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# Some properties of the tensor product of Schwartz $\varepsilon$b-space

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**Abstract.** We define the $\varepsilon$-product of an $\varepsilon$b-space by quotient bornological spaces and we show that if $G$ is a Schwartz $\varepsilon$b-space and $E|F$ is a quotient bornological space, then their $\varepsilon_c$-product $G\varepsilon_c(E|F)$ defined in [2] is isomorphic to the quotient bornological space $(G\varepsilon E)|(G\varepsilon F)$.

**Algunas propiedades del producto tensorial de $\varepsilon$b-espacios de Schwartz**

**Resumen.** Definimos el $\varepsilon$-producto de un $\varepsilon$b-espacio por un cociente de espacio bornológico y demostramos que si $G$ es un $\varepsilon$b-espacio de Schwartz y $E|F$ es un cociente de espacio bornológico, entonces su $\varepsilon_c$-producto $G\varepsilon_c(E|F)$ definido en [2], es isomorfo al cociente de espacio bornológico $(G\varepsilon E)|(G\varepsilon F)$.

## 1 Introduction and notations

It is well known that the $\varepsilon$-product by a Banach space is always a left exact functor on the category of Banach spaces, but it is not right exact in general. To solve this problem, Kaballo [9] introduced the class of $\varepsilon$-spaces, they are locally convex spaces $G$ such that the $\varepsilon$-product of the identity map of $G$ with any surjective continuous linear mapping between Banach spaces is surjective. He proved that a Banach space is an $\varepsilon$-space if and only if it is an $\mathcal{L}_\infty$-space. As a consequence, if $G$ is an $\mathcal{L}_\infty$-space, the left exact functor $G\varepsilon. : \mathbf{Ban} \to \mathbf{Ban}$, $E \to G\varepsilon E$ is exact, where $\mathbf{Ban}$ is the category of Banach spaces and bounded linear mappings. Since each b-space is an inductive limit of Banach spaces [15] and the inductive limit functor is exact on the category of b-spaces [7], it follows that the functor $G\varepsilon. : \mathbf{b} \to \mathbf{b}$, $E \to G\varepsilon E$ is exact whenever $G$ is a b-space which is an inductive limit of $\mathcal{L}_\infty$-spaces, where $\mathbf{b}$ is the category of b-spaces [15]. Now, by [17, Theorem 4.1], the last functor admits an exact extension $G\varepsilon. : \mathbf{q} \to \mathbf{q}$, $E|F \to G\varepsilon(E|F) = (G\varepsilon E)|(G\varepsilon F)$, where $\mathbf{q}$ is the category of quotient bornological spaces [17].

In [4], we have introduced the class of $\varepsilon$b-spaces that we have used to establish some isomorphisms in the category of b-spaces. The objective of this paper is to define the $\varepsilon$-product of the class of $\varepsilon$b-spaces in the category of quotient bornological spaces. It is the same as the $\varepsilon$-product defined in [2] for the class of $\mathcal{L}_\infty$-spaces. But also, we will prove that if a Schwartz b-space $G$ is a bornological inductive limit of $\mathcal{L}_\infty$-spaces and $E|F$ is a quotient bornological space, then $(G\varepsilon E)|(G\varepsilon F) = G\varepsilon_c(E|F)$. Conversely, we will show that if $G$ is a Schwartz b-space such that for each quotient bornological space $E|F$, we have $(G\varepsilon E)|(G\varepsilon F) = G\varepsilon_c(E|F)$, then $G$ is a Schwartz $\varepsilon$b-space.

To prove our results, we need to recall some definitions and notations. Let $\mathbf{EV}$ (resp. $\mathbf{Ban}$) be the category of vector spaces and linear mappings (resp. Banach spaces and bounded linear mappings).

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1. Let $(E, \|\cdot\|_E)$ be a Banach space. A Banach subspace $F$ of $E$ is a vector subspace endowing with a Banach norm $\|\cdot\|_F$ such that the inclusion map $(F, \|\cdot\|_F) \to (E, \|\cdot\|_E)$ is bounded. Observe that the norm $\|\cdot\|_F$ of $F$ is not necessarily the same as the norm induced by $\|\cdot\|_E$ on $F$, and then the Banach subspace $F$ is not necessarily closed in $E$. A quotient Banach space $E|F$ is a vector space $E/F$, where $E$ is a Banach space and $F$ a Banach subspace. It is clear that $E|F$ is not necessarily an object of the category of Banach spaces $\mathbf{Ban}$, but is one if $F$ is closed in $E$. If $E|F$ and $E_1|F_1$ are two quotient Banach spaces, a strict morphism $u : E|F \to E_1|F_1$ is a linear mapping $u : x + F \longmapsto u_1(x) + F_1$, where $u_1 : E \to E_1$ is a bounded linear mapping such that $u_1(F) \subseteq F_1$. We shall say that $u_1$ induces $u$. Two bounded linear mappings $u_1, u_2 : E \to E_1$ both inducing a strict morphism, induce the same strict morphism if and only if the linear mapping $u_1 - u_2 : E \to F_1$ is bounded. Let $E|F$ be a quotient Banach space and $E_0$ a Banach subspace of $E$ such that $F$ is a Banach subspace of $E_0$. Then the natural injection $E_0 \to E$ induces a strict morphism $E_0|F \to E|F$, and the identity mapping $Id_E : E \to E$ induces a strict morphism $E|F \to E|E_0$.

   We call $\tilde{\mathbf{q}}\mathbf{Ban}$ the category of quotient Banach spaces and strict morphisms, it is a subcategory of vector spaces $\mathbf{EV}$ and contains the category of Banach spaces $\mathbf{Ban}$ (any Banach space $E$ will be identified with the quotient Banach space $E|\{0\}$, moreover if $u_1 : E \to E_1$ is a bounded linear mapping, then $u_1$ induces a strict morphism $E|\{0\} \to E_1|\{0\}$ and every strict morphism $E|\{0\} \to E_1|\{0\}$ is inducing by a unique bounded linear mapping $u_1 : E \to E_1$). The category $\tilde{\mathbf{q}}\mathbf{Ban}$ is not abelian, in fact, if $E$ is a Banach space and $F$ a closed subspace of $E$, it would be very nice if the quotient Banach space $E|F$ where isomorphic to the quotient $(E/F)|\{0\}$. This is not the case in $\tilde{\mathbf{q}}\mathbf{Ban}$ unless $F$ is complemented in $E$.

   In [16] L. Waelbroeck introduced an abelian category $\mathbf{qBan}$ generated by $\tilde{\mathbf{q}}\mathbf{Ban}$ and inverses of pseudo-isomorphims, i.e. has the same objects as $\tilde{\mathbf{q}}\mathbf{Ban}$ and every morphism $u$ of $\mathbf{qBan}$ can be expressed as $u = v \circ s^{-1}$, where $s$ is a pseudo-isomorphism and $v$ is a strict morphism. For more information about quotient Banach spaces we refer the reader to [16].

2. Let $E$ be a real or complex vector space, and let $B$ be an absolutely convex set of $E$. Let $E_B$ be the vector space generated by $B$ i.e. $E_B = \cup_{\lambda>0}\lambda B$. The Minkowski functional of $B$ is a semi-norm on $E_B$. It is a norm, if and only if $B$ does not contain any nonzero subspace of $E$. The set $B$ is completant if its Minkowski functional is a Banach norm.

   A bounded structure $\beta$ on a vector space $E$ is defined by a set of "bounded" subsets of $E$ with the following properties:

   (1) Every finite subset of $E$ is bounded.

   (2) Every union of two bounded subsets is bounded.

   (3) Every subset of a bounded subset is bounded.

   (4) A set homothetic to a bounded subset is bounded.

   (5) Each bounded subset is contained in a completant bounded subset.

   A b-space $(E, \beta)$ is a vector space $E$ with a boundedness $\beta$. A subspace $F$ of a b-space $E$ is bornologically closed if the subspace $F \cap E_B$ is closed in $E_B$ for every completant bounded subset $B$ of $E$.

   Given two b-spaces $(E, \beta_E)$ and $(F, \beta_F)$, a linear mapping $u : E \to F$ is bounded, if it maps boundeds of $E$ into boundeds of $F$. The mapping $u$ is bornologically surjective if for every $B' \in \beta_F$, there exists $B \in \beta_E$ such that $u(B) = B'$. A Schwartz b-space $G$ is a b-space satistying the following condition: for each completant bounded disk $A$ of $G$ is there exists a completant bounded disk $B$ of $G$ such that the inclusion mapping $i_{B'B} : G_A \to G_B$ is compact. We denote by $\mathbf{b}$ the category of b-spaces and bounded linear mappings. For more information about b-spaces we refer the reader to [6], [7], and [15].

   Let $(E, \beta_E)$ be a b-space. A b-subspace of $E$ is a subspace $F$ with a boundedness $\beta_F$ such that $(F, \beta_F)$ is a b-space and $\beta_F \subseteq \beta_E$. We note that the boundedness $\beta_F$ of $F$ is not necessarily the same as the boundedness induced by $\beta_E$ on $F$, and then the b-subspace $F$ is not necessarily bornologically closed in $E$. A quotient bornological space $E|F$ is a vector space $E/F$, where $E$ is a b-space and $F$ a b-subspace of $E$. Observe that $E|F$ is not necessarily an object of the category of b-spaces $\mathbf{b}$, but it is one if $F$ is bornologically closed in $E$. If $E|F$ and $E_1|F_1$ are quotient bornological spaces, a strict morphism $u : E|F \to E_1|F_1$ is induced by a bounded linear mapping $u_1 : E \to E_1$ whose restriction to $F$ is a bounded linear mapping $F \to F_1$. Two bounded linear mappings $u_1, v_1 : E \to E_1$, both inducing a strict morphism, induce the same strict morphism $E|F \to E_1|F_1$ if and only if the linear mapping $u_1 - v_1 : E \to F_1$ is bounded.

We call $\tilde{\mathbf{q}}$ the category of quotient bornological spaces and strict morphisms. A pseudo-isomorphism $u : E|F \to E_1|F_1$ is a strict morphism induced by a bounded linear mapping $u_1 : E \to E_1$ which is bornologically surjective and such that $u_1^{-1}(F_1) = F$ i.e. $B \in \beta_F$ if $B \in \beta_E$ and $u_1(B) \in \beta_{F_1}$.

The category $\tilde{\mathbf{q}}$ is not abelian because it contains the category $\tilde{\mathbf{q}}\mathbf{Ban}$. In [17], Waelbroeck introduced an abelian category $\mathbf{q}$ generated by $\tilde{\mathbf{q}}$ and inverses of pseudo-isomorphims i.e. has the same objects as $\tilde{\mathbf{q}}$ and every morphism $u$ of $\mathbf{q}$ can be expressed as $u = v \circ s^{-1}$, where $s$ is a pseudo-isomorphism and $v$ is a strict morphism.

3. The $\varepsilon$-product of two Banach spaces $E$ and $F$ is the Banach space $E\varepsilon F$ of linear mappings $E' \to F$ whose restrictions to the unit ball of $E'$ are $\sigma(E', E)$-continuous, where $E'$ is the topological dual of $E$. It follows from Proposition 2 of [14], that the $\varepsilon$-product is symmetric. If $E_i$ and $F_i$ are Banach spaces and $u_i : E_i \to F_i$ is a bounded linear mapping, $i = 1, 2$, the $\varepsilon$-product of $u_1$ and $u_2$ is the bounded linear mapping.

$$u_1\varepsilon u_2 : E_1\varepsilon E_2 \longrightarrow F_1\varepsilon F_2, f \longmapsto u_2 \circ f \circ u_1'$$

   where $u_1'$ is the dual mapping of $u_1$. It is clear that if $G$ is a Banach space and $F$ is a Banach subspace of a Banach space $E$, then $G\varepsilon F$ is a Banach subspace of $G\varepsilon E$. For more detail about the $\varepsilon$-product we refer the reader to [8] and [14].

4. A Banach space $E$ is an $\mathcal{L}_{\infty,\lambda}$-space, $\lambda \geq 1$, if and only if every finite-dimensional subspace $F$ of $E$ is contained in a finite-dimensional subspace $F_1$ of $E$ such that $d(F_1, l_n^\infty) \leq \lambda$, where $n = \dim F_1$, $l_n^\infty$ is $\mathbb{K}^n$ ($\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$) with the norm $\sup_{1\leq i\leq n} |x_i|$, and

$$d\left(X, Y\right) = \inf\left\{\left\|T\right\| \left\|T^{-1}\right\| : \quad T : X \to Y \text{is isomorphism}\right\}$$

   is the Banach-Mazur distance of the Banach spaces $X$ and $Y$. A Banach space $E$ is an $\mathcal{L}_\infty$-space if it is an $\mathcal{L}_{\infty,\lambda}$-space for some $\lambda \geq 1$. For more information about $\mathcal{L}_\infty$-spaces we can see [12].

## 2 The $\varepsilon$-product of $\varepsilon$b-spaces

Kaballo [9] defined the class of $\varepsilon$-spaces in the category of locally convex spaces and proved that it coincides with the class of $\mathcal{L}_\infty$-spaces in the category of Banach spaces. In [2], we defined the $\varepsilon_c$-product of an $\mathcal{L}_\infty$-space by a quotient Banach space.

Recall from [4] that the $\varepsilon$-product of the b-space $G$ and the Banach space $E$ is the space $G\varepsilon E = \cup_B \left(G_B\varepsilon E\right)$, where $B$ ranges over the bounded completant subsets of the b-space $G$. On $G\varepsilon E$ we define the following bornology of b-space: a subset $C$ of $G\varepsilon E$ is bounded if there exists a completant bounded disk $A$ of $G$ such that $C$ is bounded in the Banach space $G_A\varepsilon E$. It is clear that if $F$ is a bornologically closed subspace in $G$, the subspace $F\varepsilon E$ is a bornologically closed subspace in $G\varepsilon E$.

Finally, if $G$ and $E$ are two b-spaces, the $\varepsilon$-product of $G$ and $E$ is the space $G\varepsilon E = \cup_{A,B} \left(G_A\varepsilon E_B\right)$, where $A$ (resp. $B$) ranges over the bounded completant subsets of the b-space $G$ (resp. $E$). We endow $G\varepsilon E$ with the following bornology of b-space: a subset $C$ of $G\varepsilon E$ is bounded if there exists a completant bounded disk $A$ of $G$ (resp. $B$ of $E$) such that $C$ is bounded in the Banach space $G_A\varepsilon E_B$.

Now, we define the class of $\varepsilon$b-spaces.

**Definition 1** *A b-space $G$ is an $\varepsilon$b-space if the bounded linear mapping $Id_G\varepsilon u : G\varepsilon E \to G\varepsilon F$ is bornologically surjective whenever $u : E \to F$ is a surjective bounded linear mapping between Banach spaces.*

**Example 1** *All the b-spaces $\Im(X, E)$ studied in* [1] *are the $\varepsilon$-product of the corresponding space of scalar functions $\Im(X)$ and the range space $E$. It follows from the results of the paper* [1]*, that each one of those b-spaces $\Im(X)$ is an $\varepsilon$b-space. But there exist b-spaces which are not $\varepsilon$b-spaces. In fact, if $U$ is an open subset of $\mathbb{R}^n$ and $r \in \mathbb{N}^*$, we design by $C^r(\overline{U})$ the space of $r$-continuously differentiable functions i.e. $C^r(\overline{U}) = \{f \in C^{r-1}(\overline{U}) : Df$ exists and $Df \in C(\overline{U})\}$ where $Df$ denote the derivative of $f$ and $C^0(\overline{U}) = C(\overline{U})$ is the Banach space of continuous functions on $\overline{U}$. Khenkin proved in* [10] *that if $n \geq 2$, $C^r(\overline{U})$ is not an $\varepsilon$b-space.*

On the other hand, let $U$ be a compact manifold. For each $r \in \mathbb{R}^+\backslash\mathbb{N}$, we consider the Banach space $C^r(U)$ of functions of class $C^{[r]}$ on $U$ such that for all $k \in \mathbb{N}^n$, $|k| \leq [r]$, the function $D^k f$ is continuously o-Hölderian of exponent $r - [r]$. By [5], it is an $\mathcal{L}_\infty$-space. If $r' \geq r$, we have a natural mapping $C^{r'}(U) \to C^r(U)$. Then we define the b-space $C^\infty(U)$ as the projective limit of the system $(C^r(U))_r$ in the category $\mathbf{b}$. Recall that for each $r \in \mathbb{R}^+\backslash\mathbb{N}$, the functor $C^r(U)\,\varepsilon. : \mathbf{b} \to \mathbf{b}$ is exact. But the left exact functor $C^\infty(U)\,\varepsilon. : \mathbf{b} \to \mathbf{b}$ is not necessarily right exact, and it follows that the b-space $C^\infty(U)$ is not necessarily an $\varepsilon$b-space.

Recall that a complex $F \overset{u}{\to} E \overset{v}{\to} G$ of the category $\mathbf{b}$ is exact if $v \circ u = 0$ and if for all bounded $B$ in $E$ such that $v(B) = \{0\}$, there exists a bounded $A$ of $F$ such that $u(A) = B$.

It is clear that if

$$0 \to F \to E \to G \to 0$$

is a short exact complex of the category $\mathbf{b}$, then the b-space $F$ is bornologically isomorphic to a bornologically closed subspace of $E$ and $G \simeq E/F$.

We start by recalling some basic results on $\varepsilon$b-spaces that are proved in [3].

**Proposition 1 ([3])**

1. *If $G$ is an $\varepsilon$b-space, then the functor*

$$G\varepsilon. : \mathbf{b} \to \mathbf{b} : E \to G\varepsilon E$$

*is exact.*

2. *If the b-space $G$ is an inductive limit of $\mathcal{L}_\infty$-spaces in the category $\mathbf{b}$, then $G$ is an $\varepsilon$ b-space. In particular, every nuclear b-space is an $\varepsilon$b-space.*

The next Theorem gives a characterization of $\varepsilon$b-spaces in the category $\mathbf{q}$.

**Theorem 1** *A b-space $G$ is an $\varepsilon$b-space if and only if the functor $G\widetilde{\varepsilon}. : \widetilde{\mathbf{q}} \to \mathbf{q} : E|F \to (G\varepsilon E)|(G\varepsilon F)$ can be extended to an exact functor $G\varepsilon. : \mathbf{q} \to \mathbf{q}$.*

PROOF. Let $E_1|F_1$ and $E_2|F_2$ be two quotient bornological spaces and $u : E_1|F_1 \to E_2|F_2$ a pseudo-isomorphism induced by a bounded linear mapping $u_1 : E_1 \to E_2$ which is bornologically surjective. Then

$$Id_G\varepsilon u : G\varepsilon(E_1|F_1) \longrightarrow G\varepsilon(E_2|F_2)$$

is a strict morphism induced by the bounded linear mapping

$$Id_G\varepsilon u_1 : G\varepsilon E_1 \to G\varepsilon E_2.$$

Since $G$ is an $\varepsilon$b-space the mapping $Id_G\varepsilon u_1$ is bornologically surjective. We have to show that

$$(Id_G\varepsilon u_1)^{-1}(G\varepsilon F_2) = G\varepsilon F_1.$$

Let $A$ be a bounded subset of $G\varepsilon F_1$ such that $(Id_G\varepsilon u_1)(A)$ is a bounded of $G\varepsilon F_2$. There exists a completant bounded subset $B$ in $F_2$ such that $(Id_G\varepsilon u_1)(A)$ is bounded in $G\varepsilon F_{2_B}$. Also there exists a completant bounded subset $C$ of $E_1$ such that $u_1(C) = B$ and $A$ is bounded in $G\varepsilon E_{1_C}$. Then necessarily $A$ is bounded in $G\varepsilon F_{1_C}$. This shows that the strict morphism $Id_G\varepsilon u$ is an isomorphism of $\mathbf{q}$, and it follows from Theorem 4.1 of [17], that the functor $G\widehat{\varepsilon}.$ can be extended to an exact functor $G\varepsilon. : \mathbf{q} \to \mathbf{q}$.

Conversely, let $u_1 : X \to Y$ be a surjective bounded linear mapping between Banach spaces. It induces a pseudo-isomorphism

$$u : X|u_1^{-1}(0) \to Y|\{0\}.$$

If the strict morphism

$$Id_G\varepsilon u : (G\varepsilon X)\,|\,\left(G\varepsilon(u_1^{-1}(0))\right) \to (G\varepsilon Y)\,|\,\{0\}$$

is an isomorphism, then it is epic (Theorem 7.4 of [17]) and is induced by the bounded linear mapping $Id_G\varepsilon u_1 : G\varepsilon X \to G\varepsilon Y$. Then necessarily $Id_G\varepsilon u_1$ is bornologically surjective. This proves that $G$ is an $\varepsilon$b-space.

Finally, we give the following definition

**Definition 2** *The $\varepsilon$-product of an $\varepsilon$b-space $G$ and a quotient bornological space $E|F$ is the quotient bornological space $G\varepsilon(E|F) = (G\varepsilon E)|(G\varepsilon F)$.*

## 3 On the $\varepsilon_c$-product of a Schwartz $\varepsilon$b-space

Recall from [2], that a $C(K)$-resolution of a Banach space $G$ is a sequence

$$0 \to G \xrightarrow{\Phi} C(X) \xrightarrow{\Psi} C(Y)$$

in the category $\mathbf{Ban}$ such that $\mathrm{Ker}(\Psi) = \mathrm{Im}(\Phi)$ and the range of $\Psi$ is closed in $C(Y)$.

In Proposition 3.2 of [2], it is shown the existence at least of a concrete $C(K)$-resolution for $G$ which we shall name as canonical $C(K)$-resolution of $G$.

Given an arbitrary $C(K)$-resolution of a Banach space $G$

$$0 \to G \xrightarrow{\Phi} C(X) \xrightarrow{\Psi} C(Y)$$

and a quotient bornological space $E|F$, since $C(X)$ and $C(Y)$ are $\mathcal{L}_\infty$-spaces, we define $C(X, E|F)$ and $C(Y, E|F)$ as $C(X, E)|C(X, F)$ and $C(Y, E)|C(Y, F)$ respectively. Then a strict morphism

$$\Psi\varepsilon Id_{E|F} : C(X, E)|C(X, F) \longrightarrow C(Y, E)|C(Y, F)$$

exists, it is induced by the bounded linear mapping

$$\Psi\varepsilon Id_E : C(X, E) \to C(Y, E).$$

As the category $\mathbf{q}$ is abelian, the object $\mathrm{Ker}(\Psi\varepsilon Id_{E|F})$ exists, and then we obtain the following left exact sequence in $\mathbf{q}$:

$$0 \to \mathrm{Ker}(\Psi\varepsilon Id_{E|F}) \xrightarrow{\Phi\varepsilon Id_{E|F}} C(X)\varepsilon\,(E|F) \xrightarrow{\Psi\varepsilon Id_{E|F}} C(Y)\varepsilon\,(E|F)$$

where

$$\mathrm{Ker}(\Psi\varepsilon Id_{E|F}) = (\Psi\varepsilon Id_E)^{-1}\,(C(Y)\varepsilon F)|\,(C(X)\varepsilon F)\,.$$

and the subspace $(\Psi\varepsilon Id_E)^{-1}\,(C(Y)\varepsilon F)$ of $C(X)\varepsilon E$ is a b-space for the following boundedness: a subset $B$ of $(\Psi\varepsilon Id_E)^{-1}\,(C(Y)\varepsilon F)$ is bounded if it is bounded in $C(X)\varepsilon E$ and its image $(\Psi\varepsilon Id_E)\,(B)$ is bounded in the b-space $C(Y)\varepsilon F$. It is also a b-subspace of $C(X)\varepsilon E$.

Let

$$G\varepsilon_{Res}\left(E|F\right) = \left(\Psi\varepsilon Id_E\right)^{-1}\left(C(Y)\varepsilon F\right) | \left(C(X)\varepsilon F\right),$$

then each $C\left(K\right)$-resolution of $G$ defines an object $G\varepsilon_{Res}\left(E|F\right)$. In general, this object depends on $C(K)$-resolutions of $G$. However, if $G$ is an $\mathcal{L}_\infty$-space, we can show that the strict morphism $\left(G\varepsilon E\right) | \left(G\varepsilon F\right) \to G\varepsilon_{Res}(E|F)$ is an isomorphism, and then it follows that the object $G\varepsilon_{Res}(E|F)$ is independent from $C\left(K\right)$-resolutions of $G$. To do this we need the following two Lemmas:

**Lemma 1** *If* $0 \to G_1 \to G_2 \to G_3 \to 0$ *is an exact sequence in* $\mathbf{Ban}$ *such that* $G_i$ *is an* $\mathcal{L}_\infty$*-space,* $i = 1, 2, 3$*, then for each quotient bornological space* $E|F$*, the sequence* $0 \to G_1\varepsilon(E|F) \to G_2\varepsilon(E|F) \to G_3\varepsilon(E|F) \to 0$ *is exact in* $\mathbf{q}$.

PROOF. It follows from Proposition 2.5 and Example 2.4(i) of [9] and Proposition 6.2 of [2], the exactness in $\mathbf{b}$ of the following sequences:

$$0 \to G_1\varepsilon E \to G_2\varepsilon E \to G_3\varepsilon E \to 0$$

$$0 \to G_1\varepsilon F \to G_2\varepsilon F \to G_3\varepsilon F \to 0.$$

Since $G_1$, $G_2$ and $G_3$ are $\mathcal{L}_\infty$-spaces, the sequence

$$0 \to G_i\varepsilon F \to G_i\varepsilon E \to G_i\varepsilon(E|F) \to 0, \quad i = 1, 2, 3,$$

is also exact in $\mathbf{q}$ (Theorem 1.2 of [2]). These exact sequences induce the following commutative diagram:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & G_1\varepsilon F & \longrightarrow & G_2\varepsilon F & \longrightarrow & G_3\varepsilon F & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & G_1\varepsilon E & \longrightarrow & G_2\varepsilon E & \longrightarrow & G_3\varepsilon E & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & G_1\varepsilon(E|F) & \longrightarrow & G_2\varepsilon(E|F) & \longrightarrow & G_3\varepsilon(E|F) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

where the two first lines and the three columns of the above diagram are exact. It follows from Lemma 4.3.6 of [13], that the third line is exact.

**Lemma 2** *Let* $G$ *be a Banach space,* $0 \to G \overset{\Phi}{\to} C(X) \overset{\Psi}{\to} C(Y)$ *a* $C(K)$*-resolution of* $G$ *and* $u$ : $E|F \to E_1|F_1$ *a pseudo-isomorphism between quotient bornological spaces. Then the strict morphism* $Id_G\varepsilon_{Res}u : G\varepsilon_{Res}(E|F) \to G\varepsilon_{Res}(E_1|F_1)$ *is an isomorphism.*

PROOF. By applying the functors $.\varepsilon(E|F), .\varepsilon(E_1|F_1) : \mathbf{Ban} \to \mathbf{q}$ at the $C(K)$-resolution

$$0 \to G \overset{\Phi}{\to} C(X) \overset{\Psi}{\to} C(Y)$$

we obtain the following commutatif square:

$$\begin{array}{ccc} C(X,E|F) & \longrightarrow & C(Y,E|F) \\ \downarrow {\scriptstyle Id_{C(X)}\varepsilon u} & & \downarrow {\scriptstyle Id_{C(Y)}\varepsilon u} \\ C(X,E_1|F_1) & \longrightarrow & C(Y,E_1|F_1). \end{array}$$

Let $u_1 : E \to E_1$ be a bounded linear mapping which induces $u$. The restriction of $Id_{C(X)}\varepsilon u_1$ to the b-space $(\Psi\varepsilon Id_E)^{-1}(C(Y,F))$, is a bounded linear mapping

$$(\Psi\varepsilon Id_E)^{-1}(C(Y,F)) \to (\Psi\varepsilon Id_{E_1})^{-1}(C(Y,F_1)).$$

This restriction defines a strict morphism $G\varepsilon_{Res}(E|F) \to G\varepsilon_{Res}(E_1|F_1)$ that we design by $Id_G\varepsilon_{Res}u$, it makes commutative the following diagram:

$$\begin{array}{ccccccc} 0 & \longrightarrow & G\varepsilon_{Res}(E|F) & \longrightarrow & C(X,E|F) & \longrightarrow & C(Y,E|F) \\ & & \downarrow {\scriptstyle Id_G\varepsilon_{Res}u} & & \downarrow {\scriptstyle Id_{C(X)}\varepsilon u} & & \downarrow {\scriptstyle Id_{C(Y)}\varepsilon u} \\ 0 & \longrightarrow & G\varepsilon_{Res}(E_1|F_1) & \longrightarrow & C(X,E_1|F_1) & \longrightarrow & C(Y,E_1|F_1) \end{array}$$

Since the strict morphisms $Id_{C(X)}\varepsilon u$ and $Id_{C(Y)}\varepsilon u$ are isomorphisms (Theorem 1.2 of [2]), the strict morphism $Id_G\varepsilon_{Res}u$ is also an isomorphism (Lemma 4.3.3 of [13]).

Now, we are in position to prove that if $G$ is an $\mathcal{L}_\infty$-space, then the quotient bornological space $G\varepsilon_{Res}(E|F)$ is independent on $C(K)$-resolutions of $G$. In the same way, this result gives a new characterization of the class of $\mathcal{L}_\infty$-spaces.

**Theorem 2** *A Banach space $G$ is an $\mathcal{L}_\infty$-space if and only if for each $C(K)$-resolution of $G$ and for each quotient bornological space $E|F$, we have $(G\varepsilon E)\,|\,(G\varepsilon F) = G\varepsilon_{Res}(E|F)$.*

PROOF. Let

$$0 \to G \xrightarrow{\Phi} C(X) \xrightarrow{\Psi} C(Y)$$

be a $C(K)$-resolution of $G$. This $C(K)$-resolution was constructed from the following exact sequence:

$$0 \to G \xrightarrow{\Phi} C(X) \xrightarrow{\Psi_1} C(X)/G \to 0$$

where $\Psi_1 : C(X) \to C(X)/G$ is the quotient mapping. Since the Banach spaces $C\,(X)$ and $G$ are $\mathcal{L}_\infty$-spaces, it follows from [11, Proposition 5.2 (c), p. 346]) that the cokernel of the mapping $G \xrightarrow{\Phi} C(X)$, which is $C(X)/G$, is an $\mathcal{L}_\infty$-space. Now, by Lemma 1, the sequence

$$0 \to (G\varepsilon E)\,|\,(G\varepsilon F) \to C(X,E|F) \to (C(X)/G)\varepsilon(E|F) \to 0$$

is exact.

On the other hand, the sequence

$$0 \to G\varepsilon_{Res}(E|F) \to C(X,E|F) \to C(Y,E|F)$$

is left exact. And since the bounded linear mapping $\Psi_2 : C(X)/G \to C(Y)$ is injective and of closed range, the strict morphism

$$\Psi_2\varepsilon Id_{E|F} : (C(X)/G)\varepsilon\,(E|F) \to C(Y,E|F)$$

is injective (Proposition 2.2 of [2]). As $\Psi = \Psi_2 \circ \Psi_1$, then

$$\mathrm{Ker}(\Psi\varepsilon Id_{E|F}) = \mathrm{Ker}((\Psi_2 \circ \Psi_1)\varepsilon Id_{E|F}) = \mathrm{Ker}(\Psi_1\varepsilon Id_{E|F})$$

This shows that for each quotient bornological space $E|F$, we have

$$(G\varepsilon E)\,|\,(G\varepsilon F) = G\varepsilon_{Res}(E|F).$$

Conversely, let $u_1 : X \to Y$ be a surjective bounded linear mapping between Banach spaces, the mapping $u_1$ induces a pseudo-isomorphism $u : X|u_1^{-1}(0) \to Y|\,\{0\}$. Since the morphism

$$Id_G\varepsilon_{Res}u : G\varepsilon_{Res}(X|u_1^{-1}(0)) \to G\varepsilon_{Res}(Y|\,\{0\})$$

is an isomorphism (Lemma 2) and since

$$G\varepsilon_{Res}(X|u_1^{-1}(0)) = (G\varepsilon X)\,|\,\big(G\varepsilon u_1^{-1}(0)\big)$$

and

$$G\varepsilon_{Res}(Y|\,\{0\}) = (G\varepsilon Y)|\,\{0\}$$

the bounded linear mapping $u_1\varepsilon Id_G : G\varepsilon X \to G\varepsilon Y$ is bornologically surjective and then $G$ is an $\mathcal{L}_\infty$-space.

Recall that a b-space $G$ is said to be of Schwartz if all bounded completant subset $B$ of $G$ is included in a bounded completant $A$ of $G$ such that the inclusion $i_{AB} : G_B \to G_A$ is a compact mapping. For more information about Schwartz b-spaces we refer the reader to [6].

In [2] it is shown the existence of the $\varepsilon_c$-product of a Schwartz b-space $G$ and a quotient Banach space $E|F$ as the quotient bornological space

$$G_B\varepsilon_c(E|F) = \cup_B(G_B\varepsilon_{Res}(E|F))$$

where $\cup_B$ designs the inductive limit and $G_B\varepsilon_{Res}(E|F)$ is an object of the category $\mathbf{qBan}$ such that the following sequence

$$0 \longrightarrow G_B\varepsilon_{Res}(E|F) \xrightarrow{\Phi_B\varepsilon Id_{E|F}} C(X_B)\varepsilon(E|F) \xrightarrow{\Psi_B\varepsilon Id_{E|F}} C(Y_B)\varepsilon(E|F)$$

is left exact and where

$$0 \longrightarrow G_B \xrightarrow{\Phi_B} C(X_B) \xrightarrow{\Psi_B} C(Y_B)$$

is the canonical $C\,(K)$-resolution of the Banach space $G_B$.

As an application of Theorem 2, we study some properties of Schwartz $\varepsilon$b-spaces.

**Theorem 3** 1. *If a Schwartz b-space $G$ is a bornological inductive limit of $\mathcal{L}_\infty$-spaces and $E|F$ is a quotient bornological space, then $(G\varepsilon E)|(G\varepsilon F) = G\varepsilon_c(E|F)$.*

2. *Let $G$ be a Schwartz b-space. If for every quotient bornological space $E|F$, we have $(G\varepsilon E)|(G\varepsilon F) = G\varepsilon_c(E|F)$, then $G$ is a Schwartz $\varepsilon$b-space.*

PROOF. 1. Since $G = \cup_B G_B$, where each Banach space $G_B$ is an $\mathcal{L}_\infty$-space, it follows from Theorem 2, that for each $C(K)$-resolution of $G$, we have

$$(G_B\varepsilon E)|(G_B\varepsilon F) = G_B\varepsilon_{Res}(E|F).$$

On the other hand, the quotient bornological space $(G_B\varepsilon E)|(G_B\varepsilon F)$ defines the following exact sequence

$$0 \to G_B\varepsilon F \to G_B\varepsilon E \to (G_B\varepsilon E)|(G_B\varepsilon F) \to 0.$$

By applying the exact functor $\cup_B(.)$ to the above sequence, we obtain

$$0 \to \cup_B\,(G_B\varepsilon F) \longrightarrow \cup_B\,(G_B\varepsilon E) \to \cup_B\,((G_B\varepsilon E)|(G_B\varepsilon F)) \to 0.$$

It follows that

$$\cup_B(G_B\varepsilon E)|\cup_B(G_B\varepsilon F)=\cup_B((G_B\varepsilon E)|(G_B\varepsilon F))=\cup_B(G_B\varepsilon_c(E|F))$$

and hence

$$(G\varepsilon E)|(G\varepsilon F)=G\varepsilon_c(E|F).$$

2. Let $u_1 : X \to Y$ be a surjective bounded linear mapping between Banach spaces, it induces a pseudo-isomorphism

$$u : X|u_1^{-1}(0)\to Y|\{0\}.$$

As $G=\cup_B G_B$, let

$$0\to G_B\overset{\Phi_B}{\to}C(X_B)\overset{\Psi_B}{\to}C(Y_B)$$

be a canonical $C(K)$-resolution of the Banach space $G_B$. By applying the left exact functors

$$.\varepsilon_{Res}(X|u_1^{-1}(0)) : \mathbf{Ban}\to\mathbf{q}$$

and

$$.\varepsilon_{Res}(Y|\{0\}) : \mathbf{Ban}\to\mathbf{q}$$

to the above $C(K)$-resolution of $G_B$, we obtain the following commutative diagram:

$$\begin{array}{ccccccc}
0\longrightarrow & G_B\varepsilon_{Res}(X|u_1^{-1}(0)) & \longrightarrow & C(X_B)\varepsilon(X|u_1^{-1}(0)) & \longrightarrow & C(Y_B)\varepsilon(X|u_1^{-1}(0)) \\
& \downarrow Id_{G_B}\varepsilon_{Res}u & & \downarrow Id_{C(X_B)}\varepsilon u & & \downarrow Id_{C(Y_B)}\varepsilon u \\
0\longrightarrow & G_B\varepsilon_{Res}(Y|\{0\}) & \longrightarrow & C(X_B)\varepsilon(Y|\{0\}) & \longrightarrow & C(Y_B)\varepsilon(Y|\{0\})
\end{array}$$

Since the Banach spaces $C(X)$ and $C(Y)$ are $\mathcal{L}_\infty$-spaces, then the strict morphisms $Id_{C(X_B)}\varepsilon u$ and $Id_{C(Y_B)}\varepsilon u$ are isomorphisms. It follows from Lemma 4.3.3 of [13], that the strict morphism

$$Id_{G_B}\varepsilon_{Res}u : G_B\varepsilon_{Res}(X|u_1^{-1}(0))\to G_B\varepsilon_{Res}(Y|\{0\})$$

is an isomorphism.

Now, by applying the exact functor $\cup_B(\cdot)$ to the system of isomorphisms $(Id_{G_B}\varepsilon_{Res}u)_B$, we obtain the following isomorphism:

$$\cup_B(Id_{G_B}\varepsilon_{Res}u) : \cup_B(G_B\varepsilon_{Res}(X|u_1^{-1}(0)))\to\cup_B(G_B\varepsilon_{Res}(Y|\{0\}))$$

i.e.

$$Id_G\varepsilon_c u : G\varepsilon_c(X|u_1^{-1}(0))\to G\varepsilon_c(Y|\{0\})$$

is an isomorphism. As

$$(G\varepsilon X)\,|\,\big(G\varepsilon(u_1^{-1}(0))\big)=G\varepsilon_c(X|u_1^{-1}(0))$$

and

$$G\varepsilon_c(Y|\{0\})=(G\varepsilon Y)|\{0\}$$

the bounded linear mapping $Id_G\varepsilon u : G\varepsilon X\to G\varepsilon Y$ is bornologically surjective, and hence $G$ is an $\varepsilon$b-space. This ends the proof. ■

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