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The impact of the Radon-Nikodým property on the weak bounded approximation property

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Abstract. A Banach space X is said to have the weak λ-bounded approximation property if for every separable reflexive Banach space Y and for every compact operator $T : X \to Y$, there exists a net (S_α) of finite-rank operators on X such that $\sup_\alpha \|TS_\alpha\| \leq \lambda\|T\|$ and $S_\alpha \to I_X$ uniformly on compact subsets of X.

We prove the following theorem. Let X^{**} or Y^* have the Radon-Nikodým property; if X has the weak λ-bounded approximation property, then for every bounded linear operator $T : X \to Y$, there exists a net (S_α) as in the above definition. It follows that the weak λ-bounded and λ-bounded approximation properties are equivalent for X whenever X^* or X^{**} has the Radon-Nikodým property. Relying on Johnson's theorem on lifting of the metric approximation property from Banach spaces to their dual spaces, this yields a new proof of the classical result: if X^* has the approximation property and X^* or X^{**} has the Radon-Nikodým property, then X^* has the metric approximation property.

El impacto de la propiedad de Radon-Nikodým en la propiedad de aproximación acotada débil

Resumen. Se dice que un espacio de Banach X tiene la propiedad de aproximación λ-acotada débil si para todo espacio de Banach reflexivo y separable Y, y para todo operador compacto $T : X \to Y$, existe una red (S_α) de operadores de rango finito sobre X tal que $\sup_\alpha \|TS_\alpha\| \leq \lambda\|T\|$ y $S_\alpha \to I_X$ uniformemente sobre subconjuntos compactos de X.

Probamos el siguiente teorema. Supongamos que X^{**} o Y^* tengan la propiedad de Radon-Nikodým; si X tiene la propiedad de aproximación λ-acotada débil, entonces para todo operador lineal acotado $T : X \to Y$, existe una red (S_α) como en la definición anterior. Resulta que las propiedades de aproximación λ-acotada débil y de aproximación λ-acotada son equivalentes para X, si X^*, o bien X^{**}, tiene la propiedad de Radon-Nikodým. Gracias al teorema de Johnson sobre el levantamiento de la propiedad de aproximación métrica de los espacios de Banach a sus duales, esto proporciona una nueva prueba del siguiente resultado clásico: si X^* tiene la propiedad de aproximación y X^*, o bien X^{**}, tiene la propiedad de Radon-Nikodým, entonces X^* tiene la propiedad de aproximación métrica.

1. Introduction

Let X and Y be Banach spaces. We denote by $\mathcal{L}(X, Y)$ the Banach space of bounded linear operators from X to Y, and by $\mathcal{F}(X, Y)$ and $\mathcal{K}(X, Y)$ its subspaces of finite-rank operators and compact operators.

Let I_X denote the identity operator on X. If there exists a net $(S_\alpha) \subset \mathcal{F}(X, X)$ such that $S_\alpha \to I_X$ uniformly on compact subsets of X, then X is said to have the *approximation property*. If (S_α) can be

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chosen with $\sup_\alpha \|S_\alpha\| \le \lambda$ for some $\lambda \ge 1$, then X is said to have the *λ-bounded approximation property*. If $\lambda = 1$, then X has the *metric approximation property*.

The following is a long-standing famous open problem (see, e.g., [1, page 289]).

Problem 1 *Does the approximation property of the dual space X^* of a Banach space X imply the metric approximation property of X^*?*

By an important result of Grothendieck [11, Chapter I, proof of Corollary 2 on page 182 together with Corollary 3, pages 134–135], separable dual spaces with the approximation property have the metric approximation property. The proof of this result "has always been a little mysterious" as written in [1, page 289]. The most far-reaching result in this direction is as follows.

Theorem 1 (see [5, page 246], [18, Theorem 4] and [10, Corollary 1.6]). *Let X be a Banach space such that X^* or X^{**} has the Radon-Nikodým property. If X^* has the approximation property, then X^* has the metric approximation property.*

Recently, the weak bounded approximation property was introduced and studied by Lima and Oja [16]. Let $1 \le \lambda < \infty$. Following [16, Theorem 2.4], we say that a Banach space X has the *weak λ-bounded approximation property* if for every separable reflexive Banach space Y and for every operator $T \in \mathcal{K}(X, Y)$, there exists a net $(S_\alpha) \subset \mathcal{F}(X, X)$ such that $\sup_\alpha \|TS_\alpha\| \le \lambda\|T\|$ and $S_\alpha \to I_X$ uniformly on compact subsets of X. We say that X has the *weak bounded approximation property* if X has the weak λ-bounded approximation property for some λ. We say that X has the *weak metric approximation property* if X has the weak 1-bounded approximation property.

It was proven in [16] that Problem 1 can be reformulated in terms of the weak metric approximation property in different equivalent ways. For instance (see [16, Problem 1.2]), does the weak metric approximation property of X^* imply the metric approximation property of X^*?

The following theorem is the main result of this paper.

Theorem 2 *Let X and Y be Banach spaces and let $1 \le \lambda < \infty$. Assume that X has the weak λ-bounded approximation property. If X^{**} or Y^* has the Radon-Nikodým property, then for every operator $T \in \mathcal{L}(X, Y)$, there exists a net $(S_\alpha) \subset \mathcal{F}(X, X)$ such that $\sup_\alpha \|TS_\alpha\| \le \lambda\|T\|$ and $S_\alpha \to I_X$ uniformly on compact subsets of X.*

Theorem 2 will be proven in Section 2. Section 3 presents some applications of Theorem 2. Among others a new rather unexpected proof of Theorem 1 will be given that uses Johnson's theorem [12]: if X has the metric approximation property in every equivalent norm, then X^* has the metric approximation property.

The notation we use is standard. We shall consider X as a subspace of X^{**}. The closed unit ball of X is denoted by B_X.

2. Proof of Theorem 2

The idea of the proof comes from [16, Theorem 2.4].

Let X and Y be Banach spaces and let $1 \le \lambda < \infty$. We assume that X has the weak λ-bounded approximation property and X^{**} or Y^* has the Radon-Nikodým property.

Let $T \in \mathcal{L}(X, Y)$. We may assume without loss of generality that $\|T\| = 1$. We need to show that, for every compact subset K of X and for every $\varepsilon > 0$, there is an operator $S \in \mathcal{F}(X, X)$ such that $\|TS\| \le \lambda$ and $\|Sx - x\| \le \varepsilon$ for all $x \in K$.

Let us fix a compact subset K of X and $\varepsilon > 0$. Consider the seminorm

$$p(A) = \sup\{\|Ax\| : x \in K\}, \quad A \in \mathcal{L}(X, X),$$

and put

$$C = \{TS : S \in \mathcal{F}(X,X), \quad p(S - I_X) \le \varepsilon/2\}.$$

Then C is a convex subset of $\mathcal{F}(X,Y)$. It is not empty because X has the approximation property (this is obvious if one takes $T = 0$ in the definition of the weak bounded approximation property). Let us fix $\delta > 0$ such that

$$\frac{\delta}{\lambda+\delta}\Big(\frac{\varepsilon}{2} + p(I_X)\Big) \le \frac{\varepsilon}{2}.$$

We shall show below that

$$C \cap (\lambda+\delta)B_{\mathcal{F}(X,Y)} \neq \varnothing.$$

If then $S_\delta \in \mathcal{F}(X,X)$ is such that $p(S_\delta - I_X) \le \varepsilon/2$ and $\|TS_\delta\| \le \lambda + \delta$, then

$$S = \frac{\lambda}{\lambda+\delta} S_\delta \in \mathcal{F}(X,X)$$

will be the operator that we need. Indeed, $\|TS\| \le \lambda$ and, for all $x \in K$,

$$\begin{aligned}\|Sx - x\| &\le p(S - I_X) \le p(S - S_\delta) + p(S_\delta - I_X)\\ &\le \frac{\delta}{\lambda+\delta} p(S_\delta) + \frac{\varepsilon}{2} \le \frac{\delta}{\lambda+\delta}\big(p(S_\delta - I_X) + p(I_X)\big) + \frac{\varepsilon}{2}\\ &\le \varepsilon.\end{aligned}$$

To complete the proof, suppose to the contrary that

$$C \cap (\lambda+\delta)B_{\mathcal{F}(X,Y)} = \varnothing.$$

Then there exists $f \in \big(\mathcal{F}(X,Y)\big)^*$ such that $\|f\| = 1$ and

$$\lambda + \delta = \sup\{\mathrm{Re} f(A) : \|A\| \le \lambda + \delta\} \le \inf\{\mathrm{Re} f(A) : A \in C\}.$$

By the description of $\big(\mathcal{K}(X,Y)\big)^*$ due to Feder and Saphar [7, Theorem 1] (here one needs the Radon-Nikodým property of X^{**} or Y^*), through the Hahn-Banach theorem, there exists an element u in the projective tensor product $Y^*\hat{\otimes}X^{**}$ such that $\|u\|_\pi = \|f\| = 1$ and $f(A) = \mathrm{trace}(A^{**}u)$ for all $A \in \mathcal{F}(X,Y)$.

Let $u = \sum_{n=1}^\infty y_n^* \otimes x_n^{**}$, $y_n^* \in Y^*$, $x_n^{**} \in X^{**}$, with $\sum_{n=1}^\infty \|y_n^*\|\|x_n^{**}\| < 1 + \delta/2\lambda$. Let $(\eta_n)_{n=1}^\infty$ be a sequence of positive scalars tending to ∞ such that $\sum_{n=1}^\infty \eta_n\|y_n^*\|\|x_n^{**}\| < 1 + \delta/\lambda$. We clearly may assume that $\eta_n \ge 1$ and $\|y_n^*\| = 1$ for all n.

Denote by $\mathcal{C}$ the closed absolutely convex hull in X^* of the compact set $\{0, T^*y_1^*/\eta_1, T^*y_2^*/\eta_2, \ldots\}$. Since $\mathcal{C}$ is a compact absolutely convex subset of B_{X^*}, by the isometric version of the famous Davis-Figiel-Johnson-Pełczyński factorization lemma [4] due to Lima, Nygaard, and Oja [15], there exists a separable reflexive Banach space Z, which is a linear subspace of X^*, such that the identity embedding $J : Z \to X^*$ is compact and $\|J\| \le 1$. Moreover,

$$\{T^*y_1^*/\eta_1, T^*y_2^*/\eta_2, \ldots\} \subset J(B_Z).$$

Since X has the weak λ-bounded approximation property, for the operator $J^*|_X \in \mathcal{K}(X, Z^*)$, there exists $S \in \mathcal{F}(X,X)$ such that $\|J^*|_X S\| \le \lambda$ and $p(S - I_X) \le \varepsilon/2$. Observe that $\|S^*J\| \le \lambda$, because $S^*J = S^*(J^*|_X)^* = (J^*|_X S)^*$ and $\|J^*|_X S\| \le \lambda$.

Since $TS \in C$, we have

$$\lambda + \delta \le |f(TS)|.$$

On the other hand, choosing $z_n \in B_Z$ such that $T^*y_n^* = \eta_n J z_n$ for all n, we have

$$\begin{aligned}|f(TS)| &= |\text{trace}(T^{**}S^{**}u)| = |\sum_{n=1}^{\infty} x_n^{**}(S^*T^*y_n^*)| \\ &= |\sum_{n=1}^{\infty} \eta_n x_n^{**}(S^*Jz_n)| \le \|S^*J\| \sum_{n=1}^{\infty} \eta_n \|x_n^{**}\| \\ &< \lambda(1+\delta/\lambda) = \lambda + \delta.\end{aligned}$$

This contradiction completes the proof.

3. Applications

3.1. Bounded approximation property and its weak version

The λ-bounded approximation property implies the weak λ-bounded approximation property. This is obvious from the definitions. It is not known whether the weak λ-bounded approximation property implies the λ-bounded approximation property.

Conjecture 1 (Lima-Oja [16]). *The weak λ-bounded and the λ-bounded approximation properties are, in general, different.*

Our first application concerns Conjecture 1.

Corollary 1 *Let X be a Banach space and let $1 \le \lambda < \infty$. If X^* or X^{**} has the Radon-Nikodým property, then the weak λ-bounded and the λ-bounded approximation properties are equivalent for X.*

PROOF. This is immediate from Theorem 2 applied to $Y = X$ and $T = I_X$.

Problem 2 *Are the weak λ-bounded and the λ-bounded approximation properties equivalent for a Banach space X having the Radon-Nikodým property?*

Recall that if X has the Radon-Nikodým property and is complemented in its bidual X^{**} by a projection P, then already the approximation property of X implies the $\|P\|$-bounded approximation property for X (see [18, Theorem 4] or e.g. [3, page 194]).

Corollary 1 applies, for instance, to closed subspaces of c_0. In particular, the Johnson-Schechtman space (see [13, Corollary JS]) X_{JS}, which has the 8-bounded approximation property (see [8, Theorem VI.3 and its proof]) and does not have the metric approximation property, does not have the weak metric approximation property. (This result was proven in [16, Proposition 2.3] relying on a theorem [16, Theorem 4.1] asserting that some geometric structure permits to lift the weak metric approximation property from spaces to their dual spaces.)

The famous example of a Banach space X_{FJ} which has the approximation property but fails the bounded approximation property due to Figiel and Johnson [6] can be done with X_{FJ}^* separable (see [6]). Therefore, by Corollary 1, X_{FJ} fails the weak bounded approximation property. By Corollary 1, also the Casazza-Jarchow space X_{CJ} [2, Theorem 1] fails the weak bounded approximation property. Recall that X_{CJ} has the approximation property, fails the bounded compact approximation property, and its duals $X_{CJ}^*, X_{CJ}^{**}, \dots$ are all separable and have the metric compact approximation property.

Let us recall that a Banach space X is said to have the *unique extension property* if the only operator $T \in \mathcal{L}(X^{**}, X^{**})$ such that $\|T\| \le 1$ and $T|_X = I_X$ is $T = I_{X^{**}}$. This property was introduced and deeply studied by Godefroy and Saphar in [9] (using the term "X is uniquely decomposed") and [10]. They proved in [10, Theorem 2.2] that the unique extension property of X permits to lift the metric approximation property from X to X^*. From this and Corollary 1, the following is immediate.

Corollary 2 *Let X be a Banach space having the unique extension property and let X^* or X^{**} has the Radon-Nikodým property. If X has the weak metric approximation property, then its dual X^* has the metric approximation property.*

Notice that Corollary 2 is also immediate from Theorem 1 and a recent result due to Vegard Lima [14, Theorem 2.9] that the unique extension property of X permits to lift the weak metric approximation property from X to X^*.

Concerning a possible strengthening of Corollary 2, let us mention the following special case of Problem 2.

Problem 3 *Are the weak metric and metric approximation properties equivalent for a Banach space X having the Radon-Nikodým property and the unique extension property?*

3.2. A conjecture of Defant and Floret

Consider the trace mapping V from the projective tensor product $X\hat{\otimes}Y^*$ to $(\mathcal{F}(X,Y))^*$, defined by

$$(Vu)(S) = \mathrm{trace}(Su), \quad u \in X\hat{\otimes}Y^*,\ S \in \mathcal{F}(X,Y).$$

It is well known and easy to see that $\|Vu\| \le \|u\|_\pi$, $u \in X\hat{\otimes}Y^*$. It is proven in [16] that Conjecture 1 is, in fact, equivalent to the following.

Conjecture 2 (Defant-Floret [3, page 283]). *The λ-bounded approximation property cannot be characterized using only reflexive spaces, that is, the λ-bounded approximation property of a Banach space X is not equivalent to the condition*

for all reflexive Banach spaces Y, the trace mapping $V : X\hat{\otimes}Y^* \to (F(X,Y))^*$ satisfies $\|u\|_\pi \le \lambda\|Vu\|$, $u \in X\otimes Y^*$. $(*)$

Let us mention that the λ-bounded approximation property of X is equivalent to the strengthening of condition $(*)$ with all Banach spaces Y (instead of reflexive Y). This well-known result (see, e.g., [3, page 193]) is essentially due to Grothendieck [11].

Concerning Conjecture 2, we have the next result.

Corollary 3 *Let X be a Banach space and let $1 \le \lambda < \infty$. If X^* or X^{**} has the Radon-Nikodým property, then the λ-bounded approximation property of X can be characterized using only the reflexive spaces, meaning that it is equivalent to condition $(*)$.*

PROOF. It is proven in [16, Theorem 3.2 and Remark 3.2] that $(*)$ is equivalent to the weak λ-bounded approximation property. Therefore the claim is immediate from Corollary 1.

3.3. A proof of Theorem 1

Let X be a Banach space such that X^* or X^{**} has the Radon-Nikodým property. Let us assume that X^* has the approximation property. By [16, Theorem 4.2], the approximation property of X^* is equivalent to the fact that X has the weak metric approximation property for all its equivalent renormings. Since the Radon-Nikodým property is invariant under isomorphisms, by Corollary 1, X has the metric approximation property for all its equivalent renormings. But in this case, X^* already has the metric approximation property by a well-known lifting result due to Johnson [12, Theorem 4] (see, e.g., [1, page 289]). This completes the proof of Theorem 1.

As the first step in the above proof, we applied Theorem 4.2 of [16]. For completeness, let us comment on ideas that we actually need from this theorem.

Since the approximation property is preserved under changes to equivalent norms, for the first step of the above proof, it suffices to prove that X has the weak metric approximation property. As we mentioned in the proof of Corollary 3, by [16, Theorem 3.2 and Remark 3.2], this is equivalent to condition $(*)$ with $\lambda = 1$. This condition can be verified as follows (see [16, Theorem 4.2, proof of $(c) \Rightarrow (a)$]).

Let Y be a reflexive Banach space and let $u \in X \otimes Y^*$. Since $(X \hat{\otimes} Y^*)^* = \mathcal{L}(X, Y^{**}) = \mathcal{L}(X, Y)$, there exists $T \in \mathcal{L}(X, Y)$ with $\|T\| = 1$ such that $\|u\|_\pi = \text{trace}(Tu)$. Since T is weakly compact, by a criterion of the approximation property for X^* in [17, Theorem 5], there exists a net $(T_\alpha) \subset \mathcal{F}(X, Y)$ with $\sup_\alpha \|T_\alpha\| \leq \|T\| = 1$ such that $T_\alpha^* y^* \to T^* y^*$ for all $y^* \in Y^*$. But then

$$\begin{aligned} \|u\|_\pi &= \text{trace}(Tu) = \lim_\alpha \text{trace}(T_\alpha u) \\ &\leq \sup_\alpha |\text{trace}(T_\alpha u)| = \sup_\alpha |(Vu)(T_\alpha)| \leq \|Vu\| \end{aligned}$$

as desired.

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