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Reflexivity of spaces of weakly summable sequences

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Abstract. We deal with the space of Λ-summable sequences from a locally convex space E, where Λ is a metrizable perfect sequence space. We give a characterization of the reflexivity of $\Lambda(E)$ in terms of that of Λ and E and the AK property. In particular, we prove that if Λ is an echelon sequence space and E is a Fréchet space then $\Lambda(E)$ is reflexive if and only if Λ and E are reflexive.

Reflexividad de espacios de sucesiones débilmente sumables

Resumen. Consideramos el espacio de las sucesiones Λ-sumables en un espacio localmente convexo E, donde Λ es un espacio de sucesiones perfecto y metrizable. Damos una caracterización de la reflexividad de $\Lambda(E)$ en términos de la de Λ y E y de la propiedad AK. En particular, demostramos que si Λ es un espacio escalonado y E es un espacio de Fréchet entonces $\Lambda(E)$ es reflexivo si y solo si Λ y E son reflexivos.

1 Introduction

The spaces $\ell_p[E]$ and $\ell_p\{E\}$ respectively of weakly ℓ_p-summable and absolutely ℓ_p-summable sequences in a locally convex space E were first introduced by A. Pietsch [11] in connection with the nuclearity of E. This allowed him also to introduce and study the absolutely p-summing operators. Later, in the case when E is a normed space, J. S. Cohen [2] introduced the space $\ell_p\langle E\rangle$ of strongly p-summable sequences. He used this space together with the spaces $\ell_p[E]$ and $\ell_p\{E\}$ to define the strongly and the nuclear p-summing operators. The definition of $\ell_p\langle E\rangle$ was generalized to an arbitrary locally convex space E by H. Apiola [1] in order to get new conditions for the nuclearity of E. H. Apiola studied the duality relations between the three spaces, namely $\ell_p[E]$, $\ell_p\{E\}$ and $\ell_p\langle E\rangle$. In [11], A. Pietsch introduced and studied also the space $\Lambda(E)$ of Λ-summable sequences in E, Λ being a perfect sequence space in the sense of Köthe endowed with its normal topology. M. Florencio and P. J. Paúl [4] considered the general case where Λ is no longer equipped with the normal topology, but with a general polar one. They obtained results on $\Lambda(E)$ such as the characterization of the AK property and then the relationship with the completion $\Lambda\widetilde{\otimes}_\epsilon E$ of the injective tensor product $\Lambda \otimes_\epsilon E$. In [9], the authors gave a definition of strongly Λ-summable sequences. They then reconsidered the space $\Lambda(E)$ and obtained some of its properties. They mainly described the continuous dual space of $\Lambda(E)$ in terms of strongly Λ^*-summable sequences in E', Λ^* being the α-dual of Λ and E' the dual of E. In this note, we are concerned with the reflexivity of the locally convex space $\Lambda(E)$. After a section giving preliminary results and definitions, we exhibit, in section 3, a fundamental family of bounded sets in $\Lambda(E)$. This allows us to characterize its strong dual space $\Lambda(E)'_\beta$. In section 4, we endow the space $\Lambda\langle E\rangle$ of all strongly Λ-summable sequences in E with a natural topology in the spirit of [1] for

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$\ell_p\langle E\rangle$. We then describe the continuous dual of $\Lambda\langle E\rangle$ in terms of weakly Λ^*-summable sequences of E'. The section 5 is devoted to the reflexivity of $\Lambda(E)$. We show that if Λ and E are Fréchet spaces, then $\Lambda(E)'_r = \Lambda^*\langle E'_\beta\rangle$, where $\Lambda(E)_r$ is the subspace of $\Lambda(E)$ consisting of the sequences which are the limit of their finite sections. The equality above turns out to be topological if E happens to be semi-reflexive. We then get that, for Fréchet spaces Λ and E, $\Lambda(E)$ is reflexive if and only if E and Λ are reflexive and the spaces $\Lambda(E)$ and $\Lambda^*\langle E'_\beta\rangle$ are AK. As a consequence, whenever Λ is an echelon space, $\Lambda(E)$ is reflexive if and only if E and Λ are. Using a result of [4], this gives that, in this case, $\Lambda\widetilde{\otimes}_\epsilon E$ is reflexive if and only if Λ and E are.

2 Preliminaries

Throughout this paper, Λ will be a perfect sequence space and E a sequentially complete Hausdorff locally convex space. The Köthe dual space of Λ will be denoted by Λ^* while E' will stand for the topological dual of E. The collection of all absolutely convex, $\sigma(E',E)$-closed and equicontinuous subsets of E' will be denoted by $\mathcal{M}$, while $\mathcal{S}$ will denotes a collection of closed, absolutely convex, normal and $\sigma(\Lambda^*,\Lambda)$-bounded subsets of Λ^* such that Λ^* is the union of the members of $\mathcal{S}$ and the latter is stable by homothety. We will then consider on Λ the polar topology $\tau_{\mathcal{S}}$ associated with the collection $\mathcal{S}$. This topology is generated by the seminorms

$$P_S(\alpha) := \sup\left\{\sum_n |\alpha_n\beta_n|,\ \beta = (\beta_n)_n \in S\right\}, \qquad S \in \mathcal{S}.$$

For an absolutely convex bounded subset A of a Hausdorff topological vector space F, let us denote by F_A the subspace of F generated by A. When no topology is specified on F_A, it will be endowed with the gauge $\|\cdot\|_A$ of A as a norm. We will then consider without any further mention the spaces E_B, E'_M, Λ_R and Λ^*_S, where B is a bounded subset of E, $M \in \mathcal{M}$, $S \in \mathcal{S}$ and R is a bounded absolutely convex subset of Λ. For every $M \in \mathcal{M}$, consider on E the seminorm P_M defined by

$$P_M(x) = \sup\{|a(x)|,\ a \in M\}$$

and by $E_{(M^\circ)}$ the quotient space of E by the annihilator $M^\perp$ of P_M. It is well known (see e.g. [7, Prop. 8.6.9]) that the topological dual space $(E_{(M^\circ)})'$, when $E_{(M^\circ)}$ is equipped with the associated quotient norm with P_M, is isometrically isomorphic to the Banach spaces E'_M.

A sequence $(x_n)_n \subset E$ is said to be Λ-summable if the series $\Sigma\alpha_n x_n$ converges in E for all $(\alpha_n)_n$ in Λ^*. It is weakly Λ-summable if $(a(x)_n)_n \in \Lambda$, for all $a \in E'$. The space of all Λ-summable sequences from E will be denoted by $\Lambda(E)$, while that of the weakly Λ-summable ones will be designated by $\Lambda[E]$. Similarly, $\Lambda^*_S[E'_M]$ will stand for the weakly Λ^*_S-summable sequences from E'_M, $S \in \mathcal{S}$ and $M \in \mathcal{M}$. Following [2] and [9], we will say that the sequence $(x_n)_n$ is strongly Λ-summable if for every $M \in \mathcal{M}$, the series $\Sigma a_n(x_n)$ converges for all $(a_n)_n \in \Lambda^*[E'_M]$. The space of all such sequences will be denoted by $\Lambda\langle E\rangle$. The three spaces are linear and, since Λ is perfect, the following inclusions hold: $\Lambda\langle E\rangle \subset \Lambda(E) \subset \Lambda[E]$.

Following [4], $\Lambda(E)$ will be equipped with the topology $\epsilon_{\mathcal{M},\mathcal{S}}$ generated by the family $(\epsilon_{S,M})_{S\in\mathcal{S},M\in\mathcal{M}}$ of seminorms, defined by

$$\epsilon_{S,M}(x) := \sup\left\{\sum_{n=1}^{\infty} |\alpha_n a(x_n)|,\ a \in M,\ \alpha = (\alpha_n)_{n\in\mathbb{N}} \in S\right\}, \qquad \forall x = (x_n)_n \in \Lambda(E).$$

These seminorms turn out to be defined also on $\Lambda[E]$ so that $\Lambda(E)$ is a closed topological subspace of $\Lambda[E]$. Both spaces will henceforth be equipped with this topology. The subspace $\Lambda(E)_r$ (resp. $\Lambda[E]_r$) consisting of those sequences $x = (x_n)_n$ belonging to $\Lambda(E)$ (resp. to $\Lambda[E]$) which are limits of their finite sections

$x^{(n)}$ will come in force in the sequel. Here, if e_n is the scalar sequence whose components are all zero except the n^{th} which equals 1, then

$$x^{(n)} = (x_1, x_2, \dots, x_n, 0, 0, \dots) = \sum_{i=1}^{n} x_i e_i.$$

Note that, if E and $(\Lambda, \tau_{\mathcal{S}})$ happen to be metrizable, then so is also $\Lambda[E]$. Moreover, if E and $(\Lambda, \tau_{\mathcal{S}})$ are Fréchet spaces, then so are also $\Lambda[E]$, $\Lambda(E)$ and their closed subspaces $\Lambda[E]_r$ and $\Lambda(E)_r$.

We refer the reader to Section 30 of [8] and Chapter 2 of [13] for details concerning Köthe theory of sequence spaces and to [7] for the terminology and notations concerning the general theory of locally convex spaces.

All the vector spaces considered here will be spaces on the field $\mathbb{K}$ of real or complex numbers.

3 Bounded sets of $\Lambda(E)$

If B and R are closed absolutely convex bounded subsets respectively of E and Λ, set

$$R(B) := \{(x_n)_n \in \Lambda(E) : \ \forall x' \in B^\circ, \ (x'(x_n))_n \in R\}.$$

It is easily seen that $R(B)$ is an absolutely convex subset of $\Lambda(E)$ and that

$$R(B) = \left\{ (x_n)_n \in \Lambda(E) : \ \forall \alpha = (\alpha_n)_n \in R^\circ, \ \sum_n \alpha_n x_n \in B \right\}.$$

Proposition 1 *If B and R are closed absolutely convex bounded subsets respectively of E and Λ with R normal, then $R(B)$ is a bounded subset of $\Lambda(E)$. Moreover, $R(B) \subset \Lambda_R[E_B]$.*

PROOF. It is obvious that $R(B)$ is absolutely convex. Now, if $x = (x_n)_n \in R(B)$, $M \in \mathcal{M}$ and $S \in \mathcal{S}$, then there are $r, s > 0$ so that $M \subset rB^\circ$ and $S \subset sR^\circ$. Hence

$$\begin{aligned}
\epsilon_{S,M}(x) &= \sup \left\{ \sum_{n=1}^{\infty} |\alpha_n a(x_n)|, \ a \in M, \ \alpha = (\alpha_n)_{n \in} S \right\} \\
&= s\, r\, \sup \left\{ \sum_{n=1}^{\infty} \left| \frac{\alpha_n}{s} \frac{a}{r}(x_n) \right|, \ a \in M, \ \alpha \in S \right\} \\
&\leq s\, r\, \sup \left\{ \sum_{n=1}^{\infty} |\alpha_n a(x_n)|, \ a \in B^\circ, \ \alpha \in R^\circ \right\} \\
&\leq r\, s.
\end{aligned}$$

Whereby $R(B)$ is bounded in $\Lambda(E)$. Now, let $x = (x_n)_n$ be an element of $R(B)$ and φ a continuous linear functional on E_B. Then there exists $K > 0$ such that, for all $b \in B$, $|\varphi(b)| \leq K$. Let $\alpha = (\alpha_n)_n \in \Lambda^*$ and $\mu > 0$ so that $\alpha \in \mu R^\circ$. Since R° is normal, for every $k \in \mathbb{N}$, the k^{th} finite section $\alpha^{(k)}$ of α belongs to μR°. Hence $\sum_{n=1}^{k} \alpha_n x_n = \mu \sum_{n=1}^{k} \mu^{-1} \alpha_n x_n \in \mu B$ and $\left| \sum_{n=1}^{k} \alpha_n \varphi(x_n) \right| \leq \mu K$. Therefore $(x_n)_n \subset E_B$. Let $(\epsilon_n)_n$ be a scalar sequence with $|\alpha_n \varphi(x_n)| = \epsilon_n \alpha_n \varphi(x_n)$, for all $n \in \mathbb{N}$. Thanks to the normality of R°, $(\epsilon_n \alpha_n)_n \in \mu R^\circ$ and therefore $\sum_{n=1}^{k} \epsilon_n \alpha_n x_n \in \mu B$. So,

$$\begin{aligned}
\sum_{n=1}^{k} |\alpha_n \varphi(x_n)| &= \sum_{n=1}^{k} \epsilon_n \alpha_n \varphi(x_n) \\
&= \varphi(\sum_{n=1}^{k} \epsilon_n \alpha_n x_n) \leq \mu K.
\end{aligned}$$

Thus, the series $\sum \alpha_n \varphi(x_n)$ is absolutely convergent with

$$\sum_{n=1}^{\infty} |\alpha_n \varphi(x_n)| \leq \mu K,$$

showing that $(\varphi(x_n))_n \in \Lambda$. Now, if $\alpha \in R^\circ$ then $\sum_{n=1}^{\infty} |\alpha_n \varphi(x_n)| \leq K$. That is $(\varphi(x_n))_n \in KR^{\circ\circ} = KR$. Hence $(\varphi(x_n))_n \in \Lambda_R$, whereby $x \in \Lambda_R[E_B]$. ■

The following result characterizes the bounded subsets of $\Lambda(E)$ by means of the $R(B)$'s, when E and Λ are metrizable.

Proposition 2 *If E and Λ are metrizable. Then, for every bounded subset $\mathcal{B}$ of $\Lambda(E)$, there exist closed absolutely convex bounded subsets B and R respectively of E and Λ with R normal such that $\mathcal{B} \subset R(B)$.*

PROOF. Since E and Λ are metrizable, $\mathcal{S}$ and $\mathcal{M}$ admit fundamental sequences respectively $(S_k)_{k\in\mathbb{N}}$ and $(M_p)_{p\in\mathbb{N}}$. As $\mathcal{B}$ is bounded, for every $p \in \mathbb{N}$,

$$c_{k,p} := \sup\left\{\epsilon_{S_k,M_p}(x),\ x \in \mathcal{B}\right\} < +\infty.$$

Set $B_k = \bigcap_p c_{k,p} M_p^\circ$. This is a bounded subset of E. Hence, there are $\mu_k > 0$ such that $B := \overline{\text{absconv}}(\bigcup_k \mu_k B_k)$ is still bounded. Now, consider the set

$$R_0 := \{(a(x_n))_n,\ a \in B^\circ,\ x = (x_n)_n \in \mathcal{B}\}$$

and R the normal absolutely convex hull of R_0. Obviously $\mathcal{B} \subset R(B)$. So, we only need to show that R_0, and then also R, is bounded in Λ. But for $k \in \mathbb{N}$, we have

$$\begin{aligned} P_{S_k}((a(x_n))_n) &= \sup\left\{\sum_{n=1}^{\infty} |\alpha_n a(x_n)|,\ \alpha \in S_k\right\} \\ &= \sup\left\{\left|a\left(\sum_{n=1}^{\infty} \alpha_n x_n\right)\right|,\ \alpha \in S_k\right\}. \end{aligned}$$

In order to conclude, it suffices to show that $A_k = \{\sum_{n=1}^{\infty} \alpha_n x_n,\ \alpha \in S_k,\ x \in \mathcal{B}\}$ is contained in B_k. But for every $p \in \mathbb{N}$, $\alpha \in S_k$ and $x \in \mathcal{B}$,

$$\begin{aligned} P_{M_p}\left(\sum_{n=1}^{\infty} \alpha_n x_n\right) &= \sup\left\{\left|a(\sum_{n=1}^{\infty} \alpha_n x_n)\right|,\ a \in M_p\right\} \\ &\leq \sup\left\{\sum_{n=1}^{\infty} \left|\alpha_n a(x_n)\right|,\ a \in M_p\right\} \\ &\leq c_{k,p}. \end{aligned}$$

Showing that $A_k \subset B_k$. ■

A slightly modified proof shows that, whenever the spaces Λ (resp. E) is a normed space, the result remains true without any further condition on E (resp. on Λ).

4 Dual space of $\Lambda\langle E\rangle$

We are going to define on $\Lambda\langle E\rangle$ a locally convex topology which extends that introduced by H. Apiola [1] in the ℓ_p case. We start with the following result:

Proposition 3 *Let $S \in \mathcal{S}$ and $M \in \mathcal{M}$ be given. Then*

1. $\epsilon_{S^\circ,M^\circ}$ *is a complete norm on $\Lambda_S^*\left[E'_M\right]$, where, for $a=(a_n)_n \in \Lambda_S^*\left[E'_M\right]$,*

$$\epsilon_{S^\circ,M^\circ}(a) = \sup\left\{\sum_{n=1}^{\infty}|\alpha_n a_n(x)|\,,\ x\in M^\circ,\ \alpha=(\alpha_n)_{n\in\mathbb{N}}\in S^\circ\right\}.$$

2. $\sigma_{S,M}$ *is a seminorm on $\Lambda\langle E\rangle$, where, for all $x=(x_n)_n \in \Lambda\langle E\rangle$,*

$$\sigma_{S,M}(x) = \sup\left\{\sum_{n=1}^{\infty}|a_n(x_n)|\,,\ a=(a_n)_n\in\Lambda_S^*\left[E'_M\right],\ \epsilon_{S^\circ,\,M^\circ}(a)\le 1\right\}.$$

PROOF. 1. Follows from Proposition 1 of [9], since Λ_S^* and E'_M are Banach spaces.
2. We only have to prove that $\sigma_{S,M}(x)$ is finite for every $x \in \Lambda\langle E\rangle$. Define a linear mapping T_x from $\Lambda_S^*\left[E'_M\right]$ into ℓ_1 by $T_x((a_n)_n) = (a_n(x_n))_n$. Then T_x is continuous by the closed graph theorem. Indeed, suppose that $(f^i)_{i\in\mathbb{N}} \in \Lambda_S^*\left[E'_M\right]$ converges to $f := (f_n)_n$ and $(T_x((f^i)_i)$ converges in ℓ_1 to $(\alpha_n)_n$. By the continuity of the projections, $(f_n^i)_{i\in\mathbb{N}}$ converges to f_n for every $n\in\mathbb{N}$ and then $\left(f_n^i\left(a_n\right)\right)_{i\in\mathbb{N}}$ converges to $f_n(a_n)$ as well. It follows that $\left(f_n(a_n)\right)_n = \left(\alpha_n\right)_n$ showing that the graph of T_x is closed and then that T_x is continuous. Hence, it is bounded on the unit ball of $\Lambda_S^*\left[E'_M\right]$. ∎

From now on, the space $\Lambda\langle E\rangle$ will be equipped with the topology $\sigma_{\mathcal{S},\mathcal{M}}$ generated by the seminorms $\sigma_{S,M}$, $S\in\mathcal{S}$ and $M\in\mathcal{M}$. We will also consider the subspace $\Lambda\langle E\rangle_r$ of $\Lambda\langle E\rangle$ consisting of all the sequences which are the limit of their finite sections.

The following result gives a description of the continuous dual of the subspace $\Lambda\langle E\rangle_r$.

Theorem 1

1. *For every $S\in\mathcal{S}$, $M\in\mathcal{M}$ and $a=(a_n)_n\in\Lambda_S^*\left[E'_M\right]$, the correspondence*

$$F_a : x \longmapsto \sum_{n=1}^{\infty} a_n(x_n)$$

is a continuous linear functional on $\Lambda\langle E\rangle$.

2. *Conversely, if F is a continuous linear functional on $\Lambda\langle E\rangle$, then there exist $S\in\mathcal{S}$, $M\in\mathcal{M}$ and $a=(a_n)_n\in\Lambda_S^*\left[E'_M\right]$ so that $F=F_a$ on $\Lambda\langle E\rangle_r$.*

3. *Consequently, the topological dual $\left(\Lambda\langle E\rangle_r\right)'$ of $\Lambda\langle E\rangle_r$ is isomorphic to the linear space*

$$\bigcup_{S,M}\Lambda_S^*\left[E'_M\right].$$

PROOF. 1. F_a is obviously linear and for $a=0$ there is nothing to show. Assume then that $a\neq 0$ and take $b=\dfrac{a}{\epsilon_{S^\circ,M^\circ}(a)}$. Then $\epsilon_{S^\circ,M^\circ}(b)\le 1$ and therefore

$$\begin{aligned}\left|\sum_{n=1}^{\infty}a_n(x_n)\right| &\le \sum_{n=1}^{\infty}|a_n(x_n)|\\ &\le \epsilon_{S^\circ,M^\circ}(a)\sum_{n=1}^{\infty}|b_n(x_n)|\\ &\le \epsilon_{S^\circ,M^\circ}(a)\sigma_{S,M}(x)\end{aligned}$$

whereby F_a is continuous.
2. Note first that, for every m, the linear mapping θ_m defined from E into $\Lambda\langle E\rangle$ by $\theta_m(x) = xe_m$ is continuous. Indeed, for $S \in \mathcal{S}$ and $M \in \mathcal{M}$, one has

$$\sigma_{S,M}(\theta_m(t)) = \sup\{|u_m(t)|,\ a \in \Lambda_S^*[E'_M],\ \epsilon_{S^\circ,M^\circ}(u) \leq 1\}.$$

But if $\epsilon_{S^\circ,M^\circ}(u) \leq 1$, then

$$|\alpha_m||u_m(c)| \leq 1, \qquad \forall c \in M^\circ, \quad \alpha = (\alpha_n)_n \in S^\circ.$$

Hence

$$|\alpha_m|\|u_m\|_M \leq 1, \qquad \forall \alpha = (\alpha_n)_n \in S^\circ.$$

Fix $\alpha \in S^\circ$ so that $\alpha_m \neq 0$. Then

$$\begin{aligned}\sigma_{S,M}(\theta_m(t)) &\leq \sup\{|u_m(t)|,\ |\alpha_m|\|u_m\|_M \leq 1\}\\ &\leq \sup\left\{P_M(t)\|u_m\|_M,\ \|u_m\|_M \leq \frac{1}{|\alpha_m|}\right\}\\ &\leq \frac{1}{|\alpha_m|}P_M(t).\end{aligned}$$

Whereby θ_m is continuous. Now, since F is continuous, $a_m = F \circ \theta_m$ belongs to E'. Moreover, there exist some $S \in \mathcal{S}$ and some $M \in \mathcal{M}$ such that

$$|F(x)| \leq \sigma_{S,M}(x), \qquad \forall x = (x_n)_n \in \Lambda\langle E\rangle.$$

Choosing α_m as above, we get

$$|a_m(t)| \leq \frac{1}{|\alpha_m|}P_M(t), \qquad t \in E.$$

Which means that $a_n \in E'_M$. In order to show that $a = (a_n)_n \in \Lambda_S^*[E'_M]$, let $f \in (E'_M)'$, $\alpha = (\alpha_n)_n \in \Lambda$, $n \in \mathbb{N}$ and $\delta > 0$ be given. We may and do assume that $\|f\| \leq 1$. Denote by $\widehat{E_{(M^\circ)}}$ the completion of $E_{(M^\circ)}$.

Since $(\widehat{E_{(M^\circ)}})' = (E_{(M^\circ)})'$ is isometrically isomorphic to E'_M, due to the principle of local reflexivity [3], there exists a continuous operator

$$u_n\colon \mathbb{K}.f \longrightarrow \widehat{E_{(M^\circ)}}$$

such that $\|u_k\| \leq 1 + \delta$ and $a_k(u_n f) = f(a_k)$ for all $k \in \{1, 2, \ldots, n\}$. Since every a_n is continuous and $E_{(M^\circ)}$ is dense in $\widehat{E_{(M^\circ)}}$, there exist $0 < \delta_n \leq \dfrac{\delta}{k(1 + p_S(e_n))}$ and $x_n \in E$ such that

$$\|\widehat{x_k} - u_n f\| \leq \delta_n \quad \text{and} \quad |a_k(\widehat{x_k} - u_n f)| \leq \frac{\delta}{k(|\alpha_k| + 1)},$$

$\widehat{x_n}$ being $x_n + M^\perp$.
We claim that the series $\sum \alpha_n f(a_n)$ converges absolutely. So that $(f(a_n))_n$ belongs to Λ^*. We will proceed in steps:
Step 1: Let $\rho > 0$ be such that α belongs to ρS°. We have

$$\left|\sum_{k=1}^{n} \alpha_k f(a_k)\right| \leq 2\delta + (1 + \delta)\rho, \qquad n \geq 1.$$

For

$$\begin{aligned}\left|\sum_{k=1}^{n}\alpha_k f(a_k)\right| &= \left|\sum_{k=1}^{n}a_k(u_n f_k)\right| \\ &\leq \left|\sum_{k=1}^{n}a_k(\alpha_k\widehat{x_k}-u_nf_k)\right| + \left|\sum_{k=1}^{n}a_k(\alpha_k x_k)\right| \\ &\leq \sum_{k=1}^{n}|\alpha_k||a_k(\widehat{x_k}-u_nf)| + \left|F(\sum_{k=1}^{n}\alpha_k x_k e_k)\right| \\ &\leq \delta + \sigma_{S,M}\left(\sum_{k=1}^{k}\alpha_k x_k e_k\right) \\ &= \delta + \sup\left\{\left|\sum_{k=1}^{n}x'_k(\alpha_k x_k)\right| :\ (x'_k)_k \in \Lambda_S^*[E'_M],\ \epsilon_{S^\circ,M^\circ}((x'_n)_n)\leq 1\right\}.\end{aligned}$$

But, for $(x'_k)_k \in \Lambda_S^*\left[E'_M\right]$ with $\epsilon_{S^\circ,M^\circ}((x'_n)_n) \leq 1$, we have

$$\begin{aligned}\|\alpha_k x'_k\|_M &= \sup\left\{|\alpha_k x'_k(t)| :\ t\in M^\circ\right\} \\ &= \rho\sup\left\{\left|\frac{1}{\rho}\alpha_k x'_k(t)\right| :\ t\in M^\circ\right\} \\ &\leq \rho\epsilon_{S^\circ,M^\circ}((x'_k)_k) \\ &\leq \rho.\end{aligned}$$

Whereby,

$$\begin{aligned}\left|\sum_{k=1}^{n}x'_k(\alpha_k x_k)\right| &\leq \left|\sum_{k=1}^{n}\alpha_k x'_k(\widehat{x_k}-u_nf)\right| + \left|\sum_{k=1}^{n}\alpha_k x'_n(u_nf)\right| \\ &\leq \sum_{k=1}^{n}|\alpha_k x'_k|_M\,|\widehat{x_k}-u_nf| + \left|\sum_{k=1}^{n}\alpha_k x'_n(u_nf)\right| \\ &\leq \sum_{k=1}^{n}\rho\delta_k + \left\|\sum_{k=1}^{n}\alpha_k x'_k\right\|_M \|f\|\,(1+\delta) \\ &\leq \delta + (1+\delta)\epsilon_{S^\circ,M^\circ}((x'_k)_k) \\ &\leq \delta + (1+\delta)\rho.\end{aligned}$$

Hence, for all $n\in\mathbb{N}$,

$$\left|\sum_{k=1}^{n}\alpha_k f(a_k)\right| \leq 2\delta + (1+\delta)\rho.$$

Step 2: The series $\sum\alpha_k f(a_k)$ converges absolutely.
For, since $\alpha\in\rho S^\circ$ the same holds for the sequence $\beta := (\epsilon_k\alpha_k)_k$ with $(\epsilon_n)_n$ so chosen that

$$|\alpha_n f(a_n)| = \epsilon_n\alpha_n f(a_n), \qquad n\in\mathbb{N}.$$

Then, by step 1,

$$\left|\sum_{k=1}^{n}\epsilon_k\alpha_k f(a_k)\right| \leq 2\delta + (1+\delta)\rho.$$

Therefore

$$\sum_{k=1}^{n} |\alpha_k f(a_k)| = \sum_{k=1}^{n} \epsilon_k \alpha_k f(a_k) \leq 2\delta + (1+\delta)\rho.$$

Since n is arbitrary, the series $\sum \alpha_n f(a_n)$ converges absolutely. This shows that $(f(a_n))_n$ belongs to Λ^*.
Step 3: $(a_n)_n$ belongs to $\Lambda^*_S[E'_M]$. Indeed, since δ is arbitrary in the last inequality, we get

$$\sum_{n=1}^{\infty} |\alpha_n f(a_n)| \leq \rho$$

so that $(f(a_k))_k \in \rho S^{\circ\circ} = \rho S$, whereby $(a_n)_n \in \Lambda^*_S[E'_M]$. Now, if $x = (x_n)_n \in \Lambda\langle E\rangle_r$ then $x = \sum_{m=1}^{\infty} x_m e_m$ and by the continuity of F and F_a we have

$$F(x) = \sum_{m=1}^{\infty} F(x_m e_m) = \sum_{m=1}^{\infty} a_m(x_m) = F_a(x).$$

3. By 1., the map $a \mapsto f_a$ from $\cup\{\Lambda^*_S[E'_M],\ S \in \mathcal{S}, \mathcal{M} \in \mathcal{M}\}$ into $(\Lambda\langle E\rangle_r)'$ is well defined, linear and one to one. It is onto by 2. and the definition of $\Lambda\langle E\rangle_r$. ■

According to the foregoing proof, the bilinear mapping

$$\theta : \Lambda^*_S[E'_M] \times \Lambda\langle E\rangle \to \ell_1, \langle (a_n)_n, (x_n)_n\rangle = (a_n(x_n))_n$$

is continuous in both variables.

5 Reflexivity of $\Lambda(E)$

The following lemma will be needed in the sequel:

Lemma 1 *For all $(\gamma_n)_n \in c_0$ and $x = (x_n)_n \in \Lambda[E]$, $(\gamma_n x_n)_n \in \Lambda(E)_r$.*

PROOF. For $S \in \mathcal{S}$ and $M \in \mathcal{M}$, $(\alpha_n)_n \in S$, $a \in M$ and $p \in \mathbb{N}$, one has

$$\sum_{n=p+1}^{\infty} |\alpha_n a(\gamma_n x_n)| \leq \sup_{n>p} |\gamma_n| \sum_{n=p+1}^{\infty} |\alpha_n a(x_n)| \leq \sup_{n>p} |\gamma_n|\, \epsilon_{S,M}((x_n)_n)$$

This shows that $\sum_{n=p+1}^{\infty} |\alpha_n a(\gamma_n x_n)|$ converges to 0, uniformly on $a \in M$ and $\alpha \in S$. That is $(\gamma_n x_n)_n$ is the limit in $\Lambda[E]$ of its finite sections which belong to $\Lambda(E)$. The latter being closed in $\Lambda[E]$ by Proposition 1 of [9], then $(\gamma_n x_n)_n \in \Lambda(E)_r$. ■

In the sequel, E and Λ will be a Fréchet spaces and $\mathcal{R}$ and $\mathcal{B}$ the families of all absolutely convex bounded subsets of Λ and E respectively. The members of $\mathcal{R}$ are assumed to be normal.

Theorem 2 *The equality $(\Lambda(E)_r)' = \Lambda^*\left\langle E'_\beta\right\rangle$ holds algebraically and the identity*

$$J : \left(\Lambda^*\left\langle E'_\beta\right\rangle, \sigma_{\mathcal{R},\mathcal{B}}\right) \longrightarrow (\Lambda(E)'_r, \beta(\Lambda(E)'_r, \Lambda(E)_r))$$

is continuous. If, in addition, E happens to be reflexive, then J turns out to be also open.

PROOF. By Theorem 7 of [9], we have

$$(\Lambda(E)_r)' = \bigcup_{S,M} \Lambda_S^* \langle E'_M \rangle .$$

We will then show that

$$\bigcup_{S,M} \Lambda_S^* \langle E'_M \rangle \subset \Lambda^* \left\langle E'_\beta \right\rangle \subset (\Lambda(E)_r)'.$$

Let $S \in \mathcal{S}$, $M \in \mathcal{M}$ and $(a_n)_n \in \Lambda_S^* \langle E'_M \rangle$. If H is an equicontinuous subset of $(E'_\beta)'$ and $f = (f_n)_n \in \Lambda[(E'_\beta)'_H]$, then The polar H° of H with respect to the duality $\langle (E'_\beta)', E' \rangle$ absorbs the equicontinuous (and then strongly bounded) subset M. There exists $\rho > 0$ such that $M \subset \rho H^\circ$. On the other hand, for all $n \in \mathbb{N}$, let $\epsilon_n > 0$ be such that $f_n \in \epsilon_n H$. Then, for all $x' \in M$, one has

$$|f_n(x')| = \rho\epsilon_n \left| \frac{1}{\epsilon_n} f_n(\frac{1}{\rho} x') \right| \leq \rho\epsilon_n,$$

so that each f_n is continuous on E'_M. But for $x' \in E'_M$, the mapping

$$\delta_{x'} : (E'_\beta)'_H \to \mathbb{K}, \quad \delta_{x'}(x'') = x''(x')$$

is linear and continuous. Thus, $(\delta_{x'}(f_n))_n = (f_n(x'))_n \in \Lambda \subset (\Lambda_S^*)^*$. Whereby $f \in (\Lambda_S^*)^*[(E'_M)']$. By Proposition 2 of [9], since $a \in \Lambda_S^* \langle E'_M \rangle$,

$$\sum_{n=1}^{\infty} |f_n(a_n)| < \infty.$$

Hence $(a_n)_n \in \Lambda^* \left\langle E'_\beta \right\rangle$. Since S and M were arbitrary, we obtain

$$(\Lambda(E)_r)' = \bigcup_{S,M} \Lambda_S^* \langle E'_M \rangle \subset \Lambda^* \left\langle E'_\beta \right\rangle .$$

Next, let $a = (a_n)_n \in \Lambda^* \left\langle E'_\beta \right\rangle$ and $(x_n)_n \in \Lambda(E)_r$. By Proposition 2, there exists absolutely convex bounded subsets B of E and R of Λ with R normal such that $(x_n)_n \in R(B)$. Then, by Proposition 1,

$$(x_n)_n \in \Lambda_R[E_B] \subset \Lambda_R^{**}[(E'_\beta)'_{B^{\circ\circ}}] \subset \Lambda[(E'_\beta)'_{B^{\circ\circ}}],$$

where $B^{\circ\circ}$ is the polar of B° in $(E'_\beta)'$. Since $B^{\circ\circ}$ is equicontinuous, the series $\sum_{n=1}^{\infty} |a_n(x_n)|$ is convergent by the very definition of $\Lambda^* \langle E'_\beta \rangle$. Now, consider the linear mapping defined from $\Lambda(E)_r$ into ℓ_1 by $\varphi_a((x_n)_n) = (a_n(x_n))_n$. Due to the closed graph theorem, φ_a is continuous. Then the mapping $f_a : (x_n)_n \mapsto \sum_{n=1}^{\infty} a_n(x_n)$ is continuous on $\Lambda(E)_r$ and therefore belongs to $(\Lambda(E)_r)'$. Whence $\Lambda^* \left\langle E'_\beta \right\rangle \subset (\Lambda(E)_r)'$.

For the second part of the proof, let $\mathcal{B}$ be an absolutely convex bounded subset of $\Lambda(E)_r$. By Proposition 2, there exists absolutely convex bounded subsets B of E and R of Λ with R normal such that $\mathcal{B} \subset R(B)$. We claim that the polar $\mathcal{B}^\circ$ of $\mathcal{B}$ in $\Lambda^* \langle E'_\beta \rangle = (\Lambda(E)_r)'$ contains the unit ball $V_{R,H}$ of $\sigma_{R,H}$; here $H = B^{\circ\circ}$ is the polar of B° in $(E'_\beta)'$. Let $a = (a_n)_n \in V_{R,H}$ and $x = (x_n)_n \in R(B)$. Since $B \subset H$, we have $(x_n)_n \subset (E'_\beta)'_H$. Therefore $(x'(x_n))_n \in \Lambda$ for all $x' \in E'$. By Proposition 2 of [9], we have $(x_n)_n \in \Lambda[(E'_\beta)'_H]$. But $(x'(x_n))_n \in R$, for all $x' \in B^\circ$ and

$$\epsilon_{R^\circ, B^\circ}((x_n)_n) = \sup \left\{ \sum_{n=1}^{\infty} |\alpha_n x'(x_n)|,\ x' \in H^\circ,\ \alpha = (\alpha_n)_{n\in\mathbb{N}} \in R^\circ \right\} \leq 1.$$

Hence,

$$|\langle a, x\rangle| = \left|\sum_{n=1}^{\infty} a_n(x_n)\right| \leq \sigma_{R,H}((a_n)_n) \leq 1.$$

Therefore, $B^\circ \subset V_{R,H}$. To see that J is open, let H be an absolutely convex equicontinuous subset of $(E'_\beta)'$. Then H is $\sigma((E'_\beta)', E'_\beta)$-bounded. Since E is semi-reflexive, there exists an absolutely convex bounded subset B of E such that, $H = h(B)$, where $h : E \mapsto (E'_\beta)'$ is the canonical isomorphism.

If R is a normal bounded subset of Λ, $R(B)^\circ \subset V_{R,H}$. Indeed, let $(a_n)_n \in R(B)^\circ$, $f = (f_n)_n \in \Lambda_R[(E'_\beta)'_H]$, with $\epsilon_{R^\circ,H}((f_n)_n) \leq 1$. For all $n \in \mathbb{N}$, there exists $x_n \in E_B$ such that $f_n = j(x_n)$. So that, $(x_n)_n \in \Lambda[E]$. Since $\epsilon_{R^\circ,B^\circ}((x_n)_n) \leq 1$, for all $a \in B^\circ$, $(a(x_n))_n \in R$. If $\alpha = (\alpha_n)_n \in R$, $a \in B^\circ$ with $(\gamma_n)_n \in c_0$ and $|(\gamma_n)_n|_{c_0} \leq 1$. We have

$$\sum_{n=1}^{\infty} |\alpha_n a\,(\gamma_n x_n)| \leq \sum_{n=1}^{\infty} |\alpha_n a\,(x_n)| \leq \epsilon_{R^\circ,B^\circ}((x_n)_n) \leq 1.$$

By lemma 1, $(\gamma_n x_n)_n$ is in $\Lambda(E)_r$ and then $R(B)$. Thus,

$$\left|\sum_{n=1}^{\infty} \gamma_n f_n(a_n)\right| = \left|\sum_{n=1}^{\infty} \gamma_n a_n(x_n)\right| \leq \sum_{n=1}^{\infty} |a_n(x_n)| \leq 1.$$

This shows that $\sum_{n=1}^{\infty} |f_n(a_n)| \leq 1$, and $(a_n)_n \in V_{R,H}$. ■

Next we prove our main result.

Theorem 3 *If E and Λ are Fréchet spaces, then $\Lambda(E)$ is reflexive if, and only if, the following three assertions hold :*

(i) *E and Λ are reflexive.*

(ii) *$\Lambda(E)$ is an AK-space.*

(iii) *$\Lambda^*\langle E'_\beta\rangle$ is an AK-space.*

PROOF. Suppose that $\Lambda(E)$ is reflexive, then E and Λ are reflexive as subspaces of $\Lambda(E)$. So, (i) holds. By [8, 23.5(10)], $\Lambda(E)_r$ is reflexive as a closed subspace of $\Lambda(E)$, it is then weakly quasi-complete by [8, 23.5(2)]. Thus, $\Lambda(E)_r$ is weakly sequentially complete.
Let $x = (x_n)_n \in \Lambda(E)$. Then, the sequence $(x^{(k)})_{k\in\mathbb{N}}$ consisting of the finite sections of x is contained in $\Lambda(E)_r$ and is is weakly Cauchy in it. Indeed, consider a in $(\Lambda(E)_r)'$. By Theorem 7 of [9], there exists a sequence $(a_n)_n$ in E' such that the series $\sum a_n(x_n)$ converges, $(a(x^{(k)}))_k = (\sum_{n=1}^{k} a_n(x_n))_k$ is then a Cauchy sequence, hence $(x^{(k)})_{k\in\mathbb{N}}$ converges weakly to a limit $y = (y_n)_n \in \Lambda(E)_r$ and it is obvious that $x = y$ so that (ii) holds.
Now, since $\Lambda(E)_r$ is reflexive, the same holds for its strong dual $\Lambda^*\langle E'_\beta\rangle$ and the argumentation above still works to prove (iii).
Conversely, assume that (i), (ii) and (iii) are satisfied. Then, since Λ and E are reflexive, an application of Theorem 1 and Theorem 2 gives

$$\begin{aligned}
(\Lambda(E))'' &= (\Lambda(E)_r)'', && \text{(by (ii))}\\
&= \left(\Lambda^*\left\langle E'_\beta\right\rangle\right)' = \left(\Lambda^*\left\langle E'_\beta\right\rangle_r\right)', && \text{(by (iii))}\\
&= \bigcup_{\mathcal{R},\mathcal{B}} \Lambda_R\left[\left(E'_\beta\right)'_{B^{\circ\circ}}\right], && \text{(by Theorem 1)}\\
&= \bigcup_{\mathcal{R},\mathcal{B}} \Lambda_R\left[E_B\right], && \text{(by (i))}\\
&\subset \Lambda\left[E\right] = \Lambda(E).
\end{aligned}$$

The last inclusion holds by corollary 1.4 of [5]. Hence the Fréchet space $\Lambda(E)$ is semi-reflexive and then reflexive. ■

In the sequel, Λ will stand for an echelon space defined by a Köthe matrix $(u^k)_k$. This is an increasing sequence of strictly positive sequences and

$$\Lambda := \left\{ \alpha = (\alpha_n)_n \in \mathbb{K}^{\mathbb{N}} : P_k(\alpha) = \sum_{n=1}^{\infty} u_n^k \left|\alpha_n\right| < \infty, \ \forall k \in \mathbb{N} \right\}.$$

We equip Λ with its Fréchet locally convex topology generated by the sequence $(P_k)_{k\in\mathbb{N}}$ of seminorms.

Proposition 4 *If Λ is reflexive then $\Lambda^*\langle E'_\beta\rangle$ is an AK-space.*

PROOF. Let $(a_n)_n \in \Lambda^*\langle E'_\beta\rangle$, we have to prove that $(a^{\langle k\rangle})_k$ defined by $a^{\langle k\rangle} = (0,\ldots,0,a_{k+1},a_{k+2},\ldots)$, for all $k\in\mathbb{N}$, is a null sequence. Let R be an absolutely convex normal closed and bounded subset of Λ, H an equicontinuous absolutely convex subset of $(E'_\beta)'$ and $f=(f_n)_n \in \Lambda_R[(E'_\beta)'_H]$ such that $\epsilon_{R^\circ,H}(f)\le 1$. By Theorem 2 and the remark following ([8, 45. 5. (8)]) there exist $\gamma = (\gamma_n)_n \in \Lambda^*$ and a pre-nuclear sequence $(x'_n)_n \subset E'$ such that for all $n\in\mathbb{N}$, $a_n = \gamma_n x'_n$. First we prove that $(f_n(x'_n))_n \in \Lambda$. Let $\alpha = (\alpha_n)_n \in \Lambda^*$, $\epsilon > 0$, $S\in\mathcal{S}$, such that $\beta \in S$ and $p\in\mathbb{N}$. Since $(x'_n)_n$ is pre-nuclear, there exist an equicontinuous subset $M\subset E'$, and a positive Radon measure μ on M such that

$$\sup_n \left|x'_n(x)\right| \le \int_M \left|a(x)\right| d\mu(a).$$

As, $|x_n(x)| \le \|\mu\|_1 P_M(x)$, for all $n\in\mathbb{N}$, $(x'_n)_n \subset E'_M$. Now, since M is equicontinuous, as we did in the proof of Theorem 2, $f_n \in (E'_M)'$. Now, by the principle of local reflexivity, there exists a continuous linear operator

$$T_p : \operatorname{span}\{f_1, f_2, \ldots, f_p\} \mapsto E_{(M^\circ)}$$

such that $\|T_p\| \le 1+\epsilon$ and $x'_n(T_p f_n) = f_n(x'_n)$ for all $n \in \{1,2,\ldots,p\}$. So,

$$\begin{aligned}
\sum_{n=1}^{p} \left|\alpha_n f_n(x'_n)\right| = \sum_{n=1}^{p} \left|\alpha_n x'_n(T_p f_n)\right| &\le \sum_{n=1}^{p} \int_M \left|\alpha_n a(T_p f_n)\right| d\mu(a) \\
&\le \|\mu\|_1 \sup\left\{ \sum_{n=1}^{p} \left|\alpha_n a T_p f_n\right|, \ a\in M \right\} \\
&\le \|\mu\|_1 \sup\left\{ \left|\sum_{n=1}^{p} \beta_n a T_p f_n\right|, \ a\in M, \ (\beta_n)_n \in S \right\} \\
&\le \rho_1 \rho \|\mu\|_1 \epsilon_{R^\circ,H}(f) \\
&\le \rho_1 \rho \|\mu\|_1 ,
\end{aligned}$$

where ρ_1 is such that $S \subset \rho_1 R^\circ$. Hence, $(f_n(x'_n))_n \in \Lambda$. Without loss of generality, we (may and do) assume that $(f_n(x'_n))_n \in R$. Hence,

$$\left\{ (f_n(x'_n))_n , \ f = (f_n)_n \in \Lambda_R\left[\left(E'_\beta\right)'_H \right], \ \epsilon_{R^\circ,H}(f) \le 1 \right\} \subset R,$$

which is $\sigma(\Lambda,\Lambda^*)$-compact, since Λ is reflexive. It follows from [13, 2.4.26] that

$$\lim_{k\to\infty} \sup\left\{ \sum_{n=k+1}^{\infty} \left|\gamma_n f_n(x'_n)\right|, \ f = (f_n)_n \in \Lambda_R\left[\left(E'_\beta\right)'_H \right], \ \epsilon_{R^\circ,H}(f)\le 1 \right\} = 0.$$

Thus, $\lim_{k\to\infty} \sigma_{R,H}\left(a^{\langle k\rangle}\right) = 0$. This finishes the proof. ■

Theorem 4 *Let Λ be an echelon sequence space and E a Fréchet space. Then $\Lambda(E)$ is reflexive if and only if Λ and E are reflexive.*

PROOF. It derives from Theorem 3, Proposition 4 and the fact that $\Lambda(E)$ is an AK-space by [8, 44.8 (10)]. ■

Corollary 1 *For any echelon space Λ and any Fréchet space E, the injective tensor product $\Lambda\widehat{\otimes}_\epsilon E$ is reflexive, if and only Λ and E are reflexive.*

PROOF. It follows from the Proposition 2 of [4] and the preceding theorem. ■

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