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Basis of homology adapted to the trigonal automorphism of a Riemann surface

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Abstract A closed (compact without boundary) Riemann surface S of genus g is said to be trigonal if there is a three sheeted covering (a trigonal morphism) from S to the Riemann sphere, $f : S \longrightarrow \hat{\mathbb{C}}$. If there is an automorphism of period three, ϕ, on S permuting the sheets of the covering, we shall call S cyclic trigonal and ϕ will be called trigonal automorphism.

In this paper we determine the intersection matrix on the first homology group of a cyclic trigonal Riemann surface on an adapted basis $\mathcal{B}$ to the trigonal automorphism, that is, the matrix of the trigonal automorphism is as simple as possible. We use the basis $\mathcal{B}$ to the topological classification of actions of automorphism groups on Riemann surfaces.

Base adaptada al automorfismo trigonal de una superficie de Riemann

Resumen. Una superficie de Riemann S con género g se dice que es trigonal si existe una cubierta de tres hojas (un morfismo trigonal) de S sobre la esfera de Riemann, $f : S \longrightarrow \hat{\mathbb{C}}$. Si, además, existe un automorfismo de periodo tres, ϕ, de S que permuta las hojas de la cubierta, entonces diremos que S es trigonal cíclica y ϕ será llamado el automorfismo trigonal.

En este artículo determinamos la matriz de intersección sobre el primer grupo de homología de una superficie de Riemann trigonal cíclica con respecto a una base adaptada al automorfismo trigonal, es decir, la matriz del automorfismo trigonal es lo más sencilla posible. También usamos la base anterior para la clasificación topológica de acciones de grupos de automorfismos sobre superficies de Riemann.

1 Introduction

Cyclic trigonal Riemann surfaces appear in the study of complex algebraic curves. More precisely a curve with equation $y^3 = f(x)$, $f(x) \in \mathbb{C}[X]$, give rise to a Riemann surface of such type. The trigonal Riemann surfaces constitute a subject that have been recently studied (see [1], [2] and [4]).

Let S be a cyclic trigonal Riemann surface of genus g and $f : S \longrightarrow \hat{\mathbb{C}}$ be the trigonal morphism. Let $p_1, \ldots, p_{g+2}$ be the set of branched points of the covering $f : S \longrightarrow \hat{\mathbb{C}}$. Considering some special loops in $\hat{\mathbb{C}} - \{p_1, \ldots, p_{g+2}\}$ we define, in section 3, a basis of $H_1(S, \mathbb{Z})$ adapted to the trigonal morphism of a Riemann surface and we present the matrix of the intersection form on such a basis.

In [5], there is a direct way to deal with the topological classification of $\mathbb{Z}_p^m$ actions on surfaces, where p is prime number, based on the fact that the classes of strong equivalence of a fixed point free action of $\mathbb{Z}_p^m$ on a surface provide a bilinear antisymmetric form in $\mathbb{Z}_p^m$. In section 4, using this method, we present one application to the topological classification of group actions on cyclic trigonal Riemann surfaces.

The detailed proofs and some more applications will appear in [3].

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2 Cyclic Trigonal Riemann surfaces

Let S be a closed cyclic trigonal Riemann surface of genus $g \geq 2$. The surface S can be represented as a quotient $S = \mathcal{D}/\Gamma$ of the complex unit disc $\mathcal{D}$ under the action of a Fuchsian surface group Γ, that is, a discrete subgroup of the group $\mathrm{Aut}(\mathcal{D})$ of conformal automorphisms of $\mathcal{D}$ without elliptic transformations.

The group Γ is a Fuchsian surface group with signature $(g; [-])$, and canonical presentation

$$< a_1, b_1, \ldots, a_g, b_g : \prod_{i=1}^{g} a_i b_i a_i^{-1} b_i^{-1} = 1 > .$$

A finite group G is an automorphisms group of S if and only if, it exists a Fuchsian group, Δ, and an epimorphism $\theta : \Delta \longrightarrow G$ whose kernel is Γ, see [6]. This Fuchsian group Δ is the lifting of G to the universal covering of S.

If $S = \mathcal{D}/\Gamma$ is a cyclic trigonal Riemann surface then the theory of covering spaces yields, see [4], that there is a Fuchsian group, Δ, with signature $(0, [3 \overset{g+2}{\ldots} 3])$ where Γ is contained as a subgroup of index three, and so there is an epimorphism $w : \Delta \longrightarrow \mathbb{Z}_3$ such that $\ker w = \Gamma$.

We note that Δ has a canonical presentation with generators:

$$x_i, \qquad i = 1, \ldots, g+2 \tag{1}$$

and relations

$$x_1^3 = \cdots = x_{(g+2)}^3 = 1 \tag{2}$$

$$x_1 x_2 \cdots x_{(g+2)} = 1 \tag{3}$$

3 Computing the intersection matrix of cyclic trigonal Riemann surfaces on an adapted basis

Suppose that S is a trigonal Riemann surface of genus g uniformized by Γ, $f : S \longrightarrow \hat{\mathbb{C}}$ is the trigonal morphism and ϕ is the trigonal automorphism.

Assume $\mathbb{Z}_3 =< \sigma : \sigma^3 = 1 >$ and let Δ be an NEC group with signature $(0, [3 \overset{g+2}{\ldots} 3])$ and let $w : \Delta \longrightarrow \mathbb{Z}_3$ be as constructed in the previous section. Considering a canonical presentation of Δ we assume that $w(x_1) = \sigma$.

We are going to construct a basis of $H_1(S)$ from the canonical presentation for Δ.

The trigonal morphism $f : S \longrightarrow \hat{\mathbb{C}}$ has $g+2$ points of ramification, $p_1, \ldots, p_{g+2}$. Each generator x_i, $i = 1, \ldots, g+2$, has a fixed point in $\mathbb{D}$ projecting on a branched point of f, p_i. Among them there are η points with $w(x_i) = \sigma$ and μ points with $w(x_i) = \sigma^{-1}$. Using automorphisms of $\mathbb{Z}$ and Δ we can assume that w is as follows:

$$w(x_i) = \sigma; \qquad i = 1, \ldots, \eta - \mu$$
$$w(x_i) = \sigma, \quad w(x_{i+1}) = \sigma^{-1}; \qquad i - (\eta - \mu) \equiv 1 \mod 2 \text{ and } (\eta - \mu) \leq i \leq g+2$$

Since w is an homomorphism $\eta - \mu \equiv 0 \mod 3$ and to simplify the notations let $t = \frac{\eta - \mu}{3}$. Let us consider the curves α_i, $i = 1, \ldots, t$; δ_j, $j = 1, \ldots, t-1$ and γ_k, $k = \mu + 1, \ldots, t + \mu$ as shown in the Figure 1.

For the remainder 2μ points we will consider curves δ_j, $j = t, \ldots, t + \mu - 1$, and γ_k, $k = 1, \ldots, \mu$, as we can see on the Figure 2.

Globally we have the union of the two schemes (Figure 1 and Figure 2) so we obtain t curves α_i, $t + \mu - 1$ curves δ_j, and $t + \mu$ curves γ_k, as show in the Figure 3.

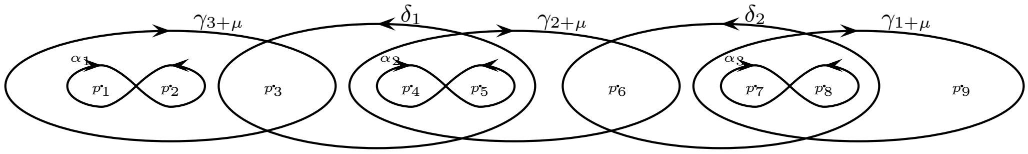

Figure 1. $\eta - \mu = 9$

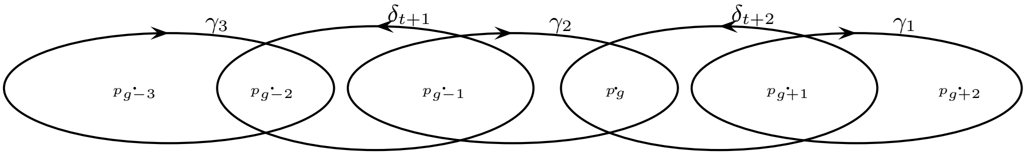

Figure 2. $\mu = 3$

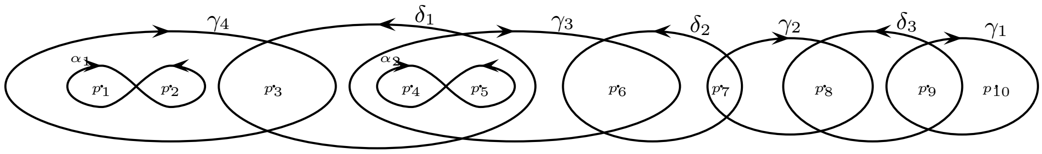

Figure 3. $g = 8; \eta = 8; \mu = 2$

Let O be a point in $\hat{\mathbb{C}} - \{p_1, \ldots, p_{g+2}\}$, and $f^{(-1)}(O) = \{O_1, O_2, O_3\}$. Each one of the curves, α_i, δ_j, γ_k represents a conjugacy class of elements in the fundamental group $\pi_1(\hat{\mathbb{C}} - \{p_1, \ldots, p_{g+2}\}, O)$. Hence lifting by the universal covering of S, in Δ, we have

$$
\begin{aligned}
\alpha_i &= [x_{3i-2}^{-1} x_{3i-1}] && \text{if } i = 1, \ldots, t \\
\delta_j &= \begin{cases} [x_{3j} x_{3j+1} x_{3j+2}] & \text{if } j = 1,\ldots,t-1 \\ [x_{2j+3} x_{2j+4}] & \text{if } j = t,\ldots, t+\mu-1 \end{cases} \\
\gamma_k &= \begin{cases} [x_{g+4-2k}^{-1} x_{g+3-2k}^{-1}] & \text{if } k = 1, \ldots, \mu \\ [x_{g+\mu+5-3k}^{-1} x_{g+\mu+4-3k}^{-1} x_{g+\mu+3-3k}^{-1}] & \text{if } k = \mu+1, \ldots, t+\mu \end{cases}
\end{aligned} \tag{4}
$$

where $[a]$ denote the conjugacy class of a.

In each conjugacy class there are three elements of Δ projecting to three closed curves of S representing three different cycles of $H_1(S)$. The elements of Δ that we are considering are in fact in Γ, we shall denote then by: x_{ij}, $i = 1, \ldots, g+1$; $j = 1, 2, 3$. This elements will be a set of generators of the subgroup $\Gamma < \Delta$ uniformizing S. From the curves α_i we get:

$$
\begin{aligned}
x_{i1} &= x_{3i-2}^{-1} x_{3i-1} \\
x_{i2} &= x_1^{-1} x_{3i-2}^{-1} x_{3i-1} x_1 \\
x_{i3} &= x_1 x_{3i-2}^{-1} x_{3i-1} x_1^{-1}
\end{aligned} \tag{5}
$$

where $i = 1, \ldots, t$; from δ_j we have:

$$
\begin{aligned}
x_{j1} &= x_{3(j-t)} x_{3(j-t)+1} x_{3(j-t)+2} \\
x_{j2} &= x_1^{-1} x_{3(j-t)} x_{3(j-t)+1} x_{3(j-t)+2} x_1 \\
x_{j3} &= x_1 x_{3(j-t)} x_{3(j-t)+1} x_{3(j-t)+2} x_1^{-1}
\end{aligned} \tag{6}
$$

if $j = t+1, \ldots, 2t-1$, or, if $j = 2t, \ldots, 2t+\mu-1$

$$\begin{aligned} x_{j1} &= x_{2j-t}x_{2j-t+1} \\ x_{j2} &= x_1^{-1}x_{2j-t}x_{2j-t+1}x_1 \\ x_{j3} &= x_1x_{2j-t}x_{2j-t+1}x_1^{-1} \end{aligned} \tag{7}$$

and the curves γ_k give rise to:

$$\begin{aligned} x_{k1} &= x^{-1}_{2(g+2)+t-2k}x^{-1}_{2(g+2)+t-2k-1} \\ x_{k2} &= x_1^{-1}x^{-1}_{2(g+2)+t-2k}x^{-1}_{2(g+2)+t-2k-1}x_1 \\ x_{k3} &= x_1x^{-1}_{2(g+2)+t-2k}x^{-1}_{2(g+2)+t-2k-1}x_1^{-1} \end{aligned} \tag{8}$$

if $k = 2t+\mu, \ldots, 2t+2\mu-1$ or

$$\begin{aligned} x_{k1} &= x^{-1}_{3(g+2)-3k}x^{-1}_{3(g+2)-3k-1}x^{-1}_{3(g+2)-3k-2} \\ x_{k2} &= x_1^{-1}x^{-1}_{3(g+2)-3k}x^{-1}_{3(g+2)-3k-1}x^{-1}_{3(g+2)-3k-2}x_1 \\ x_{k3} &= x_1x^{-1}_{3(g+2)-3k}x^{-1}_{3(g+2)-3k-1}x^{-1}_{3(g+2)-3k-2}x_1^{-1} \end{aligned} \tag{9}$$

when $k = 2t+2\mu, \ldots, g+1$.

Let $(\cdot\,,\cdot)_S$ be the intersection form of the Riemann surface S.

The projection of x_{ij} to S, $i = 1, \ldots, g+2$; $j = 1, 2, 3$, represents an element X_{ij} in $H_1(S, \mathbb{Z}_3)$.

The cycles X_{ij}, $i = 1, \ldots, g+1$; $j = 1, 2, 3$ generate $H_1(S, \mathbb{Z}_3)$ and we can compute the number of intersection between them.

The curves α_i, $i = 1, \ldots, t$ do not cut themselves and each one of the δ_j curves, $j = 1, \ldots, t+\mu-1$, cuts the curves $\gamma_{\mu+t-j+1}$ and $\gamma_{\mu+t-j}$, as we can see on Figures 1, 2 and 3.

To have a basis of $H_1(S)$ we need to consider a subset of $\{X_{ij}\}$, $i = 1, \ldots, g+1$; $j = 1, 2, 3$: $\mathcal{B} = ((X_{i1}; X_{il}), i = 1, \ldots, g)$, where $l = 2$ if $i = 1, \ldots, t$ or $i = 2t+2\mu, \ldots, g$ and $l = 3$ if $i = t+1, \ldots, 2t+2\mu-1$. The intersection form of S, respect to $\mathcal{B}$, has the following expression:

$$((X_{ik}, X_{jl})_S) = \begin{cases} A & \text{if } j = 1, \ldots, t \text{ and } i = j \\ 0 & \text{otherwise} \end{cases} \tag{10}$$

where $i = 1, \ldots, t$; $k, l = 1, 2, 3$ and A is the following matrix

$$\begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix}$$

In the next two cases we have, when $j = g+t+2-i$ that $(X_{ik}, X_{jl})_S = (X_{ik}, X_{(j-1)l})_S$,

$$((X_{ik}, X_{jl})_S) = \begin{cases} B & \text{if } j = 2t+\mu, \ldots, g+1 \text{ and } j = g+t+2-i \\ 0 & \text{otherwise} \end{cases} \tag{11}$$

where $i = t+1, \ldots, 2t+\mu-1$; $k, l = 1, 2, 3$ and B is the matrix

$$\begin{pmatrix} -1 & 0 & 1 \\ 0 & 1 & -1 \\ 1 & -1 & 0 \end{pmatrix}$$

Finally

$$((X_{(2t+\mu)k}, X_{(2t+\mu-1)l})_S) = ((X_{(g+1)k}, X_{(t+1)l})_S) = -B^t \tag{12}$$

and

$$((X_{ik}, X_{jl})_S) = \begin{cases} -B^t & \text{if } j = t+1, \ldots, 2t+\mu-1 \text{ and } j = g+t+2-i \\ 0 & \text{otherwise} \end{cases} \tag{13}$$

if $i = 2t+\mu+1, \ldots, g$; $k, l = 1, 2, 3$.

In the basis $\mathcal{B}$ the expression of the automorphism $\phi_* : H_1(S,\mathbb{Z}) \longrightarrow H_1(S,\mathbb{Z})$ is given by the fact that $\phi_*(X_{ij}) = X_{i(\sigma(j))}$. The automorphism matrix is a $g \times g$-matrix of blocks that are 2×2-matrices, where the generic block b_{ij}, $i, j = 1, \ldots, g$ is:

$$(b_{ij}) = \begin{cases} \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix} & \text{if } i = j \text{ and } i = 1, \ldots, t \text{ or } i = 2t+\mu, \ldots, g \\ \begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix} & \text{if } i = j \text{ and } i = t+1, \ldots, 2t+\mu-1 \\ 0 & \text{otherwise} \end{cases} \tag{14}$$

We have proved the following theorem:

Theorem 1 *Let S be a trigonal Riemann surface of genus $g \geq 2$. There exists a basis of homology, $\mathcal{B} = ((X_{i1}; X_{il}), i = 1, \ldots, g)$, adapted to the trigonal automorphism of S, in such a way the intersection form on S is given by* (10), (11), (12), (13) *and the action of the trigonal automorphism by* (14).

4 Applications

An action of $\bar{G}$ on a Riemann surface $\tilde{S}$ is a pair $(\tilde{S}, f)$, where f is a representation of $\bar{G}$ in the group automorphisms of $\tilde{S}$. Two actions $(\tilde{S}, f)$ and $(\tilde{S'}, f')$ are called strongly topologically equivalent if there is a homeomorphism, ψ, between $\tilde{S}$ and $\tilde{S'}$ sending the orientation of $\tilde{S}$ to the orientation of $\tilde{S'}$, such that $f'(h) = \psi \circ f(h) \circ \psi^{-1}$, for all $h \in \bar{G}$. Let $\tilde{S}$ be a Riemann surface and let G be a group isomorphic to $\mathbb{Z}_3^m$, acting without fixed points on $\tilde{S}$. Let us suppose that $\tilde{S}/G = S$ is a cyclic trigonal Riemann surface with genus g and let be $\bar{G} = G \rtimes_\phi \mathbb{Z}_3$, where ϕ is an non trivial action of $\mathbb{Z}_3$ in G, given by

$$\begin{array}{rccl} \phi_\sigma : & \mathbb{Z}_3^2 & \longrightarrow & \mathbb{Z}_3^2 \\ & a & \longmapsto & b \\ & b & \longmapsto & -(a+b) \end{array}$$

where a and b are the generators of $\mathbb{Z}_3^2$ and σ is the generator of $\mathbb{Z}_3$.

Let Δ be a Fuchsian group with signature $(0; [x_1, \ldots, x_{g+2}])$. We assume that $g+2 \equiv 0 \mod 3$ and $w(x_i) = \sigma$, $i = 1, \ldots, g+2$. Let us consider the epimorphisms

$$\begin{array}{rlcl} \theta'_1 : & \Delta & \longrightarrow & \bar{G} \\ & x_k & \longmapsto & (0,\sigma) \text{ if } k = 1, 3, \ldots, g \\ & x_l & \longmapsto & (b,\sigma) \text{ if } l = 2, g+1 \\ & x_{g+2} & \longmapsto & (a,\sigma) \end{array} ; \quad \begin{array}{rlcl} \theta'_2 : & \Delta & \longrightarrow & \bar{G} \\ & x_k & \longmapsto & (0,\sigma) \text{ if } k = 1, 4, \ldots, g+1 \\ & x_l & \longmapsto & (b,\sigma) \text{ if } l = 2, g+2 \\ & x_3 & \longmapsto & (-a-b,\sigma) \end{array}$$

Let $\mathcal{D}/\ker\theta_1$ be S_1 and $\mathcal{D}/\ker\theta_2$ be S_2. The projections $S_i = \mathcal{D}/\ker\theta_i \longrightarrow \mathcal{D}/\Delta$, $i = 1, 2$, provide actions f_i of $\tilde{G}$ on S_i, $i = 1, 2$. Using the basis in section 3 we can compute the intersection matrices of $S_i/f_i(G)$, and to conclude that the actions f_i, $i = 1, 2$, are not topological equivalent. Remark that in the above example the trigonal automorphism of $S_1/f_1(G)$ defines a topologically equivalent action to the trigonal automorphism of $S_2/f_2(G)$.

Theorem 2 *Let G be the semidirect product $\mathbb{Z}_3^2 \rtimes_\phi \mathbb{Z}_3$ where $\mathbb{Z}_3^2 = < a > \oplus < b >$ and $\phi(a) = b$; $\phi(b) = -(a+b)$. For each $g \geq 2$ there are two topologically non-equivalent actions (f_1 and f_2) of G*

on surfaces of genus g such that $f_1|_G$ and $f_2|_G$ are actions without fixed points, $S/f_1(G)$, $S/f_2(G)$ are spheres and the action of the trigonal automorphism of $S/f_1(\mathbb{Z}_3^2)$ is topologically equivalent to the action of the trigonal automorphism of $S/f_2(\mathbb{Z}_3^2)$.

Other application: we are able to compute periods matrices of cyclic trigonal Riemann surfaces using the basis of section 3, we present an example for genus $g = 4$ where the epimorphism w sends all the generators of Δ to the generator of $\mathbb{Z}_3$.

The periods matrix depending on 32 parameters is as follows:

$$\begin{pmatrix} a_1 & -b_1 & a_2 & -b_2 & a_{13} & b_{13} & a_5 & -b_5 \\ b_1 & a_1 - b_1 & b_2 & a_2 - b_2 & a_{13} + b_{13} & -a_{13} & b_5 & a_5 - b_5 \\ a_3 & -b_3 & a_4 & -b_4 & a_{14} & b_{14} & a_6 & -b_6 \\ b_3 & a_3 - b_3 & b_4 & a_4 - b_4 & a_{14} + b_{14} & -a_{14} & b_6 & a_6 - b_6 \\ a_{10} & -a_{10} - b_{10} & a_{11} & -a_{11} - b_{11} & a_{16} & b_{16} & a_{12} & -a_{12} - b_{12} \\ b_{10} & -a_{10} & b_{11} & -a_{11} & -b_{16} & a_{16} + b_{16} & b_{12} & -a_{12} \\ a_7 & -b_7 & a_8 & -b_8 & a_{15} & b_{15} & a_9 & -b_9 \\ b_7 & a_7 - b_7 & b_8 & a_8 - b_8 & a_{15} + b_{15} & -a_{15} & b_9 & a_9 - b_9 \end{pmatrix}$$

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