \to \mathbb{R}_+\}_{s\in S}$ is a partition of unity of finite multiplicity m, then ϕ is slowly oscillating if and only if it is equi-slowly oscillating.*

PROOF. Given $M, \epsilon > 0$ we can find a bounded set U such that $\mathrm{Osc}(\phi_s, M) < \epsilon/(2m)$ for all $x \in X \setminus U$ and all $s \in S$. If $a \in X\setminus U$ and $x \in B(a, M)$, then the complement F of set $T = \{\, s \in S \mid \phi_s(x)+\phi_s(a) = 0 \,\}$ contains at most $2m$ elements. Since $|\phi(x) - \phi)(a)| = \sum_{s\in F} |\phi_s(x) - \phi_s(a)| < |F| \cdot \epsilon/(2m) \leq \epsilon,\ \phi$ is slowly oscillating. ■

Lemma 11 *If $\phi = \{\phi_s\colon X \to \mathbb{R}_+\}_{s\in S}$ is an equi-slowly oscillating partition of unity of finite multiplicity m, then its carrier family $\mathrm{Carr}(\phi)$ is coarse.*

PROOF. Notice $\sup(\phi) \geq 1/m$. Given $M > 0$ we can find a bounded set U such that $\mathrm{Osc}(\phi_s, M) < 1/(2m)$ for all $x \in X \setminus U$ and all $s \in S$. By Lemma 9, $L(\phi|_{(X\setminus U)}, X \setminus U) > M$ which proves $\mathrm{Carr}(\phi)$ is coarse. ■

Remark 3 *If one drops the assumption of ϕ being of finite multiplicity, then the carrier family may not be coarse: Take a cloud C_n of $2^n + 1$ points at location 2^n with mutual distances equal 1. For each $x \in X$ define ϕ_x as taking value 0 at x and all points not in its cloud. For points $y \in \mathrm{Cloud}(x) \setminus \{x\}$ we put $\phi_x(y) = 2^{-n}$.*

Corollary 5 *If $\mathcal{U} = \{U_s\}_{s\in S}$ is a cover of X of finite multiplicity, then the following conditions are equivalent:*

1. *$\mathcal{U}$ is coarse.*

2. *There is a continuous slowly oscillating partition of unity $\phi = \{\phi_s\}_{s\in S}$ on $X \setminus A$ for some bounded subset A of X such that $\mathrm{Carr}(\phi_s) \subset U_s$ for each $s \in S$.*

3. *There is a slowly oscillating partition of unity $\phi = \{\phi_s\}_{s\in S}$ on $X \setminus A$ for some bounded subset A of X such that $\mathrm{Carr}(\phi_s) \subset U_s$ for each $s \in S$.*

PROOF. 1 $\Longrightarrow$ 2. Define $f(x) = \sum_{s\in S} \mathrm{dist}(x, X \setminus U_s)$ and $f_s(x) = \mathrm{dist}(x, X \setminus U_s)$. Notice that f is a coarsely proper Lipschitz function and Corollary 4 says that $\{f_s/f\}_{s\in S}$ is an equi-slowly oscillating partition of unity on $X \setminus A$, where A is the zero-set of f. By Lemma 10 it is a slowly oscillating partition of unity.

2 $\Longrightarrow$ 3 is obvious.

3 $\Longrightarrow$ 1 follows from Lemma 11. ■

8 Coarse dimension and Higson corona

Given a metric space X by the *Higson compactification* of X we mean a compact Hausdorff space $h(X)$ containing X as a dense subset with the property that a bounded continuous function $f\colon X \to \mathbb{R}_+$ extends over $h(X)$ if and only if f is slowly oscillating. If the metric on X is proper and X is locally compact, then X is open in $h(X)$ and the remainder $h(X) \setminus X$ is called the *Higson corona* of X and denoted by $\nu(X)$.

A metric space X is called δ*-disjoint* for some $\delta > 0$ if $d_X(x, y) \geq \delta$ for all $x \neq y$.

Theorem 3 *If X is a δ-disjoint metric space for some $\delta > 0$, then its coarse dimension equals the dimension of the Higson compactification of X.*

PROOF. Suppose $\dim^{\mathrm{coa}}_{\mathrm{rse}}(X) = m < \infty$. Given a finite open cover $\mathcal{U} = \{U_s\}_{s\in S}$ of the Higson compactification $h(X)$ of X we find a partition of unity $f = \{f_s\}_{s\in S}$ on $h(X)$ such that $\mathrm{cl}(f_s^{-1}(0,1]) \subset U_s$ for each $s \in S$ (see [7]). As $f|_X$ is slowly oscillating (see Lemma 10), the family $\{f_s^{-1}(0,1] \cap X\}_{s\in S}$ is coarse in X (see Lemma 11). By Corollary 5 there is a slowly oscillating partition of unity $g = \{g_s\}_{s\in S}$ on X whose multiplicity is at most $m+1$ and $g_s^{-1}(0,1] \subset f_s^{-1}(0,1] \cap X$ for each $s \in S$. Extend each g_s over $h(X)$ to $k_s\colon h(X) \to [0,1]$. The resulting family $k = \{k_s\}_{s\in S}$ is a partition of unity on $h(X)$. It remains to show $m(k) \leq m+1$ and $k_s^{-1}(0,1] \subset U_s$ for each $s \in S$. If there is a point $x \in h(X) \setminus X$ such that $k_s(x) > 0$ for all $s \in T$, T containing at least $m+2$ elements, then the same would be true for some neighborhood U_x of x in $h(X)$. Since $U_x \cap X \neq \emptyset$ one arrives at a contradiction with the fact that $m(g) \leq m+1$. If $k_s^{-1}(0,1]$ is not a subset of U_s for some $s \in S$, then there is $x \in h(X) \setminus X$ so that $x \in k_s^{-1}(0,1] \setminus \mathrm{cl}(f_s^{-1}(0,1])$. That means there is a neighborhood U_x of x in $h(X)$ on which f_s is identically 0. Hence $g_s|_{(U_x \setminus X)} \equiv 0$ implying $k_s(x) = 0$, a contradiction. ■

Corollary 6 *If X is a proper metric space, then the dimension of its Higson corona equals the coarse dimension of X.*

PROOF. Consider a maximal 1-disjoint subset A of X. Notice $\dim^{\mathrm{coa}}_{\mathrm{rse}}(A) = \dim^{\mathrm{coa}}_{\mathrm{rse}}(X)$ and Higson coronas $\nu(A)$ and $\nu(X)$ for both A and X are identical. Since A is 1-disjoint, $\dim^{\mathrm{coa}}_{\mathrm{rse}}(A) = \dim(h(A)) = \dim(\nu(A)) = \dim(\nu(X))$. ■

Corollary 7 *If $X = A \cup B$, then the coarse dimension of X equals maximum of the coarse dimensions of A and B.*

PROOF. Let $m = \max\left(\dim^{\mathrm{coa}}_{\mathrm{rse}}(A), \dim^{\mathrm{coa}}_{\mathrm{rse}}(B)\right)$. By Corollary 1, $\dim^{\mathrm{coa}}_{\mathrm{rse}}(X) \geq m$. By switching to maximal 1-disjoint subsets of A and B, respectively, we reduce the general case to that of X being

1-disjoint. Consider the Higson compactification $h(X)$ of X. Notice $\mathrm{cl}(A)$ is the Higson compactification of A as any slowly oscillating and bounded function $f\colon A \to \mathbb{R}_+$ extends over X to a bounded and slowly oscillating function. The same is true for B. Since $h(X) = \mathrm{cl}(A) \cup \mathrm{cl}(B)$, $\dim(h(X)) = \max\big(\dim(\mathrm{cl}(A)), \dim(\mathrm{cl}(B))\big) = \max\big(\dim{}^{\mathrm{coa}}_{\mathrm{rse}}(A), \dim{}^{\mathrm{coa}}_{\mathrm{rse}}(B)\big) = m$. ■

We plan to extend Corollary 7 to other dimensions as well. Our strategy is to show finiteness of the appropriate dimension of X first, then use Corollary 7 as well as the fact that all other dimensions are equal to the coarse dimension of X once they are finite (see Corollary 3 and Theorem 2).

Corollary 8 *If $X = A \cup B$, then the asymptotic dimension of X equals maximum of the asymptotic dimensions of A and B.*

PROOF. Let $m = \max\big(\mathrm{asdim}(A), \mathrm{asdim}(B)\big)$. Obviously $\mathrm{asdim}(X) \geq m$. Given $M > 0$ find uniformly bounded family $\mathcal{U}_A$ in A covering A and being the union of $m+1$ families, each of them $3M$-disjoint. Similarly, find uniformly bounded and $3M$-disjoint family $\mathcal{U}_B$ in B covering B and being the union of $m+1$ families, each of them $3M$-disjoint. Consider $\mathcal{U} = \mathcal{U}_A \cup \mathcal{U}_B$ and let $\mathcal{V} = \{B(U, M)\}_{U \in \mathcal{U}}$. Notice $\mathcal{V}$ is uniformly bounded in X, is of multiplicity at most $2(m+1)$, and $L(\mathcal{V}, X) \geq M$. Therefore $\mathrm{asdim}(X) \leq 2m+1$ and (see Theorem 2) $\mathrm{asdim}(X) = \dim{}^{\mathrm{coa}}_{\mathrm{rse}}(X) = m$. ■

Remark 4 *Corollary* 8 *was proved in* [1] *(see the Finite Union Theorem there) for X being a proper metric space by using totally different methods.*

Corollary 9 *If $X = A \cup B$, then the major coarse dimension of X equals maximum of the major coarse dimensions of A and B.*

PROOF. Let $m = \max\big(\dim{}^{\mathrm{coa}}_{\mathrm{rse}}(A), \dim{}^{\mathrm{coa}}_{\mathrm{rse}}(B)\big)$. By Corollary 1, $\dim{}^{\mathrm{COA}}_{\mathrm{RSE}}(X) \geq m$. Given a coarse family $\mathcal{U}$ in X put $f(x) = L_{\mathcal{U}}(x)$. If $f(x) = \infty$ for some X, then $\mathcal{U}$ has a coarse refinement of order at most 2 (see Lemma 2). Assume $f(x) < \infty$ for all $x \in X$. Pick a coarse refinement $\{V_a\}_{a \in A}$ of multiplicity at most $m+1$ of the family $\{B(a, f(a)/2)\}_{a \in A}$. Pick a coarse refinement $\{V_b\}_{b \in B}$ of multiplicity at most $m+1$ of the family $\{B(b, f(b)/2)\}_{b \in B}$. If $V_a \neq \emptyset$ define $e(V_a) = \{x \in B\big(a, f(a)\big) \mid \mathrm{dist}(x, V_a) < \mathrm{dist}(x, A \setminus V_a\}$. Observe $\bigcap_{a \in T} e(V_a) \neq \emptyset$ implies $\bigcap_{a \in T} V_a \neq \emptyset$ for every finite subset T of S. Indeed, suppose $x \in \bigcap_{a \in T} e(V_a)$ and find $\delta > 0$ such that $\mathrm{dist}(x, V_a) + \delta < \mathrm{dist}(x, A \setminus V_a\}$ for all $a \in T$. Pick $y \in A$ so that $\mathrm{dist}(x, A) + \delta > d(x, y)$. If $y \in A \setminus V_a$ for some $a \in T$, then $\mathrm{dist}(x, A) + \delta \leq \mathrm{dist}(x, V_a) + \delta < \mathrm{dist}(x, A \setminus V_a) \leq d(x, y)$, a contradiction. Therefore the multiplicity of $\{e(V_a)\}_{a \in A}$ is at most $m+1$. Do the same procedure for B and produce $\{e(V_b)\}_{b \in B}$. If $x \in a,\ M < f(a)$ and $B(x, M) \cap A \subset V_a$, then $B(x, M/2) \subset e(V_a)$. Therefore $\{e(V_a)\}_{a \in A} \cup \{e(V_b)\}_{b \in B}$ is coarse in X, refines $\mathcal{U}$, and is of multiplicity at most $2(m+1)$. Thus $\dim{}^{\mathrm{COA}}_{\mathrm{RSE}}(X) \leq 2m+1$ and (see Corollary 3) $\dim{}^{\mathrm{COA}}_{\mathrm{RSE}}(X) = \dim{}^{\mathrm{coa}}_{\mathrm{rse}}(X) = m$. ■

Corollary 10 *If $X = A \cup B$, then the minor asymptotic dimension of X equals maximum of the minor asymptotic dimensions of A and B.*

PROOF. Let $m = \max\big(\mathrm{ad}(A), \mathrm{ad}(B)\big)$. Obviously $\mathrm{ad}(X) \geq m$. Suppose $M > 0$ and find $N > 0$ such that $L^m(\mathcal{U}, A) > 2M$ for all finite covers $\mathcal{U}$ of A satisfying $L(\mathcal{U}, A) > N$. We can use the same N and claim $L^m(\mathcal{U}, B) > 2M$ for all finite covers $\mathcal{U}$ of B satisfying $L(\mathcal{U}, B) > N$. Given a finite family $\mathcal{U} = \{U_s\}_{s \in S}$ in X satisfying $L(\mathcal{U}, X) > M + N$, consider $\{B(U_s, -M)\}_{s \in S}$ and shrink it on A to a family $\{V_s\}_{s \in S}$ of multiplicity at most $m+1$ and Lebesque number at least $2M$. Do the same for B and shrink $\{B(U_s, -M)\}_{s \in S}$ on B to a family $\{W_s\}_{s \in S}$ of multiplicity at most $m+1$ and Lebesque number at least $2M$. If $V_s \neq \emptyset$ define $e(V_s) = \{\, x \in U_s \mid \mathrm{dist}(x, V_s) < \mathrm{dist}(x, A \setminus V_s\,\}$. Observe $\bigcap_{s \in T} e(V_s) \neq \emptyset$ implies $\bigcap_{s \in T} V_s \neq \emptyset$ for every finite subset T of S (see the proof of Corollary 9). Therefore the multiplicity of $\{e(V_s)\}_{s \in S}$ is at most $m+1$. Do the same procedure for B and produce $\{e(W_s)\}_{s \in S}$. Obviously $\{e(V_s)\}_{s \in S} \cup \{e(V_s)\}_{s \in S}$ refines $\mathcal{U}$ and is of multiplicity at most $2(m+1)$. If we show its Lebesque number is at least M we will demonstrate $\mathrm{ad}(X) \leq 2m+1$ and (see Theorem 2)

$\mathrm{ad}(X) = \dim^{\mathrm{coa}}_{\mathrm{rse}}(X) = m$. Suppose $x \in X$. Without loss of generality we may assume $x \in B$. There is $s \in S$ such that $B(x, 2M) \cap B \subset W_s$. Hence $B(x, M) \subset B(W_s, M) \subset U_s$ and, since any $y \in B(x, M)$ satisfies $\mathrm{dist}(y, W_s) \leq d(y, x) < M < \mathrm{dist}(y, B \setminus W_s)$, we get $y \in e(W_s)$ which completes the proof. ■

9 Coarse dimension and absolute extensors

In [2, Remark 2 on p. 1097] Dranishnikov pointed out that $\mathbb{R}_+$ is not an absolute extensor in the category of proper metric spaces and coarse functions. He characterized proper metric spaces of coarse dimension at most n as those for which $\mathbb{R}^{n+1}$ is an absolute extensor in the category of proper approximately Lipschitz functions (Definition 4 on p. 1105 and Theorem 6.6 on p. 1111). That still left the door open to the possibility of characterizing coarse dimension via $\mathbb{R}^{n+1}$ being an absolute extensor in the proper coarse category. The following result clarifies that issue in negative.

Theorem 4 *For a metric space X the following conditions are equivalent:*

1. *The coarse dimension of X is at most 0.*

2. *Y is an absolute extensor of X in the proper coarse category for all Y.*

3. *$\mathbb{R}_+$ is an absolute extensor of X in the proper coarse category.*

PROOF. $1 \Longrightarrow 2$. It suffices to show that any unbounded subset A of X is a coarsely proper and coarse retract of X. Pick $x_0 \in X$. Define by induction on n an increasing sequence M_n of natural numbers and covers $\mathcal{U}^n$ of X satisfying the following properties:

a. $M_1 = 1$.

b. $\mathcal{U}^n$ is M_n-disjoint, the diameters of its elements are smaller than M_{n+1}, and $L(\mathcal{U}^n, X) > M_n$.

For each $U \in \mathcal{U}^n$ so that $U \cap A \neq \emptyset$, pick $x_U \in U \cap A$ satisfying $d_X(x_U, x_0) > \sup\{\, d_X(x, x_0) \mid x \in U \cap A \,\} - 1/n$.

By induction on n define a sequence of subsets A_n of X and a sequence of functions $r_n \colon A_n \to A$ as follows:

i. $A_1 = A$ and $r_1 = \mathrm{id}_A$.

ii. A_{n+1} is the union of those elements of $\mathcal{U}^{n+1}$ that intersect A.

iii. If $x \in U \setminus A_n$ and $U \cap A \neq \emptyset$ for some $U \in \mathcal{U}^{n+1}$, then $r_{n+1}(x) = x_U$.

Notice $X = \bigcup_{n=1}^{\infty} A_n$ and let $r \colon X \to A$ be obtained by pasting all r_n. Observe that $x \in U \in \mathcal{U}^k$ and $U \cap A \neq \emptyset$ implies $r(x) \in U$. Indeed, for each n there is a unique element $U^n_x \in \mathcal{U}^n$ containing x and $U^i_x \subset U^j_x$ if $i < j$. Find the smallest number m so that $x \in A_m$. In that case $r(x) \in U^m_x$ by definition and k must be at least m so $U^m_x \subset U^k_x = U$.

We will show that r is coarse by proving $d_X(x, y) < M_n$ implies $d_X\big(r(x), r(y)\big) \leq M_{n+2}$. Indeed, if $d_X(x, y) < M_n$, then one of the following cases occurs:

Case 1. $U \cap A_n = \emptyset$, where U is the unique element of $\mathcal{U}^{n+1}$ containing both x_n and y_n.

Case 2. $U \cap A_n \neq \emptyset$, where U is the unique element of $\mathcal{U}^{n+1}$ containing both x_n and y_n.

In Case 1 the values $r(x)$ and $r(y)$ are identical. In Case 2 both $r(x)$ and $r(y)$ belong to $U \cap A$ and the set $U \cap A$ is of diameter at most M_{n+2}, so $d_X\big(r(x), r(y)\big) \leq M_{n+2}$ holds.

If r is not coarsely proper, then there is a sequence $x_n \to \infty$ such that $r(x_n)$ is bounded. Obviously, $x_n \notin A$ for almost all n. Consider an element $U \in \mathcal{U}^k$ containing all of $r(x_n)$. The way functions r_m were defined implies that there is a sequence of elements $U_n \in \mathcal{U}^{\alpha(n)}$ with $\alpha(n) \to \infty$ and all U_n containing U, such that $U_n \cap A$ is of almost the same diameter as $U \cap A$. That contradicts A being unbounded.

$2 \Longrightarrow 3$ is obvious.

$3 \Longrightarrow 1$. Suppose $\dim^{\text{coa}}_{\text{rse}}(X) > 0$. By Proposition 16 there exists a number $M > 0$ and a coarsely proper sequence $\{(x_n, y_n)\}_{n=1}^{\infty}$ of pairs of points in X such that $\operatorname{dist}(x_n, y_n) \to \infty$ and the points x_n and y_n can be M-scale connected in $X \setminus B(x_0, n)$ by long chain of length L_n so that $L_n \to \infty$ as $n \to \infty$. We may assume $d_X(x_{n+j}, x_n) > n$ and $d_X(y_{n+j}, y_n) > n$ for all n, $j \geq 1$. Let $B = \{x_n\} \cup \{y_n\}$. Define $f\colon B \to \mathbb{R}_+$ by sending x_n to n and y_n to $n + n \cdot L_n$. Notice f is coarsely proper and coarse. Suppose f extends to a coarse function $g\colon X \to \mathbb{R}_+$. Find $K > 0$ such that $d_X(x, y) \leq M$ implies $d\big(f(x), f(y)\big) \leq K$. Since x_n and y_n can be connected by a chain of L_n points, with consecutive points being separated by at most M, $L_n \cdot n + n - n = d\big(f(x_n), f(y_n)\big) \leq L_n \cdot K$ which leads to a contradiction for $n > K$. ■

10 Open problems

In [2, Problem 1 on p. 1126] it is asked if the asymptotic dimension of a proper metric space X equals the covering dimension of its Higson corona. Here is our version of that problem.

Problem 1 *Is there a metric space X of infinite asymptotic dimension and finite coarse dimension?*

Problem 2 *Is there a metric space X of infinite major coarse dimension and finite coarse dimension?*

Definition 20 *A metric space X is of* bounded geometry *if for every $M > 0$ there is a uniformly bounded cover $\mathcal{U}$ of X of finite multiplicity and the Lebesque number at least M.*

Definition 21 ([2, p. 1005]) *Suppose X is a metric space of bounded geometry. Given $M > 0$ let $d(M) = m(\mathcal{U}) - 1$, where $\mathcal{U}$ is a uniformly bounded cover $\mathcal{U}$ of minimal multiplicity among those of the Lebesque number at least M. X is of* slow dimension growth *if* $\lim_{M\to\infty} d(M)/M = 0$.

Just as in [2, Problem 6 on p. 1126] one can ask variants of problems 1 and 2 for spaces of bounded geometry or slow dimension growth.

Problem 3 *Suppose X is of slow dimension growth and finite coarse dimension. Is asymptotic dimension of X finite?*

Problem 4 *Suppose X is of slow dimension growth and finite coarse dimension. Is the major coarse dimension of X finite?*

The above problems remain open for minor asymptotic dimension. All of the above problems are of interest in case of X being a finitely generated group with word metric, especially $\mathrm{CAT}(0)$ groups.

Problem 5 *It is stated in* [4] *that* $\operatorname{asdim}(X \times Y) \leq \operatorname{asdim}(X) + \operatorname{asdim}(Y)$. *Are the corresponding results true for other dimensions?*

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References

[1] Bell, G. and Dranishnikov, A., (2001). On asymptotic dimension of groups, *Algebr. Geom. Topol.*, **1**, 57–71.

[2] Dranishnikov, A., (2000). Asymptotic topology, *Russian Math. Surveys*, **55**, 1085–1129.

[3] Dranishnikov, A., (2001). On asymptotic inductive dimension, *JP J. Geom. Topol.*, **1**, 239–247.

[4] Dranishnikov, A. and Januszkiewicz, T., (1999). Every Coxeter group acts amenably on a compact space, in *Proceedings of the 1999 Topology and Dynamics Conference*, Salt Lake City, UT. Topology Proc., **24**, 135–141.

[5] Dranishnikov, A., Keesling, J. and Uspenskij, V., (1998). On the Higson corona of uniformly contractible spaces, *Topology*, **37**, 791–803.

[6] Dranishnikov, A. and Zarichnyi, M., (2004). Universal spaces for asymptotic dimension, *Topology and its Appl.*, **140**, 203–225.

[7] Dydak, J., (2003). Partitions of unity, *Topology Proc.*, **27**, 125–171.

[8] Engelking, R., (1989). *General topology*, Heldermann Verlag, Berlin.

[9] Engelking, R., (1995). *Theory of Dimensions: Finite and Infinite*, Heldermann Verlag, Lemgo.

[10] Ferry, S. and Weinberger, S., (1995). A coarse approach to the Novikov Conjecture, in *Novikov Conjectures, Index Theorems, and Rigidity: Oberwolfach 1993*, S. C. Ferry, A. Ranicki and J. Rosenberg, ed., LMS Lecture Notes, **226**, 147–163.

[11] Gromov, M., (1993). Asymptotic invariants for infinite groups, in *Geometric Group Theory*, vol. **2**, 1–295, G. Niblo and M. Roller, eds., Cambridge University Press.

[12] Keesling, J., (1994). The one-dimensional Čech cohomology of the Higson compactification and its corona, *Topology Proc.*, **19**, 129–148.

[13] Roe, J., (1993). Coarse cohomology and index theory on complete Riemannian manifolds, *Memoirs Amer. Math. Soc.*, **104**, 497.

[14] Roe, J., (1996). *Index theory, coarse geometry, and topology of manifolds*, CBMS Regional Conf Series in Mathematics, American Mathematical Society, Providence, RI. **90**.

[15] Roe, J., (2003). *Lectures on coarse geometry*, University Lecture Series, **31**. American Mathematical Society, Providence, RI.

[16] Yu, G., (1998). The Novikov conjecture for groups with finite asymptotic dimension, *Ann. of Math.*, 2nd series, **147**, 2, 325–355.

[17] Yu, G., (2000). The coarse Baum-Connes conjecture for spaces which admit a uniform embedding into Hilbert space, *Invent. Math.*, **139**, 201–240.

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