Mathematical analysis of the discharge of a laminar hot gas in a colder atmosphere

S. Antontsev and J. I. Díaz

Abstract. We study the boundary layer approximation of the, already classical, mathematical model which describes the discharge of a laminar hot gas in a stagnant colder atmosphere of the same gas. We start by proving the existence and uniqueness of solutions of the nondegenerate problem under assumptions implying that the temperature T and the horizontal velocity u of the gas are strictly positive: $T \geq \delta > 0$ and $u \geq \varepsilon > 0$ (here δ and ε are given as boundary conditions in the external atmosphere). We also study the limit cases $\delta = 0$ or $\varepsilon = 0$ in which the governing system of equations become degenerate. We show that in those cases it appear some interfaces separating the zones where T and u are positive from those where they vanish.

Análisis matemático de la descarga de un gas caliente laminar en un ambiente más frio

Resumen. Se considera la aproximación de capa límite para un modelo, ya clásico, que describe la descarga de un gas caliente laminar en un ambiente estático más frio constituido por el mismo gas. Comenzamos probando la existencia y unicidad de soluciones del problema no degenerado bajo hipótesis que aseguran que la temperatura T y la componente horizontal de la velocidad u del gas son estrictamente positivas: $T \geq \delta > 0$ y $u \geq \varepsilon > 0$ (aqui δ y ε vienen dados como condiciones de contorno en el ambiente exterior). Analizamos también los casos límites $\delta = 0$ o $\epsilon = 0$ en los que el sistema de ecuaciones se hace degenerado. Mostramos que en esos casos aparecen unas interfaces que separan las zonas donde T o u son positivos de las regiones deonde se anulan.

1 Introduction

In this Preliminary Communication we present some of the results of [1] in which we consider a mathematical model for the discharge of a laminar hot gas in a stagnant colder atmosphere of the same gas. This problem can be already considered as a classical one in the dynamics of compressible fluids and has some relevance in different contexts as, for instance, in combustion. We consider the boundary layer approximation to the problem (which, among other things, allow us to combine the pressure and gravity forces in a single longitudinal buoyancy force) neglect the pressure). In the boundary layer approach, the dimensionless modeling of planar jets leads to the following system of nonlinear PDEs (see [6, 7])

$$\frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial r} = 0, \tag{1}$$

$$\rho u \frac{\partial u}{\partial x} + \rho v \frac{\partial u}{\partial r} = \frac{\partial}{\partial r} \left(\mu \frac{\partial u}{\partial r} \right) + G \left(1 - \frac{\varepsilon}{T} \right), \qquad \rho u \frac{\partial T}{\partial x} + \rho v \frac{\partial T}{\partial r} = \frac{1}{Pr} \frac{\partial}{\partial r} \left(\mu \frac{\partial T}{\partial r} \right), \qquad (2)$$

where Pr is the Prandtl number and G is the inverse squared of the relevant Froude number (both are given positive numbers). The system is completed by the constitutive laws $\rho = 1/T$ and $\mu = T^\sigma$ for some $0 < \sigma < \infty$. Here the unknowns are the planar velocity, given by the vector (v, u),

System (1)–(2) is considered in the domain $\Omega = \{(x, r) \in \mathbb{R}^2 : 0 < x < \infty, \, 0 < r < l \leq \infty\}$ with the boundary conditions

$$\frac{\partial u}{\partial r} = v = \frac{\partial T}{\partial r} = 0, \quad \text{for } r = 0 \text{ and for } x > 0, \tag{3}$$

$$u = \delta, \; T = \varepsilon, \quad \text{for } r = l \text{ and for } x > 0, \tag{4}$$

and the "initial" conditions

$$u(0, r) = u_0(r) \geq \delta, \quad T(0, r) = T_0(r) \geq \varepsilon \quad \text{for } x = 0 \text{ and for } r \in [0, l]. \tag{5}$$

We point out that the above system is only well justified for jet type initial data (i.e. with a support strictly included in [0,l] as in theorem 3), nevertheless the mathematical analysis of the system takes sense for more general initial data (as indicated before). Problem (1)–(5) was already proposed in [7] for the spacial case $l = \infty$. The cited paper also deals with the self-similar solutions and the problem of their numerical simulation. In this Note we, firstly, report the existence and uniqueness of classical solutions of the nondegenerate problem that corresponds to the assumption $\delta > 0$ and $\varepsilon > 0$. To prove it we use the von Mises variables (x, ψ), where ψ is the associated "stream function" which transforms (1)–(2), upon elimination of v, into a purely diffusive system for unknown functions (u, T).

In a second part we get some a priori estimates (independent of ε and δ) which allow us to establish the existence of a weak solution of the system when $\varepsilon = 0$ and/or $\delta = 0$. The question of uniqueness remains still open. The limit cases $\delta = 0$ or $\varepsilon = 0$ lead to the degeneracy of the system and to the formation of interfaces defined as the boundaries of the support of u or T.

Finally, we apply some suitable energy methods (in the spirit of [2]) to prove the formation of two different fronts which separate the regions where $T > 0$ from $T = 0$ and the regions where $u = 0$ from $u \neq 0$. So, our main interest, in this Note, is the study of the limit cases $\varepsilon = 0$ and/or $\delta = 0$.

2   Von Mises transformation to the variables (x, ψ)

We introduce the stream function $\psi(x, r)$ by the formulas

$$\psi_r = \rho u, \qquad \psi_x = -\rho v, \tag{6}$$

and then we define the new independent (von Mises) variables

$$(x, r) \leftrightarrow (X = x, \; \psi = \psi(x, r)), \; \frac{D(X, \psi)}{D(x, r)} = \psi_r = \rho u > 0, \tag{7}$$

which determine a homeomorphism $(x, r) \leftrightarrow (X = x, \; \psi = \psi(x, r))$. Hence, by eliminating the unknown v, we transform (1)–(2) in the system of equations for the unknown (u, T)

$$\frac{\partial u}{\partial x} = \frac{\partial}{\partial \psi} \left(a(u, T) \frac{\partial u}{\partial \psi} \right) + c(u, T), \qquad \frac{\partial T}{\partial x} = \frac{\partial}{\partial \psi} \left(b(u, T) \frac{\partial T}{\partial \psi} \right), \tag{8}$$

with

$$a = T^{\sigma - 1} u, \quad c = \frac{G}{u} T \left(1 - \frac{\varepsilon}{T} \right), \quad b = \frac{a}{Pr}.$$

Notice that for $\sigma = 1$ and $G = 0$ system (8) splits into two independent parabolic equations. If this is the case, the first equation coincides with the famous "porous medium equation" for the function u (see, e.g. [2, 3, 8, 9]). The second equation can be understood as a "generalized porous medium equation" for the function T which may also degenerate with respect to the independent variables.

3 Existence and uniqueness of solutions for the nondegenerate problems ($0 < \varepsilon \le 1$, $0 < \delta \le 1$)

We will consider two different problems associated to system (8):

Problem I: Given $0 < l < \infty$, find a solution (u, T) of system (8) in the domain $\Omega = \{(x, \psi) \in \mathbb{R}^2 : 0 < x < \infty,\ 0 < \psi < l < \infty\}$, with the boundary conditions

$$\frac{\partial u}{\partial \psi} = \frac{\partial T}{\partial \psi} = 0,\ \text{on } \psi = 0,\ x > 0 \text{ and } u = \delta, \quad T = \varepsilon,\ \text{on } \psi = l,\ x > 0, \tag{9}$$

and the "initial" conditions

$$u = u_0(\psi), T = T_0(\psi), \quad \text{on } x = 0, 0 \le \psi < l. \tag{10}$$

System (8) is of parabolic type and, so, condition (10) looks as an initial condition if we take the variable x as a "fictitious" time. Here $u_0(\psi) = u_0(r(0, \psi))$, $T_0(\psi) = T_0(r(0, \psi))$ with function $r(0, \psi)$ defined by

$$r(0, \psi) = \int_0^\psi T_0(s)/u_0(s)\,ds. \tag{11}$$

For simplicity, we assume that

$$u_0(l) = \delta, \quad T_0(l) = \varepsilon, \quad 0 < \delta \le u_0(l) \le 1, \quad 0 < \varepsilon \le T_0(l) \le 1. \tag{12}$$

Problem II: Find a solution (u, T) of system (8) in the domain $\Omega = \{(x, \psi) \in \mathbb{R}^2 |\ 0 < x < \infty, 0 < \psi < f(x) = \psi(x, l)\}$ with $r(x, f(x)) = l = \text{cst}$, under the boundary and "initial" conditions

$$\frac{\partial u}{\partial \psi} = \frac{\partial T}{\partial \psi} = 0,\ \text{on } \psi = 0,\ x > 0 \quad \text{and} \quad u = \delta,\ T = \varepsilon,\ \text{on } \psi = \psi(x, l) = f(x),\ x > 0, \tag{13}$$

$$u = u_0(\psi),\ T = T_0(\psi), \quad \text{on } x = 0, 0 \le \psi < l. \tag{14}$$

Here $\psi(x, l) = f(x)$ is an unknown function defined by the equation

$$r(x, \psi) = l, \quad 0 < l < \infty, \tag{15}$$

with a given positive constant l. Notice that both Problems I and II coincide if $l = \infty$.

3.1 Existence and uniqueness of solutions of Problem I

We start by proving (in [1]) the following maximum principle for both problems: if $G = 0$, then

$$\delta \le u(x, \psi) \le 1 \text{ and } \varepsilon \le T(x, \psi) \le 1. \tag{16}$$

Moreover, if $G > 0$, then

$$\delta \le u(x, \psi) \le \inf_{\lambda > 0} (\max\{e^{\lambda X}\sqrt{G}/\sqrt{\lambda}, 1\}) = C_0 \quad \text{and} \quad \varepsilon \le T(x, \psi) \le 1. \tag{17}$$

Estimates (16), (17) lead to the inequalities

$$0 < a_0(\varepsilon, \delta) \le a, \quad b \le b_0(\varepsilon, \delta) < \infty \quad \text{and} \quad 0 \le c \le c_0(\varepsilon, \delta) < \infty. \tag{18}$$

According to (18), system (8) is uniformely parabolic if $\varepsilon > 0$ and $\delta > 0$. Then, by applying some well known results (see, e.g. [4]), we prove in [1] the following result.

Theorem 1 *Let functions* $(u_0, T_0) \in C^\alpha[0, l]$, $0 < \alpha < 1$, *satisfying* (12) *with* $(0 \leq \varepsilon, 0 < \delta)$. *Then Problem I has at least one classical solution* $(u, T) \in C^{\alpha, \frac{\alpha}{2}}(\overline{\Omega}) \cap C^{2m+\alpha, \, m+\frac{\alpha}{2}}((\Omega)')$, $\overline{(\Omega)'} \subset \Omega$, $m \geq 1$. *Moreover, this solution is unique if* $(u_\psi, T_\psi) \in L^4(0, X; L^{2q}(0, l))$, *for some* $q > 1$.

Remark 1 *The constructed solution* $(u(x, \psi), T(x, \psi))$ *defines a homomorphism between the domain in the plane of the physical variables* $\Omega_{x, r} = \{(x, \psi) \in \mathbb{R}^2 : 0 < x < \infty, 0 < r < r(x, l)\}$ *and the domain* $\Omega_{x, l} = \{(x, \psi) \in \mathbb{R}^2 : 0 < x < \infty, 0 < \psi < l\}$. *This solution determines also a form of the stream line* $r = r(x, l) = g(x)$, $(g'(x) = v(x, l)/u(x, l))$ *in the physical domain. In this connection, the second component of the velocity* v *is defined through the formula*

$$v(x, \psi) = u(x, \psi) \frac{\partial}{\partial x} \int_0^\psi \left(\frac{ds}{\rho(x, s) u(x, s)} \right). \tag{19}$$

According to (7), (16) *and* (17), *the classical solution* $(u(x, \psi), T(x, \psi))$ *determines a classical solution* $(v(x, r), u(x, r), T(x, r))$ *of system* (1)–(2) *satisfying* (3), (4).

Remark 2 *Using* (16), (17) *and some additional integral estimates we prove that the Problem* I *has at least one solution when* $l = \infty$.

3.2 Existence and uniqueness of "local" solutions of Problem II

We introduce the new variable

$$\eta = \frac{\psi}{f(x)},$$

and reduce Problem II to the following one:

$$\frac{\partial u}{\partial x} - \frac{\eta f'(x)}{f(x)} \frac{\partial u}{\partial \eta} = \frac{1}{f^2(x)} \frac{\partial}{\partial \eta} \left(T^{\sigma-1} u \frac{\partial u}{\partial \eta} \right), \quad \frac{\partial T}{\partial x} - \frac{\eta f'(x)}{f(x)} \frac{\partial T}{\partial \eta} = \frac{1}{f^2(x)} \frac{1}{\Pr} \frac{\partial}{\partial \eta} \left(T^{\sigma-1} u \frac{\partial u}{\partial \eta} \right), \tag{20}$$

$$\frac{\partial u}{\partial \eta} = \frac{\partial T}{\partial \eta} = 0, \; \eta = 0, \; x > 0, \quad \text{and} \quad u = \delta, T = \varepsilon, \eta = 1, x > 0, \tag{21}$$

$$u = u_0(\eta f(0)), T = T_0(\eta f(0)), \quad \text{for } x = 0, 0 \leq \eta \leq 1, \tag{22}$$

in the domain $\Omega = \{(x, \eta) \in \mathbb{R}^2 | \; 0 < x < \infty, 0 < \eta < 1\}$. Here $f(0) > 0$ is a given constant and the (unknown) function $f(x)$ is defined by means of the nonlocal operator over u, T and their derivatives with respect to η by means of the relation

$$f(x) f'(x) = \left(T^{\sigma-1} u_\eta - \frac{u T^{\sigma-2} T_\eta}{\Pr} \right) \bigg|_{\eta=1} + \frac{u(x, 1)}{T(x, 1)} \int_0^1 \left(-T^{\sigma-1} T_\eta u_\eta \left(\frac{1}{u} + \frac{1}{\Pr} \right) + \frac{2}{u^2} T^{\sigma-1} u_\eta^2 \right) ds.$$

We prove (in [1]) the following result.

Theorem 2 *Let the functions* $(u_0, T_0) \in C^{2+\alpha}[0, 1]$ *(for some* $0 < \alpha < 1$*) satisfying the corresponding compatibility conditions. Then Problem* II *has a unique classical solution* $(u(x, \eta), T(x, \eta))$ *on the interval* $x \in [0, X^*]$, *where* $X^* = X^*(Y(0)) > 0$. *Moreover, the solution exists for any finite value* l, *provided that* $\int_0^1 (u_{0\eta}'^2 + T_{0\eta}'^2) d\eta$ *let sufficiently small.*

4 Existence of weak solutions for the degenerate systems

To prove the existence of solutions of the degenerate Problem I we derive (in [1]) suitable a priori estimates, independent on ε and δ, and we justify the limit passage as $\varepsilon \to 0$ or $\delta \to 0$ in suitable functional spaces. In this way, we prove the existence of weak solutions to Problem I if $\varepsilon = 0$ or $\delta = 0$.

5   Localization properties of weak solutions

5.1   Finite speed of propagation and waiting time property for the velocity u and the temperature T

Let us consider the first component u of the solution. Following the ideas of [2], we introduce the energy functions

$$E(r,\xi) = \int_0^x \int_{l_0}^{l_0+\xi} T^{\sigma-1} u u_\psi^2(t,s)\, ds\, dt, \; b(r,x) = \int_{l_0}^{l_0+\xi} u^2(x,s)\, ds.$$

We assume that $E(r,\xi) + b(r,\xi) \leq E_0 < \infty$.

Theorem 3 *Let u be the first component of any weak solution (u,T) of Problem* I *with $\delta = 0$, $0 < \varepsilon \leq 1$, $-\infty < \sigma < \infty$, and assume that*

$$u_0(\psi) = 0, \; \psi \in [l_0, l], \quad \text{for some } 0 < l_0 < l. \tag{23}$$

Then u possesses the "finite speed of propagation property"

$$u(x,\psi) = 0 \quad \text{if } \psi^{1+\alpha} \in [l_0^{1+\alpha}, l_0^{1+\alpha} + x^\lambda], \text{ where } x^\lambda \leq (l^{1+\alpha} - l_0^{1+\alpha})/C, \tag{24}$$

with $\alpha = 4/3$, $\lambda = 3/7$ and for some positive constant $C = C(E_0, \varepsilon, \sigma)$. If (additionally to (23)) the initial function $u_0(\psi)$ satisfies

$$\int_{l_0-s}^{l_0} u_0^3(\psi)\, d\psi \leq \epsilon s^{\frac{1}{1-\nu}}, \quad \text{for some } 0 \leq s \leq l_0, \tag{25}$$

with ϵ, sufficiently small then u possesses the "waiting time property": i.e., there exists $X^* > 0$ such that

$$u(x,\psi) = 0, \quad \text{for any } x \in [0, X^*], \, l_0 \leq \psi \leq l. \tag{26}$$

PROOF.   Using the method of [2], we obtain the differential ordinary inequality for the energy function

$$E^\nu(x,\xi) \leq C r^{-\alpha}\, \xi^\lambda \frac{\partial E(x,\xi)}{\partial \xi} + \epsilon^\nu C\, (\xi)_+^{\frac{\nu}{1-\nu}}. \tag{27}$$

A carefull analysis of the last inequality completes of the proof of the Theorem.   ■

A similar result is valid for the second component of the weak solution (the temperature T) of Problem I if $\varepsilon = 0, 0 < \delta$ and $1 < \sigma < \infty$.

5.2   Extinction of the temperature T for $\varepsilon = 0$, $0 < \delta \leq 1$ assumed $0 < \sigma < 1$

We prove in [1] the following result for this spacial case

Theorem 4 *Let T be the second component of a weak solution (u,T) of Problem* I *with $l < \infty$ and $0 < \sigma < 1$, $0 < \delta \leq u \leq 1$. Then $T(x,\psi) \equiv 0$, for any $x \geq X^*$, $\psi \in [0,l]$, where X^* is defined by*

$$X^* = \frac{l^\nu \nu^2 \Pr}{2\delta(1-\nu)} \int_0^l T_0^2(\psi)\, d\psi, \quad \text{with } \nu = \frac{\sigma+1}{2} < 1.$$

References

[1] Antontsev, S. N. and Díaz, J. I., (2007). On thermal and stagnation interfaces generated by the discharge of a laminar hot gas in a stagnant colder atmosphere, Manuscript. To appear.

[2] Antontsev, S. N., Díaz, J. I. and Shmarev, S. I., (2002). *Energy Methods for Free Boundary Problems: Applications to Non-linear PDEs and Fluid Mechanics*, Bikhäuser, Boston, Progress in Nonlinear Differential Equations and Their Applications, Vol. **48**.

[3] Barenblatt, G. I. and Višik, M. I.,(1956). On finite velocity of propagation in problems of non-stationary filtration of a liquid or gas, *Prikl. Mat. Meh.*, **20**X, 411–417.

[4] Ladyženskaja, O. A., Solonnikov, V. A. and Ural'tseva, N. N.,(1967). *Linear and quasilinear equations of parabolic type*, American Mathematical Society, Providence, R. I., Translated from the Russian by S. Smith. Translations of Mathematical Monographs, Vol. **23**.

[5] Pai, S., (1952). Axially symmetrical jet mixing of a compressible fluid, *Quart. Appl. Math.*, **10**, 141–148.

[6] Pai, S., (1954). *Fluid dynamics of jets*, D. Van Nostrand Company, Inc., Toronto-New York-London.

[7] Sánchez-Sanz, M., Sánchez, A. and Liñán, A., (2006). Front solutions in high-temperature laminar gas jets, *J. Fluid. Mech.*, **547**, 257–266.

[8] Vázquez, J. L., (1992). An introduction to the mathematical theory of the porous medium equation, in *Shape Optimization and Free Boundaries*, C, Mathematical and Physical Sciences, vol. **212** of Contemp. Math., Kluwer Acad. Publ., Dordrecht, Netherlands, 347–389.

[9] Zeldovič, Y. B. and Kompaneec, A. S., (1950). On the theory of propagation of heat with the heat conductivity depending upon the temperature, in *Collection in honor of the seventieth birthday of academician A. F. Ioffe*, Izdat. Akad. Nauk SSSR, Moscow, 61–71.

S. Antontsev
Dpto. de Matemática
Universidade Beira Interior
6201-001 Covilhã Portugal
and
Dpto. de Matemática Aplicada
Facultad de Matemáticas
Universidad Complutense de Madrid
28040 Madrid Spain
anton@ubi.pt and antontsevsn@mail.ru

J. I. Díaz
Dpto. de Matemática
Facultad de Matemáticas
Universidad Complutense de Madrid
28040 Madrid Spain
diaz.racefyn@insde.es