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Sharp and weighted inequalities for multilinear integral operators

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Abstract. In this paper, we prove some weighted inequalities for the multilinear operators related to certain integral operators on the generalized Morrey spaces by using the sharp estimates of the multilinear operators. The operators include Littlewood-Paley operator, Marcinkiewicz operator and Bochner-Riesz operator.

Desigualdades ajustadas y ponderadas para operadores integrales multilineales

Resumen. En este trabajo demostramos algunas desigualdades ponderadas para operadores multilineales relacionados con ciertos operadores integrales en los espacios generalizados de Morrey usando estimaciones ajustadas de los operadores multilineales. Los operadores incluyen los operadores de Littlewood-Paley, de Marcinkiewicz y de Bochner-Riesz.

1 Preliminaries and statement of main results

Throughout this paper, φ will denote a positive, increasing function on $\mathbb{R}^+$ and there exists a constant $D > 0$ such that

$$\varphi(2t) \leq D\varphi(t) \qquad \text{for } t \geq 0.$$

Let w be a non-negative weight function on $\mathbb{R}^n$ and f be a locally integrable function on $\mathbb{R}^n$. Define that, for $1 \leq p < \infty$,

$$\|f\|_{L^{p,\varphi}(w)} = \sup_{x\in\mathbb{R}^n, d>0} \left(\frac{1}{\varphi(d)} \int_{B(x,d)} |f(y)|^p w(y)\, dy \right)^{1/p},$$

where $B(x,d) = \{y \in \mathbb{R}^n : |x-y| < d\}$. The generalized weighted Morrey spaces is defined by

$$L^{p,\varphi}(\mathbb{R}^n, w) = \{f \in L^1_{\text{loc}}(\mathbb{R}^n) : \|f||_{L^{p,\varphi}(w)} < \infty\}.$$

If $\varphi(d) = d^\delta$, $\delta > 0$, then $L^{p,\varphi}(\mathbb{R}^n, w) = L^{p,\delta}(\mathbb{R}^n, w)$, which is the classical Morrey space (see [16, 17]).

In this paper, we study some integral operators as following. Suppose m_j are positive integers ($j = 1, \ldots, l$), $m_1 + \cdots + m_l = m$ and b_j are the functions on $\mathbb{R}^n$ ($j = 1, \ldots, l$). Let $F_t(x,y)$ be defined on $\mathbb{R}^n \times \mathbb{R}^n \times [0, +\infty)$, we denote

$$F_t(f)(x) = \int_{\mathbb{R}^n} F_t(x,y) f(y)\, dy$$

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and

$$F_t^b(f)(x) = \int_{\mathbb{R}^n} \frac{\prod_{j=1}^l R_{m_j+1}(b_j;x,y)}{|x-y|^m} F_t(x,y) f(y)\, dy$$

for every bounded and compactly supported function f, where

$$R_{m_j+1}(b_j;x,y) = b_j(x) - \sum_{|\alpha| \le m_j} \frac{1}{\alpha!} D^\alpha b_j(y)(x-y)^\alpha.$$

Let H be the Banach space $H = \{h : \|h\| < \infty\}$ such that, for each fixed $x \in \mathbb{R}^n$, $F_t(f)(x)$ and $F_t^b(f)(x)$ may be viewed as a mapping from $[0, +\infty)$ to H. Then, the multilinear operators related to F_t is defined by

$$T^b(f)(x) = \|F_t^b(f)(x)\|,$$

where F_t satisfies: for fixed $\varepsilon > 0$,

$$\|F_t(x,y)\| \le C|x-y|^{-n}$$

and

$$\|F_t(y,x) - F_t(z,x)\| \le C|y-z|^\varepsilon |x-z|^{-n-\varepsilon}$$

if $2|y-z| \le |x-z|$. We also define that $T(f)(x) = \|F_t(f)(x)\|$ and assume that T is bounded on $L^p(\mathbb{R}^n, w)$ for $1 < p < \infty$ and $w \in A_1$.

Note that when $m = 0$, T^b is just the multilinear commutator of T and b (see [13, 14, 23]). While when $m > 0$, T^b is non-trivial generalizations of the commutator. It is well known that multilinear operators are of great interest in harmonic analysis and have been widely studied by many authors (see [2–5]). In [7], the L^p ($p > 1$) boundedness of the multilinear singular integral operator are obtained; In [11] and [12], a variant sharp estimate for the multilinear singular integral operators are obtained; In [18–20], the authors prove some sharp estimate for the multilinear commutator. As the Morrey space may be considered as an extension of the Lebesgue space, it is natural and important to study the boundedness of the multilinear integral operator on the Morrey spaces. The purpose of this paper is two-fold. First, we establish a sharp estimate for the multilinear operator related to F_t, and second, we prove the weighted inequality for the multilinear integral operators on the generalized weighted Morrey spaces by using the sharp inequality. In section 3, we will give some applications of the theorem in this paper.

Now, let us introduce some notations. Throughout this paper, Q will denote a cube of $\mathbb{R}^n$ with sides parallel to the axes. For a ball B or a cube Q, kB or kQ will denote the ball or cube with the same center as B or Q and k times radius or edges of B or Q for $k > 0$. For any locally integrable function f, the sharp function of f is defined by

$$f^\#(x) = \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f(y) - f_Q|\, dy,$$

where, and in what follows, $f_Q = |Q|^{-1} \int_Q f(x)\, dx$. It is well-known that (see [10, 21])

$$f^\#(x) = \sup_{Q \ni x} \inf_{c \in C} \frac{1}{|Q|} \int_Q |f(y) - c|\, dy.$$

We say that f belongs to $\mathrm{BMO}(\mathbb{R}^n)$ if $f^\#$ belongs to $L^\infty(\mathbb{R}^n)$ and $\|f\|_{\mathrm{BMO}} = \|f^\#\|_{L^\infty}$. For $1 \le p < \infty$, we know $\|f\|_{\mathrm{BMO}} \approx \|f\|_{\mathrm{BMO},\, p}$ (see [21]), where

$$\|f\|_{\mathrm{BMO},\, p} = \sup_Q \left(\frac{1}{|Q|} \int_Q |f(y) - f_Q|^p\, dy \right)^{1/p}.$$

Let M be the Hardy-Littlewood maximal operator defined by

$$M(f)(x) = \sup_{Q \ni x} |Q|^{-1} \int_Q |f(y)|\, dy,$$

we write that $M_p(f) = (M(f^p))^{1/p}$ for $0 < p < \infty$. We denote the Muckenhoupt weights by A_1, that is (see [10]):

$$A_1 = \{0 < w \in L^1_{\mathrm{loc}}(\mathbb{R}^n) : M(w)(x) \le Cw(x), \text{a.e.}\}.$$

We shall prove the following theorem.

Theorem 1 *Let $1 < p < \infty$, $0 < D < 2^n$, $w \in A_1$, $D^\alpha b_j \in \mathrm{BMO}(\mathbb{R}^n)$ for all α with $|\alpha| = m_j$ and $j = 1, \ldots, l$. Then*

$$\|T^b(f)\|_{L^{p,\varphi}(w)} \le C \prod_{j=1}^{l} \left(\sum_{|\alpha_j|=m_j} \|D^{\alpha_j} b_j\|_{\mathrm{BMO}} \right) \|f\|_{L^{p,\varphi}(w)}.$$

2 Proof of Theorem

To prove the theorems, we need the following lemmas.

Lemma 1 ([4]) *Let b be a function on $\mathbb{R}^n$ and $D^\alpha b \in L^q(\mathbb{R}^n)$ for all α with $|\alpha| = m$ and some $q > n$. Then, for $x \ne y$,*

$$|R_m(b; x, y)| \le C|x-y|^m \sum_{|\alpha|=m} \left(\frac{1}{|\tilde{Q}(x,y)|} \int_{\tilde{Q}(x,y)} |D^\alpha b(z)|^q \, dz \right)^{1/q},$$

where $\tilde{Q}$ is the cube centered at x and having side length $5\sqrt{n}|x-y|$.

Lemma 2 *Let $1 < p < \infty$, $0 < D < 2^n$, $w \in A_1$. Then, for $f \in L^{p,\varphi}(\mathbb{R}^n, w)$,*

(a) $\|M(f)\|_{L^{p,\varphi}(w)} \le C\|f^\#\|_{L^{p,\varphi}(w)}$;

(b) $\|M_q(f)\|_{L^{p,\varphi}(w)} \le C\|f\|_{L^{p,\varphi}(w)}$ *for $1 < q < p$.*

PROOF. (a) Let $f \in L^{p,\varphi}(\mathbb{R}^n, w)$. For a ball $B = B(x,d) \subset \mathbb{R}^n$, note that $M(w\chi_B) \in A_1$ and by the following inequality (see [6]): for any $u \in A_1$,

$$\int_{\mathbb{R}^n} |M(f)(y)|^p u(y)\, dy \le C \int_{\mathbb{R}^n} |f^\#(y)|^p u(y)\, dy,$$

we get

$$\begin{aligned}
&\int_B |M(f)(y)|^p w(y)\, dy \\
&\quad \le \int_{\mathbb{R}^n} |M(f)(y)|^p M(w\chi_B)(y)\, dy \\
&\quad \le C \int_{\mathbb{R}^n} |f^\#(y)|^p M(w\chi_B)(y)\, dy \\
&\quad = C \left[\int_B |f^\#(y)|^p M(w\chi_B)(y)\, dy + \sum_{k=0}^{\infty} \int_{2^{k+1}B \setminus 2^k B} |f^\#(y)|^p M(w\chi_B)(y)\, dy \right] \\
&\quad \le C \left[\int_B |f^\#(y)|^p M(w)(y)\, dy + \sum_{k=0}^{\infty} \int_{2^{k+1}B \setminus 2^k B} |f^\#(y)|^p \left(\sup_{Q \ni y} \frac{1}{|Q|} \int_B w(z)\, dz \right) dy \right] \\
&\quad \le C \left[\int_B |f^\#(y)|^p M(w)(y)\, dy + \sum_{k=0}^{\infty} \int_{2^{k+1}B \setminus 2^k B} |f^\#(y)|^p \left(\frac{1}{|2^{k+1}B|} \int_B w(z)\, dz \right) dy \right]
\end{aligned}$$

$$
\begin{aligned}
&\le C\left[\int_B |f^{\#}(y)|^p w(y)\,dy + \sum_{k=0}^{\infty}\int_{2^{k+1}B}|f^{\#}(y)|^p \frac{M(w)(y)}{2^{n(k+1)}}\,dy\right]\\
&\le C\left[\int_B |f^{\#}(y)|^p w(y)\,dy + \sum_{k=0}^{\infty}\int_{2^{k+1}B}|f^{\#}(y)|^p \frac{w(y)}{2^{nk}}\,dy\right]\\
&\le C\|f^{\#}\|^p_{L^{p,\varphi}(w)}\sum_{k=0}^{\infty}2^{-nk}\varphi(2^{k+1}d)\\
&\le C\|f^{\#}\|^p_{L^{p,\varphi}(w)}\sum_{k=0}^{\infty}(2^{-n}D)^k\varphi(d)\\
&\le C\|f^{\#}\|^p_{L^{p,\varphi}(w)}\varphi(d),
\end{aligned}
$$

thus,

$$\|M(f)\|_{L^{p,\varphi}(w)} \le C\|f^{\#}\|_{L^{p,\varphi}(w)}.$$

A similar argument as in the proof of (a) will give the proof of (b), we omit the details. ■

Lemma 3 (Main Lemma) *Let $D^\alpha b_j \in \mathrm{BMO}(\mathbb{R}^n)$ for all α with $|\alpha| = m_j$ and $j = 1, \ldots, l$. Then there exists a constant $C > 0$ such that for any $f \in C_0^\infty(\mathbb{R}^n)$, $1 < q < \infty$ and $x \in \mathbb{R}^n$,*

$$(T^b(f))^{\#}(x) \le C\prod_{j=1}^{l}\left(\sum_{|\alpha_j|=m_j}\|D^{\alpha_j}b_j\|_{\mathrm{BMO}}\right)M_q(f)(x).$$

PROOF. It suffices to prove for $f \in C_0^\infty(\mathbb{R}^n)$ and some constant C_0, the following inequality holds:

$$\frac{1}{|Q|}\int_Q |T^b(f)(x) - C_0|\,dx \le C\prod_{j=1}^{l}\left(\sum_{|\alpha_j|=m_j}\|D^{\alpha_j}b_j\|_{\mathrm{BMO}}\right)M_q(f)(x).$$

Without loss of generality, we may assume $l = 2$. Fix a cube $Q = Q(x_0, d)$ and $\tilde{x} \in Q$. Let $\tilde{Q} = 5\sqrt{n}Q$ and $\tilde{b}_j(x) = b_j(x) - \sum_{|\alpha|=m_j}\frac{1}{\alpha!}(D^\alpha b_j)_{\tilde{Q}}x^\alpha$, then $R_{m_j}(b_j; x, y) = R_{m_j}(\tilde{b}_j; x, y)$ and $D^\alpha\tilde{b}_j = D^\alpha b_j - (D^\alpha b_j)_{\tilde{Q}}$ for $|\alpha| = m_j$. We write, for $f_1 = f\chi_{\tilde{Q}}$ and $f_2 = f\chi_{\mathbb{R}^n\setminus\tilde{Q}}$,

$$
\begin{aligned}
F_t^b(f)(x) &= \int_{\mathbb{R}^n}\frac{\prod_{j=1}^{2}R_{m_j+1}(\tilde{b}_j; x, y)}{|x-y|^m}F_t(x,y)f(y)\,dy\\
&= \int_{\mathbb{R}^n}\frac{\prod_{j=1}^{2}R_{m_j+1}(\tilde{b}_j; x, y)}{|x-y|^m}F_t(x,y)f_2(y)\,dy\\
&\quad + \int_{\mathbb{R}^n}\frac{\prod_{j=1}^{2}R_{m_j}(\tilde{b}_j; x, y)}{|x-y|^m}F_t(x,y)f_1(y)\,dy\\
&\quad - \sum_{|\alpha_1|=m_1}\frac{1}{\alpha_1!}\int_{\mathbb{R}^n}\frac{R_{m_2}(\tilde{b}_2; x, y)(x-y)^{\alpha_1}}{|x-y|^m}D^{\alpha_1}\tilde{b}_1(y)F_t(x,y)f_1(y)\,dy\\
&\quad - \sum_{|\alpha_2|=m_2}\frac{1}{\alpha_2!}\int_{\mathbb{R}^n}\frac{R_{m_1}(\tilde{b}_1; x, y)(x-y)^{\alpha_2}}{|x-y|^m}D^{\alpha_2}\tilde{b}_2(y)F_t(x,y)f_1(y)\,dy\\
&\quad + \sum_{\substack{|\alpha_1|=m_1\\|\alpha_2|=m_2}}\frac{1}{\alpha_1!\alpha_2!}\int_{\mathbb{R}^n}\frac{(x-y)^{\alpha_1+\alpha_2}D^{\alpha_1}\tilde{b}_1(y)D^{\alpha_2}\tilde{b}_2(y)}{|x-y|^m}F_t(x,y)f_1(y)\,dy,
\end{aligned}
$$

then

$$
\begin{aligned}
&\left|T^b(f)(x) - T^{\tilde b}(f_2)(x_0)\right| = \left|\|F_t^b(f)(x)\| - \|F_t^{\tilde b}(f_2)(x_0)\|\right| \\
&\le \|F_t^b(f)(x) - F_t^{\tilde b}(f_2)(x_0)\| \\
&\le \left\|\int_{\mathbb{R}^n} \frac{\prod_{j=1}^2 R_{m_j}(\tilde b_j; x, y)}{|x-y|^m} F_t(x,y) f_1(y)\, dy\right\| && (:= I_1(x)) \\
&\quad + \left\|\sum_{|\alpha_1|=m_1} \frac{1}{\alpha_1!} \int_{\mathbb{R}^n} \frac{R_{m_2}(\tilde b_2; x, y)(x-y)^{\alpha_1}}{|x-y|^m} D^{\alpha_1}\tilde b_1(y) F_t(x,y) f_1(y)\, dy\right\| && (:= I_2(x)) \\
&\quad + \left\|\sum_{|\alpha_2|=m_2} \frac{1}{\alpha_2!} \int_{\mathbb{R}^n} \frac{R_{m_1}(\tilde b_1; x, y)(x-y)^{\alpha_2}}{|x-y|^m} D^{\alpha_2}\tilde b_2(y) F_t(x,y) f_1(y)\, dy\right\| && (:= I_3(x)) \\
&\quad + \left\|\sum_{\substack{|\alpha_1|=m_1,\\ |\alpha_2|=m_2}} \frac{1}{\alpha_1!\alpha_2!} \int_{\mathbb{R}^n} \frac{(x-y)^{\alpha_1+\alpha_2} D^{\alpha_1}\tilde b_1(y) D^{\alpha_2}\tilde b_2(y)}{|x-y|^m} F_t(x,y) f_1(y)\, dy\right\| && (:= I_4(x)) \\
&\quad + \left|T^{\tilde b}(f_2)(x) - T^{\tilde b}(f_2)(x_0)\right| && (:= I_5(x))
\end{aligned}
$$

thus

$$
\frac{1}{|Q|}\int_Q \left|T^b(f)(x) - T^{\tilde b}(f_2)(x_0)\right| dx \le I_1 + I_2 + I_3 + I_4 + I_5,
$$

where

$$
I_k = \frac{1}{|Q|}\int_Q I_k(x)\, dx, \qquad k = 1, \dots, 5.
$$

Now, let us estimate I_1, I_2, I_3, I_4 and I_5, respectively. First, for $x \in Q$ and $y \in \tilde Q$, by Lemma 1, we get

$$
R_m(\tilde b_j; x, y) \le C|x-y|^m \sum_{|\alpha_j|=m} \|D^{\alpha_j} b_j\|_{\mathrm{BMO}},
$$

thus, by the L^q-boundedness of T for $1 < q < \infty$, we obtain

$$
\begin{aligned}
I_1 &\le C \prod_{j=1}^2 \left(\sum_{|\alpha_j|=m_j} \|D^{\alpha_j} b_j\|_{\mathrm{BMO}}\right) \frac{1}{|Q|}\int_Q |T(f_1)(x)|\, dx \\
&\le C \prod_{j=1}^2 \left(\sum_{|\alpha_j|=m_j} \|D^{\alpha_j} b_j\|_{\mathrm{BMO}}\right) \left(\frac{1}{|Q|}\int_Q |T(f_1)(x)|^q\, dx\right)^{1/q} \\
&\le C \prod_{j=1}^2 \left(\sum_{|\alpha_j|=m_j} \|D^{\alpha_j} b_j\|_{\mathrm{BMO}}\right) |Q|^{-1/q} \left(\int_Q |f_1(x)|^q\, dx\right)^{1/q} \\
&\le C \prod_{j=1}^2 \left(\sum_{|\alpha_j|=m_j} \|D^{\alpha_j} b_j\|_{\mathrm{BMO}}\right) M_q(f)(\tilde x).
\end{aligned}
$$

For I_2, denoting $q = pr$ for $1 < p < \infty$, $1 < r < \infty$ and $1/r + 1/r' = 1$, we get, by Hölder's inequality,

$$\begin{aligned}
I_2 &\le C \sum_{|\alpha_2|=m_2} \|D^{\alpha_2} b_2\|_{\mathrm{BMO}} \sum_{|\alpha_1|=m_1} \frac{1}{|Q|}\int_Q |T(D^{\alpha_1}\tilde{b}_1 f_1)(x)|\,dx \\
&\le C \sum_{|\alpha_2|=m_2} \|D^{\alpha_2} b_2\|_{\mathrm{BMO}} \sum_{|\alpha_1|=m_1} \left(\frac{1}{|Q|}\int_{\mathbb{R}^n} |T(D^{\alpha_1}\tilde{b}_1 f_1)(x)|^p\,dx\right)^{1/p} \\
&\le C \sum_{|\alpha_2|=m_2} \|D^{\alpha_2} b_2\|_{\mathrm{BMO}} \sum_{|\alpha_1|=m_1} |Q|^{-1/r}\left(\int_{\mathbb{R}^n} |D^{\alpha_1}\tilde{b}_1(x) f_1(x)|^p\,dx\right)^{1/p} \\
&\le C \sum_{|\alpha_2|=m_2} \|D^{\alpha_2} b_2\|_{\mathrm{BMO}} \sum_{|\alpha_1|=m_1} \left(\frac{1}{|Q|}\int_{\tilde{Q}} |D^{\alpha_1}\tilde{b}_1(x)|^{pr'}\,dx\right)^{1/pr'} \left(\frac{1}{|Q|}\int_{\tilde{Q}} |f(x)|^{pr}\,dx\right)^{1/pr} \\
&\le C \prod_{j=1}^{2}\left(\sum_{|\alpha|=m_j} \|D^{\alpha} b_j\|_{\mathrm{BMO}}\right) M_q(f)(\tilde{x}).
\end{aligned}$$

For I_3, similar to the proof of I_2, we get

$$I_3 \le C \prod_{j=1}^{2}\left(\sum_{|\alpha|=m_j} \|D^{\alpha} b_j\|_{\mathrm{BMO}}\right) M_q(f)(\tilde{x}).$$

Similarly, for I_4, denoting $q = pr_3$ for $1 < p < \infty$, $r_1, r_2, r_3 > 1$ and $1/r_1 + 1/r_2 + 1/r_3 = 1$, we obtain

$$\begin{aligned}
I_4 &\le C \sum_{\substack{|\alpha_1|=m1 \\ |\alpha_2|=m2}} \frac{1}{|Q|}\int_Q |T(D^{\alpha_1}\tilde{b}_1 D^{\alpha_2}\tilde{b}_2 f_1)(x)|\,dx \\
&\le C \sum_{\substack{|\alpha_1|=m1 \\ |\alpha_2|=m2}} \left(\frac{1}{|Q|}\int_{\mathbb{R}^n} |T(D^{\alpha_1}\tilde{b}_1 D^{\alpha_2}\tilde{b}_2 f_1)(x)|^p\,dx\right)^{1/p} \\
&\le C \sum_{\substack{|\alpha_1|=m1 \\ |\alpha_2|=m2}} |Q|^{-1/p}\left(\int_{\mathbb{R}^n} |D^{\alpha_1}\tilde{b}_1(x) D^{\alpha_2}\tilde{b}_2(x) f_1(x)|^p\,dx\right)^{1/p} \\
&\le C \sum_{\substack{|\alpha_1|=m1 \\ |\alpha_2|=m2}} \left(\frac{1}{|Q|}\int_{\tilde{Q}} |D^{\alpha_1}\tilde{b}_1(x)|^{pr_1}\,dx\right)^{1/pr_1} \left(\frac{1}{|Q|}\int_{\tilde{Q}} |D^{\alpha_2}\tilde{b}_2(x)|^{pr_2}\,dx\right)^{1/pr_2} \\
&\quad \times \left(\frac{1}{|Q|}\int_{\tilde{Q}} |f(x)|^{pr_3}\,dx\right)^{1/pr_3} \\
&\le C \prod_{j=1}^{2}\left(\sum_{|\alpha|=m_j} \|D^{\alpha} b_j\|_{\mathrm{BMO}}\right) M_q(f)(\tilde{x}).
\end{aligned}$$

For I_5, we write

$$F_t^{\tilde{b}}(f_2)(x) - F_t^{\tilde{b}}(f_2)(x_0) = \int_{\mathbb{R}^n}\left(\frac{F_t(x,y)}{|x-y|^m} - \frac{F_t(x_0,y)}{|x_0-y|^m}\right)\prod_{j=1}^{2} R_{m_j}(\tilde{b}_j; x, y) f_2(y)\,dy \qquad (:= I_5^{(1)})$$

$$+\int_{\mathbb{R}^n}\Big(R_{m_1}(\tilde b_1;x,y)-R_{m_1}(\tilde b_1;x_0,y)\Big)\frac{R_{m_2}(\tilde b_2;x,y)}{|x_0-y|^m}F_t(x_0,y)f_2(y)\,dy \qquad (:=I_5^{(2)})$$

$$+\int_{\mathbb{R}^n}\Big(R_{m_2}(\tilde b_2;x,y)-R_{m_2}(\tilde b_2;x_0,y)\Big)\frac{R_{m_1}(\tilde b_1;x_0,y)}{|x_0-y|^m}F_t(x_0,y)f_2(y)\,dy \qquad (:=I_5^{(3)})$$

$$-\sum_{|\alpha_1|=m_1}\frac{1}{\alpha_1!}\int_{\mathbb{R}^n}\left[\frac{R_{m_2}(\tilde b_2;x,y)(x-y)^{\alpha_1}}{|x-y|^m}F_t(x,y)-\frac{R_{m_2}(\tilde b_2;x_0,y)(x_0-y)^{\alpha_1}}{|x_0-y|^m}F_t(x_0,y)\right]$$
$$\times D^{\alpha_1}\tilde b_1(y)f_2(y)\,dy \qquad (:=I_5^{(4)})$$

$$-\sum_{|\alpha_2|=m_2}\frac{1}{\alpha_2!}\int_{\mathbb{R}^n}\left[\frac{R_{m_1}(\tilde b_1;x,y)(x-y)^{\alpha_2}}{|x-y|^m}F_t(x,y)-\frac{R_{m_1}(\tilde b_1;x_0,y)(x_0-y)^{\alpha_2}}{|x_0-y|^m}F_t(x_0,y)\right]$$
$$\times D^{\alpha_2}\tilde A_2(y)f_2(y)\,dy \qquad (:=I_5^{(5)})$$

$$+\sum_{\substack{|\alpha_1|=m_1\\|\alpha_2|=m_2}}\frac{1}{\alpha_1!\alpha_2!}\int_{\mathbb{R}^n}\left[\frac{(x-y)^{\alpha_1+\alpha_2}}{|x-y|^m}F_t(x,y)-\frac{(x_0-y)^{\alpha_1+\alpha_2}}{|x_0-y|^m}F_t(x_0,y)\right]$$
$$\times D^{\alpha_1}\tilde b_1(y)D^{\alpha_2}\tilde b_2(y)f_2(y)\,dy \qquad (:=I_5^{(6)})$$

By Lemma 1 and the following inequality (see [21])

$$|b_{Q_1}-b_{Q_2}|\le C\log(|Q_2|/|Q_1|)\|b\|_{\mathrm{BMO}} \qquad \text{for } Q_1\subset Q_2,$$

we have, for $x\in Q$ and $y\in 2^{k+1}\tilde Q\setminus 2^k\tilde Q$,

$$|R_m(\tilde b;x,y)|\le C|x-y|^m\sum_{|\alpha|=m}\left(\frac{1}{|\tilde Q(x,y)|}\int_{\tilde Q(x,y)}|D^\alpha b(z)-(D^\alpha b)_{\tilde Q}|^q\,dz\right)^{1/q}$$
$$\le C|x-y|^m\sum_{|\alpha|=m}\left(\frac{1}{|\tilde Q(x,y)|}\int_{\tilde Q(x,y)}(|D^\alpha b(z)-(D^\alpha b)_{\tilde Q(x,y)}|+|(D^\alpha b)_{\tilde Q(x,y)}-(D^\alpha b)_{\tilde Q}|^q\,dz\right)^{\frac{1}{q}}$$
$$\le C|x-y|^m\sum_{|\alpha|=m}(\|D^\alpha b\|_{\mathrm{BMO}}+\log(|\tilde Q(x,y)/\tilde Q|)\|D^\alpha b\|_{\mathrm{BMO}})$$
$$\le Ck|x-y|^m\sum_{|\alpha|=m}\|D^\alpha b\|_{\mathrm{BMO}}.$$

Note that $|x-y|\sim|x_0-y|$ for $x\in Q$ and $y\in\mathbb{R}^n\setminus\tilde Q$, we obtain, by the condition on F_t,

$$\|I_5^{(1)}\|\le C\int_{\mathbb{R}^n}\left\|\frac{F_t(x,y)}{|x-y|^m}-\frac{F_t(x,y)}{|x_0-y|^m}+\frac{F_t(x,y)}{|x_0-y|^m}-\frac{F_t(x_0,y)}{|x_0-y|^m}\right\|\prod_{j=1}^{2}|R_{m_j}(\tilde b_j;x,y)|\,|f_2(y)|\,dy$$
$$\le C\int_{\mathbb{R}^n\setminus\tilde Q}\left(\frac{|x-x_0|}{|x_0-y|^{m+n+1}}+\frac{|x-x_0|^\varepsilon}{|x_0-y|^{m+n+\varepsilon}}\right)\prod_{j=1}^{2}|R_{m_j}(\tilde b_j;x,y)|\,|f(y)|\,dy$$
$$\le C\prod_{j=1}^{2}\left(\sum_{|\alpha|=m_j}\|D^\alpha b_j\|_{\mathrm{BMO}}\right)\sum_{k=0}^{\infty}\int_{2^{k+1}\tilde Q\setminus 2^k\tilde Q}k^2\left(\frac{|x-x_0|}{|x_0-y|^{n+1}}+\frac{|x-x_0|^\varepsilon}{|x_0-y|^{n+\varepsilon}}\right)|f(y)|\,dy$$
$$\le C\prod_{j=1}^{2}\left(\sum_{|\alpha|=m_j}\|D^\alpha b_j\|_{\mathrm{BMO}}\right)\sum_{k=1}^{\infty}k^2(2^{-k}+2^{-\varepsilon k})\frac{1}{|2^k\tilde Q|}\int_{2^k\tilde Q}|f(y)|\,dy$$

$$\leq C\prod_{j=1}^{2}\left(\sum_{|\alpha|=m_j}\|D^\alpha b_j\|_{\mathrm{BMO}}\right)\sum_{k=1}^{\infty}k^2(2^{-k}+2^{-\varepsilon k})M_q(f)(\tilde{x})$$
$$\leq C\prod_{j=1}^{2}\left(\sum_{|\alpha|=m_j}\|D^\alpha b_j\|_{\mathrm{BMO}}\right)M_q(f)(\tilde{x});$$

For $I_5^{(2)}$, by the formula (see [4]):

$$R_m(\tilde{b};x,y)-R_m(\tilde{b};x_0,y)=\sum_{|\beta|<m}\frac{1}{\beta!}R_{m-|\beta|}(D^\beta\tilde{b};x,x_0)(x-y)^\beta$$

and Lemma 1, we have

$$|R_m(\tilde{b};x,y)-R_m(\tilde{b};x_0,y)|\leq C\sum_{|\beta|<m}\sum_{|\alpha|=m}|x-x_0|^{m-|\beta|}|x-y|^{|\beta|}\|D^\alpha b\|_{\mathrm{BMO}},$$

thus

$$\|I_5^{(2)}\|\leq C\prod_{j=1}^{2}\left(\sum_{|\alpha|=m_j}\|D^\alpha b_j\|_{\mathrm{BMO}}\right)\sum_{k=0}^{\infty}\int_{2^{k+1}\tilde{Q}\setminus 2^k\tilde{Q}}k\frac{|x-x_0|}{|x_0-y|^{n+1}}|f(y)|\,dy$$
$$\leq C\prod_{j=1}^{2}\left(\sum_{|\alpha|=m_j}\|D^\alpha b_j\|_{\mathrm{BMO}}\right)M_q(f)(\tilde{x});$$

Similarly,

$$\|I_5^{(3)}\|\leq C\prod_{j=1}^{2}\left(\sum_{|\alpha|=m_j}\|D^\alpha b_j\|_{\mathrm{BMO}}\right)M_q(f)(\tilde{x});$$

For $I_5^{(4)}$, we get

$$\|I_5^{(4)}\|\leq C\sum_{|\alpha_1|=m_1}\int_{\mathbb{R}^n}\left\|\frac{(x-y)^{\alpha_1}F_t(x,y)}{|x-y|^m}-\frac{(x_0-y)^{\alpha_1}F_t(x_0,y)}{|x_0-y|^m}\right\||R_{m_2}(\tilde{b}_2;x,y)|\,|D^{\alpha_1}\tilde{b}_1(y)||f_2(y)|\,dy$$
$$+C\sum_{|\alpha_1|=m_1}\int_{\mathbb{R}^n}|R_{m_2}(\tilde{b}_2;x,y)-R_{m_2}(\tilde{b}_2;x_0,y)|\frac{|(x_0-y)^{\alpha_1}|\|F_t(x_0,y)\|}{|x_0-y|^m}|D^{\alpha_1}\tilde{b}_1(y)||f_2(y)|\,dy$$
$$\leq C\sum_{|\alpha_1|=m_1}\int_{\mathbb{R}^n\setminus\tilde{Q}}\left(\frac{|x-x_0|}{|x_0-y|^{m+n+1}}+\frac{|x-x_0|^\varepsilon}{|x_0-y|^{m+n+\varepsilon}}\right)|R_{m_2}(\tilde{b}_2;x,y)|\,|D^{\alpha_1}\tilde{b}_1(y)|\,|f(y)|\,dy$$
$$\leq C\sum_{|\alpha|=m_2}\|D^\alpha b_2\|_{\mathrm{BMO}}\sum_{k=1}^{\infty}k(2^{-k}+2^{-\varepsilon k})\left(\frac{1}{|2^k\tilde{Q}|}\int_{2^k\tilde{Q}}|f(y)|^q\,dy\right)^{1/q}$$
$$\times\sum_{|\alpha_1|=m_1}\left(\frac{1}{|2^k\tilde{Q}|}\int_{2^k\tilde{Q}}|D^{\alpha_1}\tilde{b}_1(y)-(D^\alpha b_1)_{\tilde{Q}}|^{q'}\,dy\right)^{1/q'}$$
$$\leq C\prod_{j=1}^{2}\left(\sum_{|\alpha|=m_j}\|D^\alpha b_j\|_{\mathrm{BMO}}\right)M_q(f)(\tilde{x});$$

Similarly,

$$\|I_5^{(5)}\| \le C \prod_{j=1}^{2} \left(\sum_{|\alpha|=m_j} \|D^\alpha b_j\|_{\mathrm{BMO}} \right) M_q(f)(\tilde{x});$$

For $I_5^{(6)}$, taking $r_1, r_2 > 1$ such that $1/q + 1/r_1 + 1/r_2 = 1$, then

$$\begin{aligned}
\|I_5^{(6)}\| &\le C \sum_{\substack{|\alpha_1|=m_1\\|\alpha_2|=m_2}} \int_{\mathbb{R}^n} |D^{\alpha_1}\tilde{b}_1(y)||D^{\alpha_2}\tilde{b}_2(y)||f_2(y)| \\
&\quad \times \left\| \frac{(x-y)^{\alpha_1+\alpha_2} F_t(x,y)}{|x-y|^m} - \frac{(x_0-y)^{\alpha_1+\alpha_2} F_t(x_0,y)}{|x_0-y|^m} \right\| dy \\
&\le C \sum_{\substack{|\alpha_1|=m_1\\|\alpha_2|=m_2}} \int_{\mathbb{R}^n \setminus \tilde{Q}} \left(\frac{|x-x_0|}{|x_0-y|^{n+1}} + \frac{|x-x_0|^\varepsilon}{|x_0-y|^{n+\varepsilon}} \right) |D^{\alpha_1}\tilde{b}_1(y)|\,|D^{\alpha_2}\tilde{b}_2(y)|\,|f(y)|\,dy \\
&\le C \sum_{\substack{|\alpha_1|=m_1\\|\alpha_2|=m_2}} \sum_{k=1}^{\infty} k(2^{-k} + 2^{-\varepsilon k}) \left(\frac{1}{|2^k\tilde{Q}|} \int_{2^k\tilde{Q}} |f(y)|^q\,dy \right)^{1/q} \\
&\quad \times \left(\frac{1}{|2^k\tilde{Q}|} \int_{2^k\tilde{Q}} |D^{\alpha_1}\tilde{b}_1(y)|^{r_1}\,dy \right)^{1/r_1} \left(\frac{1}{|2^k\tilde{Q}|} \int_{2^k\tilde{Q}} |D^{\alpha_2}\tilde{A}_2(y)|^{r_2}\,dy \right)^{1/r_2} \\
&\le C \prod_{j=1}^{2} \left(\sum_{|\alpha|=m_j} \|D^\alpha b_j\|_{\mathrm{BMO}} \right) M_q(f)(\tilde{x});
\end{aligned}$$

Thus

$$\|I_5\| \le C \prod_{j=1}^{2} \left(\sum_{|\alpha|=m_j} \|D^\alpha b_j\|_{\mathrm{BMO}} \right) M_q(f)(\tilde{x}).$$

This completes the proof of Main Lemma. ■

PROOF OF THEOREM 1. Taking $1 < q < p$ in Main Lemma, by Lemma 2, we obtain

$$\begin{aligned}
\|T^b(f)\|_{L^{p,\varphi}(w)} &\le C\|M(T^b(f))\|_{L^{p,\varphi}(w)} \\
&\le C\|(T^A(f))^\#\|_{L^{p,\varphi}(w)} \\
&\le C \prod_{j=1}^{2} \left(\sum_{|\alpha|=m_j} \|D^\alpha b_j\|_{\mathrm{BMO}} \right) \|M_q(f)\|_{L^{p,\varphi}(w)} \\
&\le C \prod_{j=1}^{2} \left(\sum_{|\alpha|=m_j} \|D^\alpha b_j\|_{\mathrm{BMO}} \right) \|f\|_{L^{p,\varphi}(w)}.
\end{aligned}$$

This finishes the proof. ■

3 Applications

In this section we shall apply the theorem of the paper to several operators such as the Littlewood-Paley operator, Marcinkiewicz operator and Bochner-Riesz operator.

3.1 Application 1: Littlewood-Paley operator

Let $\varepsilon > 0$ and ψ be a fixed function which satisfies the following properties:

1. $\int_{\mathbb{R}^n} \psi(x)\,dx = 0$,
2. $|\psi(x)| \le C(1+|x|)^{-(n+1)}$,
3. $|\psi(x+y) - \psi(x)| \le C|y|^\varepsilon(1+|x|)^{-(n+1+\varepsilon)}$ when $2|y| < |x|$;

The multilinear Littlewood-Paley operator is defined by

$$g_\psi^b(f)(x) = \left(\int_0^\infty |F_t^b(f)(x)|^2 \,\frac{dt}{t}\right)^{1/2},$$

where

$$F_t^b(f)(x) = \int_{\mathbb{R}^n} \frac{\prod_{j=1}^l R_{m_j+1}(b_j; x, y)}{|x-y|^m} \psi_t(x-y) f(y)\,dy$$

and $\psi_t(x) = t^{-n}\psi(x/t)$ for $t > 0$. Set $F_t(f) = \psi_t * f$. We also define that

$$g_\psi(f)(x) = \left(\int_0^\infty |F_t(f)(x)|^2 \,\frac{dt}{t}\right)^{1/2},$$

which is the Littlewood-Paley operator (see [22]).

Let H be the space $H = \left\{h : \|h\| = \left(\int_0^\infty |h(t)|^2\,dt/t\right)^{1/2} < \infty\right\}$, then, for each fixed $x \in \mathbb{R}^n$, $F_t^A(f)(x)$ may be viewed as a mapping from $[0, +\infty)$ to H, and it is clear that

$$g_\psi(f)(x) = \|F_t(f)(x)\| \qquad \text{and} \qquad g_\psi^b(f)(x) = ||F_t^b(f)(x)||.$$

We know that g_ψ is bounded on $L^p(\mathbb{R}^n, w)$ for $1 < p < \infty$ and $w \in A_1$ (see [21]). Thus

$$(1) \qquad (g_\psi^b(f))^\#(x) \le C \prod_{j=1}^l \left(\sum_{|\alpha_j| = m_j} \|D^{\alpha_j} b_j\|_{\mathrm{BMO}}\right) M_q(f)(x)$$

for any $f \in C_0^\infty(\mathbb{R}^n)$ and $1 < q < \infty$;

$$(2) \qquad \|g_\psi^b(f)\|_{L^{p,\varphi}(w)} \le C \prod_{j=1}^l \left(\sum_{|\alpha_j| = m_j} \|D^{\alpha_j} b_j\|_{\mathrm{BMO}}\right) \|f\|_{L^{p,\varphi}(w)}$$

for $0 < D < 2^n$ and any $w \in A_1$, $1 < p < \infty$.

3.2 Application 2: Marcinkiewicz operator

Let $\delta \ge 0$, $0 < \gamma \le 1$ and Ω be homogeneous of degree zero on $\mathbb{R}^n$ with $\int_{S^{n-1}} \Omega(x')\,d\sigma(x') = 0$. Assume that $\Omega \in Lip_\gamma(S^{n-1})$, that is there exists a constant $M > 0$ such that for any x, $y \in S^{n-1}$, $|\Omega(x) - \Omega(y)| \le M|x-y|^\gamma$. The multilinear Marcinkiewicz operator is defined by

$$\mu_\Omega^b(f)(x) = \left(\int_0^\infty |F_t^b(f)(x)|^2 \,\frac{dt}{t^3}\right)^{1/2},$$

where

$$F_t^b(f)(x) = \int_{|x-y|\le t} \frac{\prod_{j=1}^l R_{m_j+1}(b_j; x, y)}{|x-y|^m} \frac{\Omega(x-y)}{|x-y|^{n-1}} f(y)\,dy,$$

set

$$F_t(f)(x) = \int_{|x-y|\le t} \frac{\Omega(x-y)}{|x-y|^{n-1}} f(y)\,dy.$$

We also define that

$$\mu_\Omega(f)(x) = \left(\int_0^\infty |F_t(f)(x)|^2 \frac{dt}{t^3}\right)^{1/2}$$

which is the Marcinkiewicz operator (see [23]).

Let H be the space $H = \left\{h : \|h\| = \left(\int_0^\infty |h(t)|^2\,dt/t^3\right)^{1/2} < \infty\right\}$. Then, it is clear that

$$\mu_\Omega(f)(x) = \|F_t(f)(x)\| \qquad \text{and} \qquad \mu_\Omega^b(f)(x) = \|F_t^b(f)(x)\|.$$

We know that μ_Ω is bounded on $L^p(\mathbb{R}^n, w)$ for $1 < p < \infty$ and $w \in A_1$ (see [22]). Thus

$$(1) \qquad (\mu_\Omega^b(f))^{\#}(x) \le C \prod_{j=1}^{l} \left(\sum_{|\alpha_j|=m_j} \|D^{\alpha_j} b_j\|_{\mathrm{BMO}}\right) M_q(f)(x)$$

for any $f \in C_0^\infty(\mathbb{R}^n)$ and $1 < q < \infty$;

$$(2) \qquad \|\mu_\Omega^b(f)\|_{L^{p,\varphi}(w)} \le C \prod_{j=1}^{l} \left(\sum_{|\alpha_j|=m_j} \|D^{\alpha_j} b_j\|_{\mathrm{BMO}}\right) \|f\|_{L^{p,\varphi}(w)}$$

for $0 < D < 2^n$ and any $w \in A_1$, $1 < p < \infty$.

3.3 Application 3: Bochner-Riesz operator

Let $\delta > (n-1)/2$, $B_t^\delta \widehat{(f)(\xi)} = (1 - t^2|\xi|^2)_+^\delta \hat{f}(\xi)$. Denote that

$$B_{\delta,t}^b(f)(x) = \int_{\mathbb{R}^n} \frac{\prod_{j=1}^l R_{m_j+1}(b_j; x, y)}{|x-y|^m} B_t^\delta(x-y) f(y)\,dy,$$

where $B_t^\delta(z) = t^{-n} B^\delta(z/t)$ for $t > 0$. The maximal multilinear Bochner-Riesz operator is defined by

$$B_{\delta,*}^b(f)(x) = \sup_{t>0} |B_{\delta,t}^b(f)(x)|;$$

We also define that

$$B_{\delta,*}(f)(x) = \sup_{t>0} |B_t^\delta(f)(x)|$$

which is the Bochner-Riesz operator (see [15]).

Let H be the space $H = \{h : \|h\| = \sup_{t>0} |h(t)| < \infty\}$, then it is clear that

$$B_*^\delta(f)(x) = \|B_t^\delta(f)(x)\| \qquad \text{and} \qquad B_{\delta,*}^b(f)(x) = \|B_{\delta,t}^b(f)(x)\|.$$

We know that $B_{\delta,*}$ is bounded on $L^p(\mathbb{R}^n, w)$ for $1 < p < \infty$ and $w \in A_1$ (see [14]). Thus

$$(1) \qquad (B_{\delta,*}^b(f))^{\#}(x) \le C \prod_{j=1}^{l} \left(\sum_{|\alpha_j|=m_j} \|D^{\alpha_j} b_j\|_{\mathrm{BMO}}\right) M_q(f)(x)$$

for any $f \in C_0^\infty(\mathbb{R}^n)$ and $1 < q < \infty$;

$$(2) \qquad \|B_{\delta,*}^b(f)\|_{L^{p,\varphi}(w)} \le C \prod_{j=1}^{l} \left(\sum_{|\alpha_j|=m_j} \|D^{\alpha_j} b_j\|_{\mathrm{BMO}} \right) \|f\|_{L^{p,\varphi}(w)}$$

for $0 < D < 2^n$ and any $w \in A_1$, $1 < p < \infty$.

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