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Weighted Herz type spaces estimates of multilinear singular integral operators for the extreme cases

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Abstract. We prove some weighted endpoint estimates for some multilinear operators related to certain singular integral operators on Herz and Herz type Hardy spaces.

Estimaciones ponderadas de operadores integrales singulares multilineales en casos extremos en espacios de tipo Herz

Resumen. Demostramos algunas estimaciones ponderadas de puntos finales para algunos operadores multilineales relacionados con ciertos operadores integrales singulares en espacios de Herz y Herz tipo Hardy.

1 Introduction

Let T be the Calderón-Zygmund singular integral operator. A classical result of Coifman, Rochberg and Weiss (see [5]) states that the commutator $[b, T](f) = T(bf) - bT(f)$ (where $b \in \mathrm{BMO}(\mathbb{R}^n)$) is bounded on $L^p(\mathbb{R}^n)$ for $1 < p < \infty$. In [10], the boundedness properties of the commutators for the extreme values of p are obtained. In recent years, the theory of Herz space and Herz type Hardy space, as a local version of Lebesgue space and Hardy space, has been developed (see [7, 8, 11, 12]). The main purpose of this paper is to establish the weighted endpoint continuity properties of some multilinear operators related to certain non-convolution type singular integral operators on Herz and Herz type Hardy spaces.

2 Notation and Statement of theorems

Throughout this paper, we denote the Muckenhoupt weights by A_p for $1 \leq p < \infty$ (see [9]). $Q(x, r)$ will denote a cube of $\mathbb{R}^n$ with r side centered at x and with sides parallel to the axes. For a cube Q and a locally integrable function f, let $f_Q = |Q|^{-1} \int_Q f(x)\,dx$ and $f^{\#}(x) = \sup_{x \in Q} |Q|^{-1} \int_Q |f(y) - f_Q|\,dy$. Moreover, f is said to belong to $\mathrm{BMO}(\mathbb{R}^n)$ if $f^{\#} \in L^\infty(\mathbb{R}^n)$ and define that $\|f\|_{\mathrm{BMO}} = \|f^{\#}\|_{L^\infty}$.

For a non-negative weight functions w, we define the weighted central BMO space by $\mathrm{CMO}(w)$, which is the space of those functions $f \in L_{\mathrm{loc}}(\mathbb{R}^n)$ such that

$$\|f\|_{\mathrm{CMO}(w)} = \sup_{r>1} w(Q(0, r))^{-1} \int_Q |f(x) - f_Q| w(x)\,dx < \infty,$$

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where, and in what follows, $f(Q) = \int_Q f(x)\,dx$ for a cube Q and a locally integrable function f. It is well-known that (see [8, 9])

$$\|f\|_{\mathrm{CMO}(w)} \approx \sup_{r>1} \inf_{c\in\mathbf{C}} w(Q(0,r))^{-1} \int_Q |f(x) - c| w(x)\,dx,$$

where, and in what follows, $A \approx B$ means there exist two positives C_1 and C_2 such that $A \le C_1 B \le C_2 A$ and **C** is the collections of all number.

Let $S(\mathbb{R}^n)$ be the Schwartz class and $S'(\mathbb{R}^n)$ be the spaces of tempered distributions which are the collections of all continuous linear functionals on $S(\mathbb{R}^n)$ (see [13, p. 262]). For $k \in \mathbb{Z}$, define $B_k = \{x \in \mathbb{R}^n : |x| \le 2^k\}$ and $C_k = B_k \setminus B_{k-1}$. Denote by χ_k the characteristic function of C_k and by $\tilde{\chi}_k$ the characteristic function of C_k for $k \ge 1$ and $\tilde{\chi}_0$ the characteristic function of B_0.

Definition 1 *Let $1 < p < \infty$ and w_1, w_2 be two non-negative weight functions on $\mathbb{R}^n$.*

1. *The homogeneous weighted Herz space is defined by*

$$\dot{K}_p(w_1, w_2; \mathbb{R}^n) = \{f \in L^p_{\mathrm{loc}}(\mathbb{R}^n \setminus \{0\}) : \|f\|_{\dot{K}_p(w_1,w_2)} < \infty\},$$

where

$$\|f\|_{\dot{K}_p(w_1,w_2)} = \sum_{k=-\infty}^{\infty} [w_1(B_k)]^{1-1/p} \|f\chi_k\|_{L^p(w_2)};$$

2. *The nonhomogeneous weighted Herz space is defined by*

$$K_p(w_1, w_2; \mathbb{R}^n) = \{f \in L^p_{\mathrm{loc}}(\mathbb{R}^n) : \|f\|_{K_p(w_1,w_2)} < \infty\},$$

where

$$\|f\|_{K_p(w_1,w_2)} = \sum_{k=0}^{\infty} [w_1(B_k)]^{1-1/p} \|f\tilde{\chi}_k\|_{L^p(w_2)};$$

3. *The homogeneous weighted Herz type Hardy space is defined by*

$$H\dot{K}_p(w_1, w_2; \mathbb{R}^n) = \{f \in S'(\mathbb{R}^n) : G(f) \in \dot{K}_p(w_1, w_2; \mathbb{R}^n)\},$$

where

$$\|f\|_{H\dot{K}_p(w_1,w_2)} = \|G(f)\|_{\dot{K}_p(w_1,w_2)};$$

4. *The nonhomogeneous weighted Herz type Hardy space is defined by*

$$HK_p(w_1, w_2; \mathbb{R}^n) = \{f \in S'(\mathbb{R}^n) : G(f) \in K_p(w_1, w_2; \mathbb{R}^n)\},$$

where

$$\|f\|_{HK_p(w_1,w_2)} = \|G(f)\|_{K_p(w_1,w_2)};$$

where $G(f)$ is the grand maximal function of f.

The Herz type Hardy spaces have the atomic decomposition characterization.

Definition 2 *Let $1 < p < \infty$ and $w_1, w_2 \in A_1$. A function $a(x)$ on $\mathbb{R}^n$ is called a central $(n(1 - 1/p), p; w_1, w_2)$-atom (or a central $(n(1 - 1/p), p; w_1, w_2)$-atom of restrict type), if*

1. Supp $a \subset B(0, r)$ *for some $r > 0$ (or for some $r \ge 1$);*
2. $\|a\|_{L^p(w_2)} \le [w_1(B(0,r))]^{1/p-1}$,

3. $\int_{\mathbb{R}^n} a(x)\,dx = 0$.

Lemma 1 (see [8, 12]) *Let $w_1, w_2 \in A_1$ and $1 < p < \infty$. A temperate distribution f belongs to $H\dot{K}_p(w_1, w_2; \mathbb{R}^n)$ (or $HK_p(w_1, w_2; \mathbb{R}^n)$) if and only if there exist central $(n(1-1/p), p; w_1, w_2)$-atoms (or central $(n(1-1/p), p; w_1, w_2)$-atoms of restrict type) a_j supported on $B_j = B(0, 2^j)$ and constants λ_j, $\sum_j |\lambda_j| < \infty$ such that $f = \sum_{j=-\infty}^{\infty} \lambda_j a_j$ (or $f = \sum_{j=0}^{\infty} \lambda_j a_j$) in the $S'(\mathbb{R}^n)$ sense, and*

$$\|f\|_{H\dot{K}_p(w_1,w_2)} (\text{or } \|f\|_{HK_p(w_1,w_2)}) \approx \sum_j |\lambda_j|.$$

Definition 3 *Let $1 < p < \infty$ and w be a non-negative weight functions on $\mathbb{R}^n$. We shall call $B_p(w)$ the space of those functions f on $\mathbb{R}^n$ such that*

$$\|f\|_{B_p(w)} = \sup_{r>1} [w(Q(0,r))]^{-1/p} \|f\chi_{Q(0,r)}\|_{L^p(w)} < \infty.$$

In this paper, we will consider a class of multilinear operators related to some non-convolution type singular integral operators, whose definition are the following.

Let m be a positive integer and A be a function on $\mathbb{R}^n$. Set

$$R_{m+1}(A; x, y) = A(x) - \sum_{|\alpha| \le m} \frac{1}{\alpha!} D^\alpha A(y)(x-y)^\alpha$$

and

$$Q_{m+1}(A; x, y) = R_m(A; x, y) - \sum_{|\alpha| = m} \frac{1}{\alpha!} D^\alpha A(x)(x-y)^\alpha.$$

Definition 4 *Let $T : S \to S'$ be a linear operator. T is called a singular integral operator if there exists a locally integrable function $K(x, y)$ on $\mathbb{R}^n \times \mathbb{R}^n$ such that*

$$T(f)(x) = \int_{\mathbb{R}^n} K(x, y) f(y)\,dy$$

for every bounded and compactly supported function f, where K satisfies: for a fixed $\varepsilon > 0$,

$$|K(x, y)| \le C|x-y|^{-n}$$

and

$$|K(y, x) - K(z, x)| + |K(x, y) - K(x, z)| \le C|y-z|^\varepsilon |x-z|^{-n-\varepsilon}$$

if $2|y-z| \le |x-z|$. The multilinear operator related to the singular integral operator T is defined by

$$T^A(f)(x) = \int_{\mathbb{R}^n} \frac{R_{m+1}(A; x, y)}{|x-y|^m} K(x, y) f(y)\,dy.$$

We also consider the variant of T^A, which is defined by

$$\tilde{T}^A(f)(x) = \int_{\mathbb{R}^n} \frac{Q_{m+1}(A; x, y)}{|x-y|^m} K(x, y) f(y)\,dy.$$

Note that when $m = 0$, T^A is just the commutators of T and A (see [5, 10]). It is well known that this multilinear operator, as a non-trivial extension of the commutator, is of great interest in harmonic analysis and has been widely studied by many authors (see [2, 3, 4]). In [6], the weighted L^p-boundedness ($p > 1$) of the multilinear operator related to some singular integral operator is obtained. In [1], the weak (H^1, L^1)-boundedness of the multilinear operator related to some singular integral operator are obtained. In this paper, we will study the weighted endpoint continuity properties of the multilinear operators T^A and $\tilde{T}^A$ on Herz and Herz type Hardy spaces.

We shall prove the following theorems in Section 3.

Theorem 1 *Let $1 < p < \infty$, $w \in A_1$ and $D^\alpha A \in \mathrm{BMO}(\mathbb{R}^n)$ for all α with $|\alpha| = m$. Suppose that T^A is the same as in Definition 4 and that T is bounded on $L^p(w)$ for any $1 < p < \infty$ and $w \in A_1$. Then T^A is bounded from $B_p(w)$ to $\mathrm{CMO}(w)$.*

Theorem 2 *Let $1 < p < \infty$, $w_1, w_2 \in A_1$ and $D^\alpha A \in \mathrm{BMO}(\mathbb{R}^n)$ for all α with $|\alpha| = m$. Suppose that $\tilde{T}^A$ is the same as in Definition 4 and that $\tilde{T}^A$ is bounded on $L^p(w)$ for any $1 < p < \infty$ and $w \in A_1$. Then $\tilde{T}^A$ is bounded from $\dot{H}K_p(w_1, w_2; \mathbb{R}^n)$ (or $HK_p(w_1, w_2; \mathbb{R}^n)$) to $\dot{K}_p(w_1, w_2; \mathbb{R}^n)$ (or $HK_p(w_1, w_2; \mathbb{R}^n)$).*

Theorem 3 *Let $1 < p < \infty$, $w \in A_1$ and $D^\alpha A \in \mathrm{BMO}(\mathbb{R}^n)$ for all α with $|\alpha| = m$. Suppose that $\tilde{T}^A$ is the same as in Definition 4 and that $\tilde{T}^A$ is bounded on $L^p(w)$ for any $1 < p < \infty$ and $w \in A_1$. Then the following two statements are equivalent:*

(i) *$\tilde{T}^A$ is bounded from $B_p(w)$ to $\mathrm{CMO}(w)$;*

(ii) *for any cube Q and $z \in 3Q \setminus 2Q$, we have*

$$\frac{1}{w(Q)} \int_Q \left| \sum_{|\alpha|=m} \frac{1}{\alpha!} |D^\alpha A(x) - (D^\alpha A)_Q| \int_{(4Q)^c} K_\alpha(z, y) f(y) dy \right| w(x)\, dx \le C \|f\|_{B_p(w)},$$

where $K_\alpha(z, y) = \frac{(z-y)^\alpha}{|z-y|^m} K(z, y)$ for $|\alpha| = m$.

3 Proofs of Theorems

To prove the theorem, we need the following lemma.

Lemma 2 (see [4]) *Let A be a function on $\mathbb{R}^n$ and $D^\alpha A \in L^q(\mathbb{R}^n)$ for $|\alpha| = m$ and some $q > n$. Then*

$$|R_m(A; x, y)| \le C|x - y|^m \sum_{|\alpha|=m} \left(\frac{1}{|\tilde{Q}(x, y)|} \int_{\tilde{Q}(x,y)} |D^\alpha A(z)|^q\, dz \right)^{1/q},$$

where $\tilde{Q}(x, y)$ is the cube centered at x and having side length $5\sqrt{n}|x - y|$.

Lemma 3 (see [4, p. 454 (28)] and [11, p. 222]) *Let Q be a cube and*

$$\tilde{A}(x) = A(x) - \sum_{|\alpha|=m} \frac{1}{\alpha!} (D^\alpha A)_Q x^\alpha.$$

Then $R_{m+1}(A; x, y) = R_{m+1}(\tilde{A}; x, y)$.

PROOF OF THEOREM 1. It is only to prove that there exists a constant C_Q such that

$$\frac{1}{w(Q)} \int_Q |T^A(f)(x) - C_Q| w(x)\, dx \le C \|f\|_{B_p(w)}$$

holds for any cube $Q = Q(0, d)$ with $d > 1$. Fix a cube $Q = Q(0, d)$ with $d > 1$. Let $\tilde{Q} = 5\sqrt{n}Q$ and $\tilde{A}(x) = A(x) - \sum_{|\alpha|=m} \frac{1}{\alpha!} (D^\alpha A)_{\tilde{Q}} x^\alpha$, then $R_{m+1}(A; x, y) = R_{m+1}(\tilde{A}; x, y)$ and $D^\alpha \tilde{A} = D^\alpha A - (D^\alpha A)_{\tilde{Q}}$ for all α with $|\alpha| = m$ by Lemma 3 and induction. We write, for $f_1 = f\chi_{\tilde{Q}}$ and $f_2 = f\chi_{\mathbb{R}^n \setminus \tilde{Q}}$,

$$\begin{aligned} T^A(f)(x) &= \int_{\mathbb{R}^n} \frac{R_{m+1}(\tilde{A}; x, y)}{|x - y|^m} K(x, y) f(y)\, dy \\ &= \int_{\mathbb{R}^n} \frac{R_m(\tilde{A}; x, y)}{|x - y|^m} K(x, y) f_1(y)\, dy - \sum_{|\alpha|=m} \frac{1}{\alpha!} \int_{\mathbb{R}^n} \frac{K(x, y)(x - y)^\alpha}{|x - y|^m} D^\alpha \tilde{A}(y) f_1(y)\, dy \\ &\quad + \int_{\mathbb{R}^n} \frac{R_{m+1}(\tilde{A}; x, y)}{|x - y|^m} K(x, y) f_2(y)\, dy, \end{aligned}$$

then

$$\frac{1}{w(Q)}\int_Q \left|T^A(f)(x)-T^{\tilde{A}}(f_2)(0)\right| w(x)\,dx \le$$

$$\frac{1}{w(Q)}\int_Q \left|T\left(\frac{R_m(\tilde{A};x,\cdot)}{|x-\cdot|^m}f_1\right)(x)\right| w(x)\,dx \qquad \text{(I)}$$

$$+\sum_{|\alpha|=m}\frac{1}{\alpha!}\frac{1}{|Q|}\int_Q \left|T\left(\frac{(x-\cdot)^\alpha}{|x-\cdot|^m}D^\alpha \tilde{A} f_1\right)(x)\right| w(x)\,dx \qquad \text{(II)}$$

$$+\frac{1}{w(Q)}\int_Q |T^{\tilde{A}}(f_2)(x)-T^{\tilde{A}}(f_2)(0)| w(x)\,dx \qquad \text{(III)}$$

For (I), note that for $x\in Q$ and $y\in\tilde{Q}$, using Lemma 1, we get

$$R_m(\tilde{A};x,y)\le C|x-y|^m\sum_{|\beta|=m}\|D^\beta A\|_{\mathrm{BMO}},$$

thus, by the $L^p(w)$-boundedness of T and $w(Q)\approx w(\tilde{Q})$ (see [9]), we get

$$\begin{aligned}
\text{(I)} &\le \frac{C}{w(Q)}\int_Q \left|T\Big(\sum_{|\alpha|=m}||D^\alpha A||_{\mathrm{BMO}} f_1\Big)(x)\right| w(x)\,dx\\
&\le C\sum_{|\alpha|=m}\|D^\alpha A\|_{\mathrm{BMO}}\left(\frac{1}{w(Q)}\int_Q |T(f_1)(x)|^p w(x)\,dx\right)^{1/p}\\
&\le C\sum_{|\alpha|=m}\|D^\alpha A\|_{\mathrm{BMO}} w(\tilde{Q})^{-1/p}\|f\chi_{\tilde{Q}}\|_{L^p(w)}\\
&\le C\sum_{|\alpha|=m}\|D^\alpha A\|_{\mathrm{BMO}}\|f\|_{B_p(w)}.
\end{aligned}$$

For (II), noting that $w\in A_1$, w satisfies the reverse of Hölder's inequality:

$$\left(\frac{1}{|Q|}\int_Q w(x)^r\,dx\right)^{1/r}\le \frac{C}{|Q|}\int_Q w(x)\,dx$$

for all cube Q and some $1<r<\infty$ (see [9]), taking q, $s>1$ such that $qs<p$ and $r=(ps-qs)/(p-qs)$, then by the $L^q(w)$-boundedness of T and Hölder's inequality, denoting $1/s+1/s'=1$, we gain

$$\begin{aligned}
\text{(II)} &\le \frac{C}{w(Q)}\int_Q \left|T\Big(\sum_{|\alpha|=m}(D^\alpha A-(D^\alpha A)_{\tilde{Q}})f_1\Big)(x)\right| w(x)\,dx\\
&\le C\sum_{|\alpha|=m}\left(\frac{1}{w(Q)}\int_Q \left|T((D^\alpha A-(D^\alpha A)_{\tilde{Q}})f_1)(x)\right|^q w(x)\,dx\right)^{1/q}\\
&\le C\sum_{|\alpha|=m}\left(\frac{1}{w(Q)}\int \left|(D^\alpha A(x)-(D^\alpha A)_{\tilde{Q}})f_1(x)\right|^q w(x)\,dx\right)^{1/q}\\
&\le C\sum_{|\alpha|=m} w(Q)^{-1/q}\left(\int_{\tilde{Q}}\left|D^\alpha A(x)-(D^\alpha A)_{\tilde{Q}}\right|^{qs'}dx\right)^{1/qs'}\left(\int_{\tilde{Q}}\left|f_1(x)\right|^{qs} w(x)^s\,dx\right)^{1/qs}
\end{aligned}$$

$$
\begin{aligned}
&\le C\sum_{|\alpha|=m}\|D^\alpha A\|_{\mathrm{BMO}}|Q|^{1/(qs')}w(Q)^{-1/q}\left(\int_{\tilde{Q}}|f_1(x)|^p w(x)\,dx\right)^{1/p}\left(\int_{\tilde{Q}}w(x)^r dx\right)^{(p-q)/rpq}\\
&\le C\sum_{|\alpha|=m}\|D^\alpha A\|_{\mathrm{BMO}}|Q|^{1/qs'}w(\tilde{Q})^{-1/q}\|f\chi_{\tilde{Q}}\|_{L^p(w)}\left(\frac{w(Q)}{|Q|}\right)^{(p-q)/pq}|Q|^{(p-q)/pqr}\\
&\le C\sum_{|\alpha|=m}\|D^\alpha A\|_{\mathrm{BMO}}w(\tilde{Q})^{-1/p}\|f\chi_{\tilde{Q}}\|_{L^p(w)}\\
&\le C\|f\|_{B_p(w)}.
\end{aligned}
$$

To estimate (III), we write

$$
\begin{aligned}
&T^{\tilde{A}}(f_2)(x)-T^{\tilde{A}}(f_2)(0)\\
&\qquad=\int_{\mathbb{R}^n}\left[\frac{K(x,y)}{|x-y|^m}-\frac{K(0,y)}{|y|^m}\right]R_m(\tilde{A};x,y)f_2(y)\,dy && (\mathrm{III}_1)\\
&\qquad\quad+\int_{\mathbb{R}^n}\frac{K(0,y)f_2(y)}{|y|^m}[R_m(\tilde{A};x,y)-R_m(\tilde{A};0,y)]\,dy && (\mathrm{III}_2)\\
&\qquad\quad-\sum_{|\alpha|=m}\frac{1}{\alpha!}\int_{\mathbb{R}^n}\left(\frac{K(x,y)(x-y)^\alpha}{|x-y|^m}-\frac{K(0,y)(-y)^\alpha}{|y|^m}\right)D^\alpha\tilde{A}(y)f_2(y)\,dy && (\mathrm{III}_3)
\end{aligned}
$$

By Lemma 2 and the following inequality (see [13])

$$
|b_{Q_1}-b_{Q_2}|\le C\log(|Q_2|/|Q_1|)\|b\|_{\mathrm{BMO}}\qquad \text{for } Q_1\subset Q_2,
$$

we know that, for $x\in Q$ and $y\in 2^{k+1}\tilde{Q}\setminus 2^k\tilde{Q}$,

$$
\begin{aligned}
|R_m(\tilde{A};x,y)|&\le C|x-y|^m\sum_{|\alpha|=m}(\|D^\alpha A\|_{\mathrm{BMO}}+|(D^\alpha A)_{\tilde{Q}(x,y)}-(D^\alpha A)_{\tilde{Q}}|)\\
&\le Ck|x-y|^m\sum_{|\alpha|=m}\|D^\alpha A\|_{\mathrm{BMO}}.
\end{aligned}
$$

Note that $|x-y|\approx|y|$ for $x\in Q$ and $y\in\mathbb{R}^n\setminus\tilde{Q}$ and $w\in A_1\subset A_p$, we obtain, by the condition on K and Hölder's inequality,

$$
\begin{aligned}
|(\mathrm{III}_1)|&\le\int_{\mathbb{R}^n}\left(\left|\frac{K(x,y)}{|x-y|^m}-\frac{K(x,y)}{|y|^m}\right|+\left|\frac{K(x,y)}{|y|^m}-\frac{K(0,y)}{|y|^m}\right|\right)|R_m(\tilde{A};x,y)|\,|f_2(y)|\,dy\\
&\le C\int_{\mathbb{R}^n}\left(\frac{|x|}{|y|^{m+n+1}}+\frac{|x|^\varepsilon}{|y|^{m+n+\varepsilon}}\right)|R_m(\tilde{A};x,y)|\,|f_2(y)|\,dy\\
&\le C\sum_{|\alpha|=m}\|D^\alpha A\|_{\mathrm{BMO}}\sum_{k=0}^\infty\int_{2^{k+1}\tilde{Q}\setminus 2^k\tilde{Q}}k\left(\frac{|x|}{|y|^{n+1}}+\frac{|x|^\varepsilon}{|y|^{n+\varepsilon}}\right)|f(y)|\,dy\\
&\le C\sum_{|\alpha|=m}\|D^\alpha A\|_{\mathrm{BMO}}\sum_{k=1}^\infty k(2^{-k}+2^{-\varepsilon k})w(2^k\tilde{Q})^{-1/p}\left(\int_{2^k\tilde{Q}}|f(y)|^p w(y)dy\right)^{1/p}\\
&\qquad\times\left(\frac{1}{|2^k\tilde{Q}|}\int_{2^k\tilde{Q}}w(y)\,dy\right)^{1/p}\left(\frac{1}{|2^k\tilde{Q}|}\int_{2^k\tilde{Q}}w(y)^{-\frac{1}{p-1}}\,dy\right)^{(p-1)/p}\\
&\le C\sum_{|\alpha|=m}\|D^\alpha A\|_{\mathrm{BMO}}\sum_{k=1}^\infty k(2^{-k}+2^{-\varepsilon k})w(2^k\tilde{Q})^{-1/p}\|f\chi_{2^k\tilde{Q}}\|_{L^p(w)}
\end{aligned}
$$

$$\leq C\sum_{|\alpha|=m}\|D^\alpha A\|_{\mathrm{BMO}}\sum_{k=1}^{\infty}k(2^{-k}+2^{-\varepsilon k})\|f\|_{B_p(w)}$$
$$\leq C\sum_{|\alpha|=m}\|D^\alpha A\|_{\mathrm{BMO}}\|f\|_{B_p(w)}.$$

For (III_2), by the formula (see [4]):

$$R_m(\tilde{A};x,y)-R_m(\tilde{A};x_0,y)=\sum_{|\beta|<m}\frac{1}{\beta!}R_{m-|\beta|}(D^\beta\tilde{A};x,x_0)(x-y)^\beta$$

and Lemma 2, we have

$$|R_m(\tilde{A};x,y)-R_m(\tilde{A};x_0,y)|\leq C\sum_{|\beta|<m}\sum_{|\alpha|=m}|x-x_0|^{m-|\beta|}|x-y|^{|\beta|}\|D^\alpha A\|_{\mathrm{BMO}}.$$

Then in a similar way to the estimates of (III_1), we get

$$(III_2)\leq C\sum_{|\alpha|=m}\|D^\alpha A\|_{\mathrm{BMO}}\sum_{k=0}^{\infty}\int_{2^{k+1}\tilde{Q}\setminus 2^k\tilde{Q}}\frac{|x|}{|y|^{n+1}}|f(y)|\,dy\leq C\|D^\alpha A\|_{\mathrm{BMO}}\|f\|_{B_p(w)}.$$

For (III_3), taking $1<s<p$ and $r>1$ such that $1/r+1/s=1$, by Hölder's inequality and noting that $w\in A_1\subset A_{p/s}$, in a similar way to the estimates of (III_1), we obtain

$$|(III_3)|\leq C\sum_{|\alpha|=m}\sum_{k=0}^{\infty}\int_{2^{k+1}\tilde{Q}\setminus 2^k\tilde{Q}}\left(\frac{|x|}{|y|^{n+1}}+\frac{|x|^\varepsilon}{|y|^{n+\varepsilon}}\right)|D^\alpha\tilde{A}(y)|\,|f(y)|\,dy$$
$$\leq C\sum_{|\alpha|=m}\sum_{k=1}^{\infty}\left(\frac{d}{(2^kd)^{n+1}}+\frac{d^\varepsilon}{(2^kd)^{n+\varepsilon}}\right)\left(\int_{2^k\tilde{Q}}|f(y)|^s\,dy\right)^{1/s}$$
$$\times(2^kd)^{n/r}\left(|2^k\tilde{Q}|^{-1}\int_{2^k\tilde{Q}}|D^\alpha A(y)-(D^\alpha A)_{\tilde{Q}}|^r\,dy\right)^{1/r}$$
$$\leq C\sum_{|\alpha|=m}\|D^\alpha A\|_{\mathrm{BMO}}\sum_{k=1}^{\infty}k\left(\frac{d}{(2^kd)^{n+1}}+\frac{d^\varepsilon}{(2^kd)^{n+\varepsilon}}\right)(2^kd)^{n(1-1/s)}$$
$$\times\left(\int_{2^k\tilde{Q}}|f(y)|^pw(y)\,dy\right)^{1/p}\left(\int_{2^k\tilde{Q}}w(y)^{-s/(p-s)}\,dy\right)^{(p-s)/ps}$$
$$\leq C\sum_{|\alpha|=m}\|D^\alpha A\|_{\mathrm{BMO}}\sum_{k=1}^{\infty}k(2^{-k}+2^{-\varepsilon k})w(2^k\tilde{Q})^{-1/p}\|f\chi_{2^k\tilde{Q}}\|_{L^p(w)}$$
$$\times\left(\frac{1}{|2^k\tilde{Q}|}\int_{2^k\tilde{Q}}w(y)\,dy\right)^{1/p}\left(\frac{1}{|2^k\tilde{Q}|}\int_{2^k\tilde{Q}}w(y)^{-s/(p-s)}\,dy\right)^{(p-s)/ps}$$
$$\leq C\sum_{|\alpha|=m}\|D^\alpha A\|_{\mathrm{BMO}}\sum_{k=1}^{\infty}k(2^{-k}+2^{-\varepsilon k})w(2^k\tilde{Q})^{-1/p}\|f\chi_{2^k\tilde{Q}}\|_{L^p(w)}$$
$$\leq C\sum_{|\alpha|=m}\|D^\alpha A\|_{\mathrm{BMO}}\|f\|_{B_p(w)}.$$

Thus

$$(III)\leq C\sum_{|\alpha|=m}\|D^\alpha A\|_{\mathrm{BMO}}\|f\|_{B_p(w)}.$$

This finishes the proof of Theorem 1. ■

PROOF OF THEOREM 2. Let $f \in H\dot{K}_p(w_1, w_2; \mathbb{R}^n)$, by Lemma 1, $f = \sum_{j=-\infty}^{\infty} \lambda_j a_j$, where $a_j's$ are central $(n(1-1/p), p; w_1, w_2)$-atoms with $\operatorname{supp} a_j \subset B_j = B(0, 2^j)$ and $\|f\|_{H\dot{K}_p(w_1,w_2)} \approx \sum_j |\lambda_j|$. We write, by Minkowski' inequality,

$$\begin{aligned}
\|\tilde{T}^A(f)\|_{\dot{K}_p(w_1,w_2)} &= \sum_{k=-\infty}^{\infty} [w_1(B_k)]^{1-1/p} \|\chi_k \tilde{T}^A(f)\|_{L^p(w_2)} \\
&\le \sum_{k=-\infty}^{\infty} [w_1(B_k)]^{1-1/p} \sum_{j=-\infty}^{k-1} |\lambda_j| \|\chi_k \tilde{T}^A(a_j)\|_{L^p(w_2)} && \text{(J)} \\
&\quad + \sum_{k=-\infty}^{\infty} [w_1(B_k)]^{1-1/p} \sum_{j=k}^{\infty} |\lambda_j| \|\chi_k \tilde{T}^A(a_j)\|_{L^p(w_2)} && \text{(JJ)}
\end{aligned}$$

For (JJ), by the $L^p(w)$-boundedness of $\tilde{T}^A$ for $1 < p < \infty$ and $w \in A_1$, we get, note that $j \ge k$,

$$\begin{aligned}
\text{(JJ)} &\le C \sum_{k=-\infty}^{\infty} [w_1(B_k)]^{1-1/p} \sum_{j=k}^{\infty} |\lambda_j| \|a_j\|_{L^p(w_2)} \\
&\le C \sum_{k=-\infty}^{\infty} [w_1(B_k)]^{1-1/p} \sum_{j=k}^{\infty} |\lambda_j|^p [w_1(B_j)]^{-(1-1/p)} \\
&\le C \sum_{j=-\infty}^{\infty} |\lambda_j| \sum_{k=-\infty}^{j} \left[\frac{w_1(B_k)}{w_1(B_j)}\right]^{1-1/p} \\
&\le C \sum_{j=-\infty}^{\infty} |\lambda_j| \\
&\le C \|f\|_{H\dot{K}_p(w_1,w_2)}.
\end{aligned}$$

To obtain the estimate of (J), we denote that

$$\tilde{A}(x) = A(x) - \sum_{|\alpha|=m} \frac{1}{\alpha!} (D^\alpha A)_{2B_j} x^\alpha.$$

Then $Q_{m+1}(A; x, y) = Q_{m+1}(\tilde{A}; x, y)$ and $Q_{m+1}(A; x, y) = R_m(A; x, y) - \sum_{|\alpha|=m} \frac{1}{\alpha!}(x-y)^\alpha D^\alpha A(x)$. We write, by the vanishing moment of a_j and for $x \in B_k$ with $k \ge j+1$,

$$\begin{aligned}
\tilde{T}^A(a_j)(x) &= \int_{\mathbb{R}^n} \frac{K(x,y) R_m(A; x, y)}{|x-y|^m} a_j(y)\, dy - \sum_{|\alpha|=m} \frac{1}{\alpha!} \int_{\mathbb{R}^n} \frac{K(x,y) D^\alpha \tilde{A}(x)(x-y)^\alpha}{|x-y|^m} a_j(y)\, dy \\
&= \int_{\mathbb{R}^n} \left[\frac{K(x,y)}{|x-y|^m} - \frac{K(x,0)}{|x|^m}\right] R_m(\tilde{A}; x, y) a_j(y)\, dy \\
&\quad + \int_{\mathbb{R}^n} \frac{K(x,0)}{|x|^m} [R_m(\tilde{A}; x, y) - R_m(\tilde{A}; x, 0)] a_j(y)\, dy \\
&\quad - \sum_{|\alpha|=m} \frac{1}{\alpha!} \int_{\mathbb{R}^n} \left[\frac{K(x,y)(x-y)^\alpha}{|x-y|^m} - \frac{K(x,0)x^\alpha}{|x|^m}\right] D^\alpha \tilde{A}(x) a_j(y)\, dy.
\end{aligned}$$

Similar to the proof of Theorem 1, we obtain

$$
\begin{aligned}
|\tilde{T}^A(a_j)(x)| &\le C\int_{\mathbb{R}^n}\left[\frac{|y|}{|x|^{m+n+1}}+\frac{|y|^\varepsilon}{|x|^{m+n+\varepsilon}}\right]|R_m(\tilde{A};x,y)|\,|a_j(y)|\,dy\\
&\quad + C\sum_{|\alpha|=m}\int_{\mathbb{R}^n}\left[\frac{|y|}{|x|^{n+1}}+\frac{|y|^\varepsilon}{|x|^{n+\varepsilon}}\right]|D^\alpha\tilde{A}(x)|\,|a_j(y)|\,dy\\
&\le C\sum_{|\alpha|=m}\|D^\alpha A\|_{\mathrm{BMO}}\left[\frac{2^j}{2^{k(n+1)}}+\frac{2^{j\varepsilon}}{2^{k(n+\varepsilon)}}\right]\|a_j\|_{L^p(w_2)}\left(\int_{B_j}w_2(y)^{-\frac{1}{p-1}}\,dy\right)^{\frac{p-1}{p}}\\
&\quad + C\sum_{|\alpha|=m}\left[\frac{2^j}{2^{k(n+1)}}+\frac{2^{j\varepsilon}}{2^{k(n+\varepsilon)}}\right]|D^\alpha\tilde{A}(x)|\|a_j\|_{L^p(w_2)}\left(\int_{B_j}w_2(y)^{-\frac{1}{p-1}}\,dy\right)^{\frac{p-1}{p}}\\
&\le C\sum_{|\alpha|=m}\|D^\alpha A\|_{\mathrm{BMO}}\left[\frac{2^j}{2^{k(n+1)}}+\frac{2^{j\varepsilon}}{2^{k(n+\varepsilon)}}\right][w_1(B_j)]^{-(1-\frac{1}{p})}\left(\int_{B_j}w_2(y)^{-\frac{1}{p-1}}\,dy\right)^{\frac{p-1}{p}}\\
&\quad + C\sum_{|\alpha|=m}\left[\frac{2^j}{2^{k(n+1)}}+\frac{2^{j\varepsilon}}{2^{k(n+\varepsilon)}}\right]|D^\alpha\tilde{A}(x)|[w_1(B_j)]^{-(1-\frac{1}{p})}\left(\int_{B_j}w_2(y)^{-\frac{1}{p-1}}\,dy\right)^{\frac{p-1}{p}}.
\end{aligned}
$$

Notice that if $w\in A_1$, then $\frac{w(B_2)}{|B_2|}\frac{|B_1|}{w(B_1)}\le C$ for all balls B_1, B_2 with $B_1\subset B_2$ and satisfies the reverse Hölder's inequality (see [13]):

$$
\left(\frac{1}{|B|}\int_B w(x)^r\,dx\right)^{1/r}\le\frac{C}{|B|}\int_B w(x)\,dx
$$

for all balls B and some $1<r<\infty$. Thus

$$
\begin{aligned}
(\mathrm{J}) &\le C\sum_{k=-\infty}^{\infty}[w_1(B_k)]^{1-\frac{1}{p}}\sum_{j=-\infty}^{k-1}|\lambda_j|\left[\frac{2^j}{2^{k(n+1)}}+\frac{2^{j\varepsilon}}{2^{k(n+\varepsilon)}}\right][w_1(B_j)]^{-\left(1-\frac{1}{p}\right)}\\
&\quad\times\left(\int_{B_j}w_2(y)^{-\frac{1}{p-1}}\,dy\right)^{\frac{p-1}{p}}\left[[w_2(B_k)]^{\frac{1}{p}}+\sum_{|\alpha|=m}\left(\int_{B_k}|D^\alpha\tilde{A}(x)|^p w_2(x)\,dx\right)^{\frac{1}{p}}\right]\\
&\le C\sum_{k=-\infty}^{\infty}[w_1(B_k)]^{1-\frac{1}{p}}\sum_{j=-\infty}^{k-1}|\lambda_j|\left[\frac{2^j}{2^{k(n+1)}}+\frac{2^{j\varepsilon}}{2^{k(n+\varepsilon)}}\right]\left(\int_{B_j}w_2(y)^{-\frac{1}{p-1}}\,dy\right)^{\frac{p-1}{p}}\\
&\quad\times[w_1(B_j)]^{-(1-\frac{1}{p})}\Bigg[[w_2(B_k)]^{\frac{1}{p}}\\
&\qquad+\sum_{|\alpha|=m}\left(\frac{1}{|B_k|}\int_{B_k}|D^\alpha\tilde{A}(x)|^{r'p}\,dx\right)^{\frac{1}{r'p}}\left(\frac{1}{|B_k|}\int_{B_k}w_2(x)^r\,dx\right)^{\frac{1}{rp}}|B_k|^{\frac{1}{p}}\Bigg]\\
&\le C\sum_{k=-\infty}^{\infty}\sum_{j=-\infty}^{k-1}|\lambda_j|\left[\frac{2^j}{2^{k(n+1)}}+\frac{2^{j\varepsilon}}{2^{k(n+\varepsilon)}}\right]\left[\frac{w_1(B_k)}{w_1(B_j)}\right]^{1-\frac{1}{p}}\left(\int_{B_j}w_2(x)^{-\frac{1}{p-1}}\,dx\right)^{\frac{p-1}{p}}[w_2(B_k)]^{\frac{1}{p}}\\
&\le C\sum_{k=-\infty}^{\infty}\sum_{j=-\infty}^{k-1}|\lambda_j|\left[\frac{2^j}{2^{k(n+1)}}+\frac{2^{j\varepsilon}}{2^{k(n+\varepsilon)}}\right]\left[\frac{w_1(B_k)}{w_1(B_j)}\right]^{1-\frac{1}{p}}\left[\frac{w_2(B_k)}{w_2(B_j)}\right]^{\frac{1}{p}}
\end{aligned}
$$

$$
\begin{aligned}
&\times \left(\frac{1}{|B_j|} \int_{B_j} w_2(x)\, dx \right)^{\frac{1}{p}} \left(\frac{1}{|B_j|} \int_{B_j} w_2(y)^{-\frac{1}{p-1}}\, dy \right)^{\frac{p-1}{p}} |B_j| \\
&\le C \sum_{j=-\infty}^{\infty} |\lambda_j| \sum_{k=j+1}^{\infty} \left[\frac{2^j}{2^{k(n+1)}} + \frac{2^{j\varepsilon}}{2^{k(n+\varepsilon)}} \right] \left[\frac{w_1(B_k)}{w_1(B_j)} \frac{|B_j|}{|B_k|} \right]^{1-\frac{1}{p}} \left[\frac{w_2(B_k)}{w_2(B_j)} \frac{|B_j|}{|B_k|} \right]^{\frac{1}{p}} |B_k| \\
&\le C \sum_{j=-\infty}^{\infty} |\lambda_j| \sum_{k=j+1}^{\infty} [2^{j-k} + 2^{(j-k)\varepsilon}] \\
&\le C \sum_{j=-\infty}^{\infty} |\lambda_j| \\
&\le C \|f\|_{H\dot{K}_p(w_1,w_2)}.
\end{aligned}
$$

This completes the proof of Theorem 2. ■

PROOF OF THEOREM 3. For any cube $Q = Q(0,d)$ with $d > 1$, let $f \in B_p(w)$ and $\tilde{A}(x) = A(x) - \sum_{|\alpha|=m} \frac{1}{\alpha!}(D^\alpha A)_{\tilde{Q}} x^\alpha$. We write, for $f = f\chi_{4Q} + f\chi_{(4Q)^c} = f_1 + f_2$ and $z \in 3Q \setminus 2Q$,

$$
\begin{aligned}
\tilde{T}^A(f)(x) = \tilde{T}^A(f_1)(x) &+ \int_{\mathbb{R}^n} \frac{R_m(\tilde{A}; x, y)}{|x-y|^m} K(x,y) f_2(y)\, dy \\
&- \sum_{|\alpha|=m} \frac{1}{\alpha!} (D^\alpha A(x) - (D^\alpha A)_Q)(T_\alpha(f_2)(x) - T_\alpha(f_2)(z)) \\
&- \sum_{|\alpha|=m} \frac{1}{\alpha!} (D^\alpha A(x) - (D^\alpha A)_Q) T_\alpha(f_2)(z) \\
= I_1(x) &+ I_2(x) + I_3(x,z) + I_4(x,z),
\end{aligned}
$$

where T_α is the singular integral operator with the kernel $\dfrac{(x-y)^\alpha}{|x-y|^m} K(x,y)$ for $|\alpha| = m$. Note that $(I_4(\cdot, z))_Q = 0$, so that we have

$$
\begin{aligned}
&\tilde{T}^A(f)(x) - (\tilde{T}^A(f))_Q = \\
&\qquad I_1(x) - (I_1(\cdot))_Q + I_2(x) - I_2(z) - [I_2(\cdot) - I_2(z)]_Q - I_3(x,z) + (I_3(x,z))_Q - I_4(x,z).
\end{aligned}
$$

By the $L^p(w)$-boundedness of $\tilde{T}^A$, we get

$$
\begin{aligned}
\frac{1}{w(Q)} \int_Q |I_1(x)| w(x)\, dx &\le C \left(\frac{1}{w(Q)} \int_Q |T^A(f_1)(x)|^p w(x)\, dx \right)^{\frac{1}{p}} \\
&\le C w(Q)^{-\frac{1}{p}} \|f_1\|_{L^p(w)} \\
&\le C \|f\|_{B_p(w)}.
\end{aligned}
$$

In a similar way to the proof of Theorem 1, we obtain

$$
|I_2(x) - I_2(z)| \le C||f||_{B_p(w)}
$$

and

$$
\frac{1}{w(Q)} \int_Q |I_3(x,z)| w(x)\, dx \le C \|f\|_{B_p(w)}.
$$

Then integrating in x on Q and using the above estimates, we obtain the equivalence of the estimate

$$\frac{1}{w(Q)}\int_Q |\tilde{T}^A(x) - (\tilde{T}^A)_Q| w(x)\, dx \le C\|f\|_{B_p(w)}$$

and the estimate

$$\frac{1}{w(Q)}\int_Q |I_4(x,z)| w(x)\, dx \le C\|f\|_{B_p(w)}.$$

This completes the proof of Theorem 3. ■

Finally, we apply Theorems 1, 2 and 3 to the Calderón-Zygmund singular integral operator.
Let T be the Calderón-Zygmund operator defined by (see [9, 13])

$$T(f)(x) = \int K(x,y) f(y)\, dy,$$

the multilinear operator related to T is defined by

$$T^A(f)(x) = \int \frac{R_{m+1}(A;x,y)}{|x-y|^m} K(x,y) f(y)\, dy.$$

We know that T satisfies the conditions in Theorem 1, 2 and 3, thus, the conclusions of Theorem 1, 2 and 3 hold for T^A and $\tilde{T}^A$.

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