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Embedding into Banach spaces with finite dimensional decompositions

Edward W. Odell and Thomas Schlumprecht

Abstract. This paper deals with the following types of problems: Assume a Banach space X has some property (P). Can it be embedded into some Banach space Z with a finite dimensional decomposition having property (P), or more generally, having a property related to (P)? Secondly, given a class of Banach spaces, does there exist a Banach space in this class, or in a closely related one, which is universal for this class?

Inclusión en espacios de Banach con descomposiciones finito-dimensionales

Resumen. Este artículo trata los siguientes tipos de problemas: se supone que un espacio de Banach X tiene cierta propiedad (P), ¿puede incluirse en un espacio de Banach Z con una descomposición finito-dimensional que verifique (P), o, en general, una propiedad relacionada con (P)? En segundo lugar, dada una clase de espacios de Banach, ¿existe en dicha clase, o en una próxima, un espacio de Banach universal para la clase?

1. Introduction

The fact that every separable infinite dimensional real Banach space X embeds into $C[0,1]$ dates back to the early days of Banach space theory [3, Théorème 9, page 185]. This result has inspired two types of problems. First, given a space X in a certain class can it be embedded isomorphically into a space Y of the same class with a basis or, more generally, a finite dimensional decomposition (FDD)? Secondly, given a class of spaces does there exist a universal space X for that class which is in the class or in a closely related one? By saying X is universal for a class C we mean that each $Y \in C$ embeds into X. As it happens these two types of problems are often related in that solving a problem of the first type can lead to a solution to the analogous problem of second type.

For example, J. Bourgain [4] asked if there exists a separable reflexive space X which is universal for the class of all separable superreflexive Banach spaces. This question arose from his result that if X contains an isomorph of all separable reflexive spaces then X is universal, i.e., contains an isomorph of $C[0,1]$. This improved an earlier result of Szlenk [29] who showed X^* was not separable. Work by S. Pruss [28] showed that it sufficed to prove that for a separable superreflexive space Y there exists $1 < q \leq p < \infty$, $C < \infty$

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and a space Z with an FDD $E = (E_i)$ satisfying C-(p, q)-estimates,

$$C^{-1}\Big(\sum \|z_i\|^p\Big)^{1/p} \le \|\sum z_i\| \le C\Big(\sum \|z_i\|^q\Big)^{1/q}$$

for all block sequences (z_i) of Z w.r.t. (E_i). Such a space Z is automatically reflexive and thus we have the problem of given p, q, when does a reflexive space Y embed into such a space Z.

An earlier version of this problem (when $p = q$) was raised by W.B. Johnson [9] resulting from his work on L_p and earlier work with M. Zippin [14, 15]. The problem addressed in [9] was to characterize when a subspace X of L_p, $1 < p < 2$, embeds into ℓ_p. In [12] it was shown that if a subspace X of L_p, with $2 < p < \infty$, embeds into ℓ_p if (and only if by [16]) X does not contain an isomorph of ℓ_2 (later improved to X almost isometrically embeds into ℓ_p [18]). This characterization does not work in L_p, $1 \le p < 2$, since L_q embeds into L_p if $p \le q \le 2$, but the $p > 2$ characterization is equivalent (again by [16]) to every normalized basic sequence in X has a subsequence 2-equivalent to the unit vector basis of ℓ_p. Johnson showed that this criterion (with "2-equivalent" replaced by C-equivalent for some $C < \infty$) characterized when $X \subseteq L_p$, $1 < p < 2$, embeds into ℓ_p. His argument showed that X embedded into $(\sum H_n)_{\ell_p}$ for some blocking of the Haar basis into an $\mathrm{FDD}(H_n)$ and of course $(\sum H_n)_{\ell_p}$ embeds into ℓ_p. Johnson also considered the dual problem which brought quotient characterizations into the picture. These had appeared earlier [15] when it was shown that X embeds into $(\sum E_n)_{\ell_p}$, where (E_n) is a sequence of finite dimensional Banach spaces iff X is a quotient of such a space.

It turns out that the characterization required to ensure that a reflexive space X embeds into one with an FDD satisfying (p, q)-estimates is not a subsequence criterion in the general setting, i.e. if we do not assume X to be a subspace of L_p, but rather one that can be expressed in terms of weakly null trees in S_X, the unit sphere of X. This can be viewed as an infinite version of the notion of asymptotic structure [23]. If X is a Banach space then, for $n \in \mathbb{N}$, a normalized monotone basis is said to be in the n^{th}-*asymptotic structure of* X, and we write $(e_i)_{i=1}^n \in \{X\}_n$, if for all $\varepsilon > 0$ the following holds ($\mathrm{cof}(X)$ will denote the set of all closed subspaces of X having finite codimension):

$$\begin{aligned}&\forall X_1 \in \mathrm{cof}(X)\, \exists x_1 \in S_{X_1}\, \forall X_2 \in \mathrm{cof}(X)\, \exists x_2 \in S_{X_2}\, \ldots \forall X_n \in \mathrm{cof}(X)\, \exists x_n \in S_{X_n} \\ &\qquad (x_i)_{i=1}^n \text{ is } (1+\varepsilon)\text{-equivalent to } (e_i)_{i=1}^n.\end{aligned} \tag{1..1}$$

The fact that some normalized monotone basis $(e_i)_{i=1}^n$ is a member of $\{X\}_n$ can be, maybe more intuitively, described by a game between two players. Player I chooses $X_1 \in \mathrm{cof}(X)$, then Player II chooses $x_1 \in S_{X_1}$. This procedure is repeated until a sequence $(x_i)_{i=1}^n$ is obtained. Player II is declared winner of the game if $(x_i)_{i=1}^n$ is $(1+\varepsilon)$-equivalent to $(e_i)_{i=1}^n$. Condition (1..1) means that Player II has a winning strategy.

It is not hard to show that $\{X\}_n$ is a compact subset of $\mathcal{M}_n$, the set of all such normalized monotone bases $(e_i)_{i=1}^n$ under the metric $\log d_b(\cdot,\cdot)$ where $d_b((e_i)_{i=1}^n, (f_i)_{i=1}^n)$ is the basis equivalence constant between the bases. Lembergs [20] proof of Krivine's theorem shows that there is a $1 \le p \le \infty$, so that the unit vector basis of ℓ_p^n is in $\{X\}_n$ for all $n \in \mathbb{N}$. In [23] it is shown that $\{X\}_n$ is also the smallest closed subset $\mathcal{C}$ of $\mathcal{M}_n$ with the property that, for all $\varepsilon > 0$, player I has a winning strategy for forcing player II to select $(x_i)_{i=1}^n$ with $d_b((x_i)_{i=1}^n, \mathcal{C}) < 1 + \varepsilon$. This does not generalize to produce say $\{X\}_\infty$ since we lose compactness. However we can still consider a class $\mathcal{A}$ of normalized monotone bases with the property that in the infinite game player I has a winning strategy for forcing II to select $(x_i)_{i=1}^\infty \in \mathcal{A}$.

These notions can be restated in terms of weakly null trees when X^* is separable. Indeed $\{X\}_n$ is the smallest class such that every weakly null tree of length n in S_X admits a branch $(x_i)_{i=1}^n$ with $d_b((x_i)_{i=1}^n, \{X\}_n) < 1 + \varepsilon$. Precise definitions of weakly null trees and other terminology appear in Section 2.

If $\mathcal{A}$ is as above for X we can also restate the winning strategy for player I in terms of weakly null trees (of infinite level) but there are some difficulties. First given plays $X_1, X_2, \ldots$ by player I we cannot select a branch (x_i) with $x_i \in X_i$ for all i but only that x_i is close to an element of S_{X_i}. Secondly not all games are determined so we need a fattening $\mathcal{A}_\varepsilon$ of $\mathcal{A}$ and then need to close it to $\overline{\mathcal{A}}_\varepsilon$ in the product of the discrete topology on S_X to obtain a determined game. This will lead to the property that if every weakly null tree in X admits a branch in $\mathcal{A}$ then if $X \subseteq Z$, a space with an appropriate FDD (E_i), one can find a blocking (F_i) of (E_i) and $\bar{\delta} = (\delta)$, $\delta_i \downarrow 0$, so that every $(x_i) \subseteq S_X$ which is a $\bar{\delta}$-skipped block sequence w.r.t. (F_i) is in $\overline{\mathcal{A}}_\varepsilon$. These will be defined precisely in Section 2.

An application will be the solution of Johnson's problem (when does a reflexive space X embed into an ℓ_p-FDD?)), Pruss' problem (when does a reflexive space X embed into one with an FDD satisfying (p, q)-estimates) and, as a consequence, Bourgain's problem. These solutions will be given in Sections 4 and 5. Among other characterizations we will show that if for some $C < \infty$ every weakly null tree in a reflexive space X admits a branch C-dominating the unit vector basis of ℓ_p and a branch C-dominated by the unit vector basis of ℓ_q then X embeds into a space with an FDD satisfying (p, q)-estimates.

The machinery developed in Section 2 also has applications in the nonreflexive setting. In Section 3 we consider and characterize spaces X of Szlenk index ω, the smallest possible. $S_z(X)$ is an ordinal index which is less than ω_1 iff X^* is separable. For $\varepsilon > 0$ set $K_0(X, \varepsilon) = B_{X^*}$ and for $\alpha < \omega_1$ we recursively define

$$K_{\alpha+1}(X,\varepsilon) = \left\{ x^* \in K_\alpha(X,\varepsilon) : \begin{array}{c} \exists\, (x_n^*) \subseteq K_\alpha(x,\varepsilon) \text{ with} \\ w^* - \lim_{n\to\infty} x_n^* = x^* \text{ and } \liminf_{n\to\infty} \|x_n^* - x^*\| \geq \varepsilon \end{array} \right\}$$

If α is a limit ordinal,

$$K_\alpha(X,\varepsilon) = \bigcap_{\beta<\alpha} K_\beta(X,\varepsilon) \ .$$

$S_Z(X, \varepsilon)$ is the smallest α with $K_\alpha(X, \varepsilon) = \emptyset$ or ω_1 otherwise.

$$S_Z(X) = \sup\{S_Z(X,\varepsilon) : 0 < \varepsilon < 1\} \ .$$

We will show that $S_z(X) = \omega$ iff X^* can be embedded as a w^*-closed subspace of a space Z with an FDD satisfying 1-$(p, 1)$-estimates. A long list of further equivalent conditions (Theorem 3.4) will be given including that X can be renormed to be w^*-uniform Kadec Klee and X can be renormed to be asymptotically uniformly smooth (of power type q for some $q > 1$).

Asymptotic uniformly smooth (a.u.s.) and asymptotic uniformly convex (a.u.c) norms, defined in Section 3, are asymptotic versions of uniformly smooth and uniformly convex due to [11] based upon modulii of V.D. Milman [22]. Theorem 3.4, mentioned above, gives the result that X can be given an a.u.s. norm iff it can be given one of power type q for some $q > 1$. We obtain a similar result for a.u.c. for reflexive spaces. Recall that Pisier [27] proved that a superreflexive (equivalently, uniformly convex) space can be renormed to be uniformly convex of power type p for some $2 \leq p < \infty$ and similarly for uniformly smooth with $1 < p \leq 2$.

In Section 3 we also give a proof of Kalton's theorem [17] that a Banach space X embeds into c_0 if for some $C < \infty$ every weakly null tree in S_X admits a branch $(x_i)_{i=1}^\infty$ satisfying $\sup_n \|\sum_1^n x_i\| \leq C$. This proof fits nicely into our Section 2 machinery.

In Section 5 we discuss applications of our results to universal problems. In regard to Bourgain's problem we show the space constructed is universal for the class

$$\{X : X \text{ is reflexive, } S_z(X) = S_z(X^*) = \omega\} \ ,$$

which includes all superreflexive spaces. We also discuss the universal problem for reflexive a.u.s. (or a.u.c.) spaces.

A central theme of the problems we have presented is coordinatization. A coordinate-free property is considered and we wish to embed a space X with this property into a space Z with an FDD which realizes this property w.r.t. its "coordinates". The tools we use, in addition to the ones mentioned above, are several. There are the blocking arguments of Johnson and Zippin [9], [14, 15] and some known embedding theorems which we cite now.

1.1 [5]

If X^* is separable then X is a quotient of a space with a shrinking basis.

1.2 [30]

If X^* is separable then X embeds into a space with a shrinking basis.

1.3 [30]

If X is reflexive then X embeds into a reflexive space with a basis.

We will often begin with $X \subseteq Z$, one of the spaces given by 1.2, 1.3 or with X a quotient of Z (as in 1.1) and the problem will be to put a new norm on Z which reflects the structure of X that we wish to coordinatize and maintains that X is a subspace of Z (or a quotient).

All of our Banach spaces in this paper are real and separable. We will use $X, Y, Z, \ldots$ for infinite dimensional spaces and $E, F, G, \ldots$ for finite dimensional spaces.

Most of the results we will present have appeared in a number of recent papers ([24], [25], [26] [19], [17], [7], [11]). As the theory has developed the proofs and results have been better understood, generalized and improved. Our aim is to give a unified presentation of these improvements and in several cases present easier proofs. New results are also included.

2. A general combinatorial result

In this section we state and prove three general combinatorial results (Theorem 1 and Corollaries 1 and 2). There statement are reformulations and improvements of results in [24]. We will present a different, possibly more accessible, proof.

We first introduce some notation.

Let Z be a Banach space with an FDD $E = (E_n)$. For $n \in \mathbb{N}$ we denote the n-th *coordinate projection* by P_n^E, i.e. $P_n^E : Z \to E_n, \quad \sum z_i \mapsto z_n$. For finite $A \subset \mathbb{N}$ we put $P_A^E = \sum_{n\in A} P_n^E$. The *projection constant* of (E_n) (in Z) is defined by

$$K = K(E, Z) = \sup_{m\le n} \|P_{[m,n]}^E\|.$$

Recall that K is always finite and, as in the case of bases, we call (E_n) *bimonotone (in Z)* if $K = 1$. By passing to the equivalent norm

$$|||\cdot||| : Z \to \mathbb{R}, \quad z \mapsto \sup_{m\le n} \|P_{[m,n]}^E(z)\|,$$

we can always renorm Z, so that $K = 1$.

For a sequence (E_i) of finite dimensional spaces we define the vector space

$$c_{00}(\oplus_{i=1}^\infty E_i) = \big\{(z_i) : z_i \in E_i, \text{ for } i \in \mathbb{N}, \text{ and } \{i \in \mathbb{N} : z_i \neq 0\} \text{ is finite}\big\},$$

which is dense in each Banach space for which (E_n) is an FDD. For $A \subset \mathbb{N}$ we denote by $\oplus_{i\in A} E_i$ the linear subspace of $c_{00}(\oplus E_i)$ generated by the elements of $(E_i)_{i\in A}$ and we denote its closure in Z by $(\oplus E_i)_Z$. As usual we denote the vector space of sequences in $\mathbb{R}$ which are eventually zero by c_{00} and its unit vector basis by (e_i).

The vector space $c_{00}(\oplus_{i=1}^\infty E_i^*)$, where E_i^* is the dual space of E_i, for $i \in \mathbb{N}$, is a w^*-dense subspace of Z^*. We denote the norm closure of $c_{00}(\oplus_{i=1}^\infty E_i^*)$ in Z^* by $Z^{(*)}$. $Z^{(*)}$ is w^*-dense in Z^*, the unit ball $B_{Z^{(*)}}$ norms Z and (E_i^*) is an FDD of $Z^{(*)}$ having a projection constant not exceeding $K(E, Z)$. If $K(E, Z) = 1$ then $B_{Z^{(*)}}$ is 1-norming and $Z^{(*)(*)} = Z$.

For $z \in c_{00}(\oplus E_i)$ we define the *E-support of z* by

$$\mathrm{supp}_E(z) = \big\{ i \in \mathbb{N} : P_i^E(z) \neq 0 \big\}.$$

A non-zero sequence (finite or infinite) $(z_j) \subset c_{00}(\oplus E_i)$ is called a *block sequence of* (E_i) if

$$\max \mathrm{supp}_E(z_n) < \min \mathrm{supp}_E(z_{n+1}), \text{ whenever } n \in \mathbb{N} \text{ (or } n < \text{length}(z_j)),$$

and it is called a *skipped block sequence of* (E_i) if

$$1 < \min \mathrm{supp}_E(z_1) < \max \mathrm{supp}_E(z_n) < \min \mathrm{supp}_E(z_{n+1}) - 1, \text{ whenever } n \in \mathbb{N} \text{ (or } n < \text{length}(z_i)).$$

Let $\overline{\delta} = (\delta_n) \subset (0, 1]$. A (finite or infinite) sequence $(z_j) \subset S_Z = \{z \in Z : \|z\| = 1\}$ is called a *$\overline{\delta}$-block sequence of* (E_n) or a *$\overline{\delta}$-skipped block sequence of* (E_n) if there are $1 \leq k_1 < \ell_1 < k_2 < \ell_2 < \dots$ in $\mathbb{N}$ so that

$$\|z_n - P^E_{[k_n,\ell_n]}(z_n)\| < \delta_n, \text{ or } \|z_n - P^E_{(k_n,\ell_n]}(z_n)\| < \delta_n, \text{ respectively,}$$

for all $n \in \mathbb{N}$ (or $n \leq$ length(z_j)). Of course one could generalize the notion of $\overline{\delta}$- block and $\overline{\delta}$-skipped block sequences to more general sequences, but we prefer to introduce this notion only for normalized sequences. It is important to note that in the definition of $\overline{\delta}$-skipped block sequences $k_1 \geq 1$, and that therefore the E_1-coordinate of z_1 is small (depending on δ_1).

A sequence of finite-dimensional spaces (G_n) is called a *blocking of* (E_n) if there are $0 = k_0 < k_1 < k_2 < \dots$ in $\mathbb{N}$ so that $G_n = \oplus_{i=k_{n-1}+1}^{k_n} E_i$, for $n = 1, 2 \dots$.

We denote the sequences in S_Z of length $n \in \mathbb{N}$ by S_Z^n and the infinite sequences in S_Z by S_Z^ω. For $m, n \in \mathbb{N}$, for $x = (x_1, x_2, \dots x_m) \in S_Z^m$ and $y = (y_1, y_2, \dots y_n) \in S_Z^n$ or $y = (y_i) \in S_Z^\omega$ we denote the concatenation of x and y by (x, y), i.e.

$$(x, y) = (x_1, x_2 \dots x_m, y_1, \dots y_m), \text{ or } (x, y) = (x_1, x_2 \dots x_m, y_1, y_2 \dots) \text{ respectively .}$$

We also allow the case $x = \emptyset$ or $y = \emptyset$ and let $(\emptyset, y) = y$ and $(x, \emptyset) = x$.

Let $\mathcal{A} \subset S_Z^\omega$ be given. We denote the closure of $\mathcal{A}$ with respect to the product topology of the discrete topology on S_Z by $\overline{\mathcal{A}}$. Note that if $\mathcal{A}$ is closed it follows for $x = (x_i) \in S_Z^w$,

$$x \in \mathcal{A} \iff \forall n \in \mathbb{N}\, \exists z \in S_Z^\omega \quad (x_1, x_2, \dots x_n, z) \in \mathcal{A} \tag{2..1}$$

If $\overline{\varepsilon} = (\varepsilon_i)$ is a sequence in $[0, \infty)$ we write

$$\mathcal{A}_{\overline{\varepsilon}} = \big\{ (z_i) \in S_Z^\omega : \exists (\tilde{z}_i) \in \mathcal{A}, \quad \|z_i - \tilde{z}_i\| \leq \varepsilon_i \big\}$$

and call the set $\mathcal{A}_{\overline{\varepsilon}}$ *the $\overline{\varepsilon}$-fattening of* $\mathcal{A}$. For $\ell \in \mathbb{N}$ and $\overline{\varepsilon} = (\varepsilon_i)_{i=1}^\ell \subset [0, \infty)$ we let $\mathcal{A}_{\overline{\varepsilon}} = \mathcal{A}_{\overline{\delta}}$, where $\overline{\delta} = (\delta_i)$ and $\delta_i = \varepsilon_i$, for $i = 1, 2 \dots \ell$ and $\delta_i = 0$ if $i > \ell$.

If $\ell \in \mathbb{N}$ and $x_1, x_2, \dots x_\ell \in S_Z$ we let

$$\mathcal{A}(x_1, x_2, \dots x_\ell) = \big\{ z = (z_i) \in S_Z^\omega : (x_1, x_2, \dots x_\ell, z) \in \mathcal{A} \big\}.$$

Let $\mathcal{A} \subset S_Z^\omega$ and $\mathcal{B} = \prod_{i=1}^\infty B_i$, where $B_n \subset S_Z$ for $n \in \mathbb{N}$.

We consider the following $(\mathcal{A}, \mathcal{B})$*-game* between two players: Assume that $E = (E_i)$ is an FDD for Z.

Player I chooses $n_1 \in \mathbb{N}$

Player II chooses $z_1 \in c_{00}\big(\oplus_{i=n_1+1}^\infty E_i \big) \cap B_1$,

Player I chooses $n_2 \in \mathbb{N}$

Player II chooses $z_2 \in c_{00}\big(\oplus_{i=n_2+1}^\infty E_i \big) \cap B_2$,

$\vdots$

Player I wins the $(\mathcal{A},\mathcal{B})$-game if the resulting sequence (z_n) lies in $\mathcal{A}$. If Player I has a winning strategy (forcing the sequence (z_i) to be in $\mathcal{A}$) we will write $WI(\mathcal{A},\mathcal{B})$ and if Player II has a winning strategy (being able to choose (z_i) outside of $\mathcal{A}$) we write $WII(\mathcal{A},\mathcal{B})$. If $\mathcal{A}$ is a Borel set with respect to the product of the discrete topology on S_Z^ω (note that $\mathcal{B}$ is always closed in the product of the discrete topology on S_Z^ω), it follows from the main theorem in [21] that the game is determined, i.e. either $WI(\mathcal{A},\mathcal{B})$ or $WII(\mathcal{A},\mathcal{B})$.

Let us define $WII(\mathcal{A},\mathcal{B})$ formally. We will need to introduce trees in Banach spaces.

We define

$$T_\infty = \bigcup_{\ell\in\mathbb{N}} \big\{(n_1,n_2,\ldots,n_\ell) : n_1 < n_2 < \ldots n_\ell \text{ are in } \mathbb{N}\big\}.$$

If $\alpha=(m_1,m_2,\ldots m_\ell)\in T_\infty$, we call ℓ the *length of* α and denote it by $|\alpha|$, and $\beta=(n_1,n_2,\ldots n_k)\in T_\infty$ is called an *extension of* α, or α is called *a restriction of* β, if $k\ge\ell$ and $n_i=m_i$, for $i=1,2,\ldots,\ell$. We then write $\alpha\le\beta$ and with this order $(T_\infty,\le)$ is a tree.

In this work *trees* in a Banach space X are families in X indexed by T_∞, thus they are countable infinitely branching trees of countably infinite length.

For a tree $(x_\alpha)_{\alpha\in T_\infty}$ in a Banach space X, and $\alpha=(n_1,n_2,\ldots,n_\ell)\in T_\infty\cup\{\emptyset\}$ we call the sequences of the form $(x_{(\alpha,n)})_{n>n_\ell}$ *nodes of* $(x_\alpha)_{\alpha\in T_\infty}$. The sequences (y_n), with $y_i=x_{(n_1,n_2,\ldots,n_i)}$, for $i\in\mathbb{N}$, for some strictly increasing sequence $(n_i)\subset\mathbb{N}$, are called *branches of* $(x_\alpha)_{\alpha\in T_\infty}$. Thus, branches of a tree $(x_\alpha)_{\alpha\in T_\infty}$ are sequences of the form (x_{α_n}) where (α_n) is a maximal linearly ordered (with respect to extension) subset of T_∞.

If $(x_\alpha)_{\alpha\in T_\infty}$ is a tree in X and if $T'\subset T_\infty$ is closed under taking restrictions so that for each $\alpha\in T'\cup\{\emptyset\}$ infinitely many direct successors of α are also in T' then we call $(x_\alpha)_{\alpha\in T'}$ a *full subtree of* $(x_\alpha)_{\alpha\in T_\infty}$. Note that $(x_\alpha)_{\alpha\in T'}$ could then be relabelled to a family indexed by T_∞ and note that the branches of $(x_\alpha)_{\alpha\in T'}$ are branches of $(x_\alpha)_{\alpha\in T_\infty}$ and that the nodes of $(x_\alpha)_{\alpha\in T'}$ are subsequences of certain nodes of $(x_\alpha)_{\alpha\in T_\infty}$.

We call a tree $(x_\alpha)_{\alpha\in T_\infty}$ in a Banach space X *normalized* if $\|x_\alpha\|=1$, for all $\alpha\in T_\infty$ and *weakly null* if every node is weakly null. More generally if $\mathcal{T}$ is a topology on X and a tree $(x_\alpha)_{\alpha\in T_\infty}$ in a Banach space X is called $\mathcal{T}$*-null* if every node converges to 0 with respect to $\mathcal{T}$.

If $(x_\alpha)_{\alpha\in T_\infty}$ is a tree in a Banach space Z which has an FDD (E_n) we call it a *block tree of* (E_n) if every node is a block sequence of (E_n).

We will also need to consider trees of finite length. For $\ell\in\mathbb{N}$ we call a family $(x_\alpha)_{\alpha\in T_\infty,|\alpha|\le\ell}$ in X a *tree of length* ℓ. Note that the notions nodes, branches, $\mathcal{T}$-null and block trees can be defined analogously for trees of finite length.

Definition 1. *Assume that Z is a Banach space with an FDD (E_i), $\mathcal{A}\subset S_Z^\omega$ and $\mathcal{B}=\prod_{i=1}^\infty B_i$, with $B_i\subset S_Z$ for $i\in\mathbb{N}$. We say that* Player II has a winning strategy for the $(\mathcal{A},\mathcal{B})$-game *if*

($WII(\mathcal{A},\mathcal{B})$) There exists a block tree $(x_\alpha)_{\alpha\in T_\infty}$ of (E_i) in S_Z all of whose branches are in $\mathcal{B}$ but none of its branches are in $\mathcal{A}$.

In case that the $(\mathcal{A},\mathcal{B})$-game is determined $WI(\mathcal{A},\mathcal{B})$ can be therefore stated as follows.

($WI(\mathcal{A},\mathcal{B})$) Every block tree $(x_\alpha)_{\alpha\in T_\infty}$ of (E_i) in S_X all of whose branches are in $\mathcal{B}$ has a branch in $\mathcal{A}$.

The proof of the following Proposition is easy.

Proposition 1. *Let $\mathcal{A},\widetilde{\mathcal{A}}\subset S_Z^\omega$, $\mathcal{B}=\prod_{i=1}^\infty B_i$, with $B_i\subset S_Z$ for $i\in\mathbb{N}$. Assume that the $(\mathcal{A},\mathcal{B})$-game and the $(\widetilde{\mathcal{A}},\mathcal{B})$-game are determined.*

a) If $\mathcal{A}\subset\widetilde{\mathcal{A}}$, then

$$WI(\mathcal{A},\mathcal{B})\Rightarrow WI(\widetilde{\mathcal{A}},\mathcal{B}) \text{ and } WII(\widetilde{\mathcal{A}},\mathcal{B})\Rightarrow WII(\mathcal{A},\mathcal{B}).$$

b) $WI(\mathcal{A},\mathcal{B}) \iff \exists n\in\mathbb{N}\,\forall x\in\big(\oplus_{i=n+1}^\infty E_i\big)\cap B_1 \quad WI((\mathcal{A}(x),\prod_{i=2}^\infty B_i)$

c) If $\ell \in \mathbb{N}$, $\overline{\varepsilon} = (\varepsilon_i)_{i=1}^{\ell} \subset [0,\infty)$ and $x_i, y_i \in B_i$ with $\|x_i - y_i\| \le \varepsilon_i$ for $i = 1, 2 \dots \ell$ then

$$WI\Big(\mathcal{A}(x_1, x_2, \dots, x_\ell), \prod_{i=\ell+1}^{\infty} B_i\Big) \Rightarrow WI\Big(\mathcal{A}_{\overline{\varepsilon}}(y_1, y_2, \dots, y_\ell), \prod_{i=\ell+1}^{\infty} B_i\Big).$$

Lemma 1. *Let $\mathcal{A}$, and $\overline{\varepsilon} = (\varepsilon_i), \overline{\delta} = (\delta_i) \subset [0,\infty)$ Then*

$$\overline{(\overline{\mathcal{A}_{\overline{\varepsilon}}})_{\overline{\delta}}} \subset \overline{\mathcal{A}_{\overline{\varepsilon}+\overline{\delta}}}.$$

PROOF. We observe

$$\begin{aligned}
u = (u_i) \in \overline{(\overline{\mathcal{A}_{\overline{\varepsilon}}})_{\overline{\delta}}}
&\iff \forall n \in \mathbb{N}\, \exists v^{(n)} \in S_Z^{\omega} \quad (u_1, \dots, u_n, v^{(n)}) \in \big(\overline{\mathcal{A}_{\overline{\varepsilon}}}\big)_{\overline{\delta}} \\
&\Rightarrow \forall n \in \mathbb{N}\, \exists x_1, x_2, \dots x_n \in S_Z \text{ and } w^{(n)} \in S_Z^{\omega} \\
&\qquad \|x_i - u_i\| \le \delta_i, \text{ for } i = 1, \dots n, \text{ and } (x_1, \dots, x_n, w^{(n)}) \in \overline{\mathcal{A}_{\overline{\varepsilon}}}, \\
&\Rightarrow \forall n \in \mathbb{N}\, \exists x_1, x_2, \dots x_n \in S_Z \text{ and } w^{(n)} \in S_Z^{\omega}\, \forall m \in \mathbb{N} \exists y^{(m)} \in S_Z^{\omega} \\
&\qquad \|x_i - u_i\| \le \delta_i \text{ for } i = 1, 2 \dots n \text{ and } (x_1, \dots, x_n, w_1^{(n)}, w_2^{(n)} \dots w_m^{(n)}, y^{(m)}) \in \mathcal{A}_{\overline{\varepsilon}} \\
&\Longrightarrow \forall n \in \mathbb{N}\, \exists x_1, x_2, \dots x_n \in S_Z\, \exists y^{(n)} \in S_Z^{\omega} \\
&\qquad \|x_i - u_i\| \le \delta_i \text{ for } i = 1, 2 \dots n \text{ and } (x_1, \dots, x_n, y^{(n)}) \in \mathcal{A}_{\overline{\varepsilon}} \\
&\Longrightarrow \forall \ell \in \mathbb{N}\, \exists z^{(\ell)} \in \mathcal{A} \quad \|u_i - z_i^{(\ell)}\| \le \delta_i + \varepsilon_i, \text{ for } i = 1, 2 \dots \ell \\
&\iff u \in \overline{\mathcal{A}_{\overline{\varepsilon}+\overline{\delta}}}.
\end{aligned}$$

Now we can state one of our main combinatorial principles.

Theorem 1. *Let Z have an FDD (E_i) and let $B_i \subset S_Z$, for $i = 1, 2 \dots$. Put $\mathcal{B} = \prod_{i=1}^{\infty} B_i$ and let $\mathcal{A} \subset S_Z^{\omega}$.*

Assume that for all $\overline{\varepsilon} = (\varepsilon_i) \subset (0,1]$ we have $WI(\overline{\mathcal{A}_{\overline{\varepsilon}}}, \mathcal{B})$.

Then for all $\overline{\varepsilon} = (\varepsilon_i) \subset (0,1]$ there exists a blocking (G_i) of (E_i) so that every skipped block sequence (z_i) of (G_i), with $z_i \in B_i$, for $i \in \mathbb{N}$, is in $\overline{\mathcal{A}_{\overline{\varepsilon}}}$.

PROOF. Let $\overline{\varepsilon} = (\varepsilon_i) \subset (0,1]$ be given. For $k = 0, 1, 2 \dots$ put $\overline{\varepsilon}^{(k)} = (\varepsilon_i^{(k)})$ with $\varepsilon_i^{(k)} = \varepsilon_i(1 - 2^{-k})/2$ for $i \in \mathbb{N}$.

We put $\widetilde{\mathcal{A}} = \overline{\mathcal{A}_{\overline{\varepsilon}/2}}$.

For $\ell \in \mathbb{N}$ we write $\mathcal{B}^{(\ell)} = \prod_{i=\ell+1}^{\infty} B_i$.

By induction we choose for $k \in \mathbb{N}$ numbers $n_k \in \mathbb{N}$ so that $0 = n_0 < n_1 < n_2 < \dots$, and so that for any $k \in \mathbb{N}$, putting with $G_k = \oplus_{i=n_{k-1}}^{n_k} E_i$,

$$WI\big(\overline{\widetilde{\mathcal{A}}_{\overline{\varepsilon}^{(k)}}}(\sigma, x), \mathcal{B}^{(\ell+1)}\big) \text{ for any } 0 \le \ell < k \text{ and any normalized skipped block} \tag{2..2}$$

$$\sigma = (x_1, x_2, \dots x_\ell) \in \prod_{i=1}^{\ell} B_i \text{ of } (G_i)_{i=1}^{k-1}\ (\sigma = \emptyset \text{ if } \ell = 0)$$

$$\text{and } x \in S_{\oplus_{i=n_k+1}^{\infty} E_i} \cap B_{\ell+1}$$

$$WI\big(\overline{\widetilde{\mathcal{A}}_{\overline{\varepsilon}^{(k)}}}(\sigma), \mathcal{B}^{(\ell)}\big) \text{ for any } 0 \le \ell < k \text{ and any normalized skipped block} \tag{2..3}$$

$$\sigma = (x_1, x_2, \dots x_\ell) \in \prod_{i=1}^{\ell} B_i \text{ of } (G_i)_{i=1}^{k}$$

For $k = 1$ we deduce from Proposition 1 (b) that there is an $n_1 \in \mathbb{N}$ so that $WI\big(\overline{\widetilde{\mathcal{A}}}_{\overline{\varepsilon}^{(1)}}(x), \mathcal{B}^{(1)}\big)$ for any $x \in S_{\oplus_{i=n_1+1}^{\infty} E_i} \cap B_i$. This implies (2..2) and (2..3) (note that for $k = 1$ σ can only be chosen to be $\emptyset$ in (2..2) and (2..3)).

Assume $n_1 < n_2 < \dots n_k$ have been chosen for some $k \in \mathbb{N}$. We will first choose n_{k+1} so that (2..2) is satisfied. In the case that $k = 1$ we simply choose $n_2 = n_1 + 1$ and note that (2..2) for $k = 2$ follows from (2..2) for $k = 1$ since in both cases $\sigma = \emptyset$ is the only choice. If $k > 1$ we can use the compactness of the sphere of a finite dimensional space and choose a finite set $\mathcal{F}$ of normalized skipped blocks $(x_1, x_2, \dots, x_\ell) \in \prod_{i=1}^{\ell} B_i$, of $(G_i)_{i=1}^{k}$ so that for any $\ell \le k$ and any normalized skipped block with length ℓ, $\sigma = (x_1, x_2, \dots, x_\ell) \in \prod_{i=1}^{\ell} B_i$ of $(G_i)_{i=1}^{k}$, there is a $\sigma' = (x_1', x_2', \dots, x_\ell') \in \mathcal{F}$ with $\|x_i - x_i'\| < \varepsilon_i 2^{-k-2}$, for $i = 1, 2, \dots, \ell$. Then, using the induction hypothesis (2..3) for k, and Proposition 1 (b), we choose $n_{k+1} \in \mathbb{N}$ large enough so that $WI\big(\overline{\widetilde{\mathcal{A}}}_{\overline{\varepsilon}^{(k)}}(\sigma, x), \mathcal{B}^{(\ell+1)}\big)$ for any $\sigma \in \mathcal{F}$ and $x \in S_{\oplus_{i=n_{k+1}+1}^{\infty} E_i} \cap B_{\ell+1}$. From Proposition 1 (c) and our choice of $\mathcal{F}$ we deduce $WI\big(\overline{\widetilde{\mathcal{A}}}_{\overline{\varepsilon}^{(k+1)}}(\sigma, x), \mathcal{B}^{(\ell+1)}\big)$ for any $0 \le \ell < k$, any normalized skipped block σ of $(G_i)_{i=1}^{k}$ of length ℓ in $\prod_{i=1}^{\ell} B_i$ and any $x \in S_{\oplus_{i=n_{k+1}+1}^{\infty} E_i} \cap B_{\ell+1}$, and, thus, (using the induction hypothesis for $\sigma = \emptyset$) we deduce (2..2) for $k + 1$.

In order to verify (2..3) let $\sigma = (x_1, x_2, \dots, x_\ell) \in \prod_{i=1}^{\ell} B_i$ be a normalized skipped block of $(G_i)_{i=1}^{k+1}$ (the case $\sigma = \emptyset$ follows from the induction hypothesis). Then $\sigma' = (x_1, x_2, \dots, x_{\ell-1})$ is empty or a normalized skipped block sequence of $(G_i)_{i=1}^{k-1}$ in $\prod_{i=1}^{\ell-1} B_i$. In the second case $WI\big(\overline{\widetilde{\mathcal{A}}}_{\overline{\varepsilon}^{(k+1)}})(\sigma), \mathcal{B}^{(\ell)}\big) = WI\big(\overline{\widetilde{\mathcal{A}}}_{\overline{\varepsilon}^{(k+1)}})(\sigma', x_\ell), \mathcal{B}^{(\ell)}\big)$ follows from (2..2) for k and from Proposition 1 (a). This finishes the recursive definition of the n_k's and G_k's.

Let (z_n) any normalized skipped block sequence of (G_i) which lies in $\mathcal{B}$. For any $n \in \mathbb{N}$ it follows from (2..3) for $\sigma = (z_i)_{i=1}^{n}$ that $WI(\overline{\widetilde{\mathcal{A}}_{\overline{\varepsilon}/2}}(\sigma), \mathcal{B})$, and, thus, $\widetilde{\mathcal{A}}_{\overline{\varepsilon}/2}(\sigma) \neq \emptyset$, which means that σ is extendable to a sequence in $\widetilde{\mathcal{A}}_{\overline{\varepsilon}/2}$ (note that $\lim_{n\to\infty} \varepsilon_i^{(n)} = \varepsilon_i$). Thus, any normalized skipped block sequence which is element of $\mathcal{B}$ lies in $\overline{\widetilde{\mathcal{A}}_{\overline{\varepsilon}/2}}$ and, thus, by Lemma 1, in $\overline{\mathcal{A}_{\overline{\varepsilon}}}$.

Now let X be a closed subspace of Z having an FDD (E_i) and $\mathcal{A} \subset S_X^{\omega}$. We consider the following game

Player I chooses $n_1 \in \mathbb{N}$
Player II chooses $x_1 \in \big(\oplus_{i=n_1+1}^{\infty} E_i\big)_Z \cap X$, $\|x_1\| = 1$,
Player I chooses $n_2 \in \mathbb{N}$
Player II chooses $x_2 \in \big(\oplus_{i=n_2+1}^{\infty} E_i\big)_Z \cap X$, $\|x_2\| = 1$,
$\vdots$

As before, Player I has won if $(x_i) \in \mathcal{A}$. Since the game does not only depend on $\mathcal{A}$ but on the superspace Z in which X is embedded and its FDD (E_I) we denote the game $(\mathcal{A}, Z)$-game.

Definition 2. *Assume that X is the subspace of a space Z which has an FDD (E_i) and that $\mathcal{A} \subset S_X^{\omega}$. Define for $n \in \mathbb{N}$*

$$X_n = X \cap \big(\oplus_{i=n+1}^{\infty} E_i\big)_Z = \{x \in X : \forall z^* \in \oplus_{i=1}^{n} E_i^* \quad z^*(x) = 0\}\big),$$

a closed subspace with finite codimension in X.

We say that Player II has a winning strategy in the $(\mathcal{A}, Z)$-game if

$WII(\mathcal{A}, Z)$ there is a tree $(x_\alpha)_{\alpha \in T_\infty}$ in S_X so that for any $\alpha = (n_1, \dots n_\ell) \in T_\infty \cup \emptyset$ $x_{(\alpha,n)} \in X_n$ whenever $n > n_\ell$, and so that no branch lies in $\mathcal{A}$.
In case that the $(\mathcal{A}, Z)$-game is determined Player I has a winning strategy in the $(\mathcal{A}, Z)$-game if *the negation of $WII(\mathcal{A}, Z)$ is true and thus*

$WI(\mathcal{A},Z)$ For any tree $(x_\alpha)_{\alpha\in T_\infty}$ in S_X so that for any $\alpha=(n_1,\dots n_\ell)\in T_\infty\cup\emptyset$ $x_{(\alpha,n)}\in X_n$ whenever $n>n_\ell$, there is branch in $\mathcal{A}$.

For $\mathcal{A}\subset S_X^\omega\subset S_Z^\omega$ and a sequence $\overline{\varepsilon}=(\varepsilon_i)$ in $[0,\infty)$ we understand by $\mathcal{A}_{\overline{\varepsilon}}$ the $\overline{\varepsilon}$-fattening of $\mathcal{A}$ as a subset of S_Z^ω. In case we want to restrict ourselves to S_X we write $\mathcal{A}^X_{\overline{\varepsilon}}$, i.e.

$$\mathcal{A}^X_{\overline{\varepsilon}}=\mathcal{A}_{\overline{\varepsilon}}\cap S_X^\omega=\big\{(x_i)\in S_X^\omega:\exists(y_i)\in\mathcal{A}\quad\|x_i-y_i\|\le\varepsilon_i \text{ for all } i\in\mathbb{N}\big\}.$$

Since S_X^ω is closed in S_Z^ω with respect to the product of the discrete topology, we deduce that $\overline{\mathcal{A}}^X=\overline{\mathcal{A}^X}$ for $\mathcal{A}\subset S_X^\omega$.

The following Proposition reduces the $(\mathcal{A},Z)$-game to a game we treated before. In order to be able to do so we need some technical assumption on the embedding of X into Z (see condition (2..4) below).

Proposition 2. *Let $X\subset Z$, a space with an FDD (E_i). Assume the following condition on X, Z and the embedding of X into Z is satisfied:*

$$\begin{gathered}\textit{There is a } C>0 \textit{ so that for all } m\in\mathbb{N} \textit{ and } \varepsilon>0 \textit{ there is an } n=n(\varepsilon,m)\ge m\\ \|x\|_{X/X_m}\le C\big[\|P^E_{[1,n]}(x)\|+\varepsilon\big] \textit{ whenever } x\in S_X.\end{gathered}\tag{2..4}$$

Assume that $\mathcal{A}\subset S_X^\omega$ and that for all null sequences $\overline{\varepsilon}\subset(0,1]$ we have $WI(\overline{\mathcal{A}^X_{\overline{\varepsilon}}},Z)$.

Then it follows for all null sequences $\overline{\varepsilon}=(\varepsilon_i)\subset(0,1]$ that $WI(\overline{\mathcal{A}_{\overline{\varepsilon}}},(S_X^\omega)_{\overline{\delta}})$ holds, where $\overline{\delta}=(\delta_i)$ with $\delta_i=\varepsilon_i/28CK$ for $i\in\mathbb{N}$, with C satisfying (2..4) and K being the projection constant of (E_i) in Z.

Proof. Let $\mathcal{A}\subset S_X^\omega$ and assume that $WI(\overline{\mathcal{A}^X_{\overline{\eta}}},Z)$ is satisfied for all null sequences $\overline{\eta}=(\eta_i)\subset(0,1]$. For a null sequence $\overline{\varepsilon}=(\varepsilon_i)\subset(0,1]$ we need to verify $WI(\overline{\mathcal{A}_{\overline{\varepsilon}}},(S_X^\omega)_{\overline{\delta}})$ (with $\delta_i=\varepsilon_i/28KC$ for $i\in\mathbb{N}$) and so we let $(z_\alpha)_{\alpha\in T_\infty}$ be a block tree of (E_i) in S_Z all of whose branches lie in $(S_X^\omega)_{\overline{\delta}}=\{(z_i)\in S_Z^\omega:\mathrm{dist}(z_i,S_X)\le\delta_i \text{ for } i=1,2\dots\}$.

After passing to a full subtree of (z_α) we can assume that for any $\alpha=(m_1,\dots m_\ell)$ in T_∞

$$z_\alpha\in\oplus_{j=1+n(\delta_\ell,m_\ell)}^\infty E_j\tag{2..5}$$

(where $n(\varepsilon,m)$ is chosen as in (2..4)).

For $\alpha=(m_1,m_2,\dots m_\ell)\in T_\infty$ we choose $y_\alpha\in S_X$ with $\|y_\alpha-z_\alpha\|<2\delta_\ell$ and, thus, by (2..5)

$$\|P^E_{[1,n(\delta_\ell,m_\ell)]}(y_\alpha)\|=\|P^E_{[1,n(\delta_\ell,m_\ell)]}(y_\alpha-z_\alpha)\|\le 2K\delta_\ell.$$

Using (2..4) we can therefore choose an $x'_\alpha\in X_{m_\ell}$ so that

$$\|x'_\alpha-y_\alpha\|\le C(2K\delta_\ell+\delta_\ell)\le 3CK\delta_\ell,$$

and thus

$$1-3CK\delta_\ell\le\|x'_\alpha\|\le 1+3CK\delta_\ell.$$

Letting $x_\alpha=x'_\alpha/\|x'_\alpha\|$ we deduce that

$$\begin{aligned}\|y_\alpha-x_\alpha\|&\le\|y_\alpha-x'_\alpha\|+\|x'_\alpha-x_\alpha\|\\&\le 3CK\delta_\ell+(1+3CK\delta_\ell)3CK\delta_\ell/(1-3CK\delta_\ell)\le 12CK\delta_\ell\end{aligned}$$

(the last inequality follows from the fact that $(1+3CK\delta_\ell)/(1-3CK\delta_\ell)\le 3$) and, thus,

$$\|z_\alpha-x_\alpha\|\le 14CK\delta_\ell=\varepsilon_\ell/2.$$

Using $WI(\overline{\mathcal{A}^X_{\overline{\varepsilon}/2}},Z)$ and noting that $x_\alpha\in X_{m_\ell}$, for $\alpha=(m_1,m_2,\dots m_\ell)\in T_\infty$ we can choose a branch of (x_α) which is in $\overline{\mathcal{A}^X_{\overline{\varepsilon}/2}}$. Thus, the corresponding branch of (z_α) lies in $\overline{\mathcal{A}_{\overline{\varepsilon}}}$.

From [24, Lemma 3.1] it follows that every separable Banach space X is a subspace of a space Z with an FDD satisfying the condition (2..4) (with $n(m) = m$). The following Proposition exhibits two general situations in which (2..4) is automatically satisfied.

Proposition 3. *Assume X is a subspace of a space Z having an FDD (E_i). In the following two cases (2..4) holds:*

a) If (E_i) is a shrinking FDD for Z. In that case C in (2..4) can be chosen arbitrarily close to 1.

b) If (E_i) is boundedly complete for Z (i.e. Z is the dual of $Z^{(*)}$) and the ball of X is a w^*-closed subset of Z. In that case C can be chosen to be the projection constant K of (E_i) in Z.

PROOF. In order to prove (a) we will show that for any $m \in \mathbb{N}$ and any $0 < \varepsilon < 1$ there is an $n = n(\varepsilon, m)$ so that

$$\|x\|_{X/X_m} \le (1+\varepsilon)\big[\|P^E_{[1,n]}(x)\| + \varepsilon\big], \text{ whenever } x \in S_X$$

(i.e. C in (2..4) can be chosen arbitrarily close to 1).

Since X/X_m is finite dimensional and

$$(X/X_m)^* = X_m^\perp = \big\{x^* \in X^* : x^*|_{X_m} \equiv 0\big\},$$

we can choose a finite set $A_m \subset S_{X_m^\perp} \subset S_{X^*}$ for which

$$\|x\|_{X/X_m} \le (1+\varepsilon) \max_{f\in A_m} |f(x)| \text{ whenever } x \in X.$$

By the Theorem of Hahn Banach we can extend each $f \in A_m$ to an element $g \in S_{Z^*}$. Let us denote the set of all of these extensions B_m. Since B_m is finite and since (E_i^*) is an FDD of Z^* we can choose an $n = n(\varepsilon, m)$ so that $\|P^{E^*}_{[1,n(m)]}(g) - g\| < \varepsilon$ for all $g \in B_m$. Since $P^{E^*}_{[1,n(m)]}$ is the adjoint operator of $P^E_{[1,n(m)]}$ (consider $P^{E^*}_{[1,n(m)]}$ to be an operator from Z^* to Z^* and $P^E_{[1,n(m)]}$ to be an operator from Z to Z), it follows for $x \in S_X$, that

$$\begin{aligned}
\|x\|_{X/X_m} &\le (1+\varepsilon) \max_{g\in B_m} |g(x)| \\
&\le (1+\varepsilon) \max_{g\in B_m} \big[\big|P^{E^*}_{[1,n(m)]}(g)(x)\big| + \|P^{E^*}_{[1,n(m)]}(g) - g\|\big] \\
&\le (1+\varepsilon)\big[\max_{g\in B_m} |g\big(P^E_{[1,n(m)]}(x)\big)| + \varepsilon\big] \le (1+\varepsilon)\big[\|P^E_{[1,n(m)]}(x)\| + \varepsilon\big],
\end{aligned}$$

which proves our claim and finishes the proof of part (a).

In order to show (b) we assume that X is a subspace of a space Z which has a boundedly complete FDD (E_i) and the unit ball of X is a w*-closed subset of Z, which is the dual of $Z^{(*)}$.

For $m \in \mathbb{N}$ and $\varepsilon > 0$ we will show that the inequality in (2..4) holds for some n and $C = K$. Assume that this was not true. and we could choose a sequence $(x_n) \subset S_X$ so that for any $n \in \mathbb{N}$

$$\|x_n\|_{X/X_m} > K\big[\|P^E_{[1,n]}(x_n)\| + \varepsilon\big].$$

By the compactness of B_X in the w* topology we can choose a subsequence x_{n_k} which converges in w* to some $x \in B_X$. For fixed ℓ it follows that $(P^E_{[1,\ell]}(x_{n_k}))$ converges in norm to $P^E_{[1,\ell]}(x)$. Secondly, since X/X_m is finite dimensional it follows that $\lim_{k\to\infty} \|x_{n_k}\|_{X/X_m} = \|x\|_{X/X_m}$, and, thus, it follows that

$$\begin{aligned}
\|x\| &= \lim_{\ell\to\infty} \|P^E_{[1,\ell]}(x)\| \\
&= \lim_{\ell\to\infty} \lim_{k\to\infty} \|P^E_{[1,\ell]}(x_{n_k})\| \\
&\le K \limsup_{k\to\infty} \|P^E_{[1,n_k]}(x_{n_k})\| \\
&\le \limsup_{k\to\infty} \|x_{n_k}\|_{X/X_m} - K\varepsilon = \|x\|_{X/X_m} - K\varepsilon,
\end{aligned}$$

which is a contradiction since $\|x\| \ge \|x\|_{X/X_m}$.

By combining Theorem 1 and Proposition 2 we deduce

Corollary 1. *Let X be a subspace of a space Z with an FDD (E_i) and assume that this embedding satisfies condition (2..4). Let $K \geq 1$ be the projection constant of (E_i) in Z and let $C \geq 1$ be chosen so that (2..4) holds.*

For $\mathcal{A} \subset S_X^\omega$ the following conditions are equivalent

a) For all null sequences $\overline{\varepsilon} = (\varepsilon_n) \subset (0,1]$, $WI(\overline{\mathcal{A}^X_{\overline{\varepsilon}}}, Z)$ holds.

b) For all null sequences $\overline{\varepsilon} = (\varepsilon_n) \subset (0,1]$, every $\overline{\varepsilon}/420CK$-skipped block sequence $(z_n) \subset X$ is in $\mathcal{A}_{\overline{\varepsilon}}$.

In the case that X has a separable dual (a) and (b) are equivalent to the following condition

c) For all null sequences $\overline{\varepsilon} = (\varepsilon_n) \subset (0,1]$ every weakly null tree in S_X has a branch in $\overline{\mathcal{A}_{\overline{\varepsilon}}}$

In the case that (E_i) is a boundedly complete FDD of Z and B_X is w^*-closed in $Z = (Z^{(*)})^*$ the conditions (a) and (b) are equivalent to

d) For all null sequences $\overline{\varepsilon} = (\varepsilon_n) \subset (0,1]$ every w^*-null tree in S_X has a branch in $\overline{\mathcal{A}_{\overline{\varepsilon}}}$

PROOF. (a)⇒(b) Let $\overline{\varepsilon} = (\varepsilon_i) \subset (0,1]$ be a null sequence, choose $\overline{\eta} = (\eta_i)$ with $\eta_i = \varepsilon_i/3$, for $i \in \mathbb{N}$, and $\overline{\delta} = (\delta_i)$ with $\delta_i = \eta_i/140CK = \varepsilon_i/420CK$

We deduce from Proposition 2 that $WI(\overline{\mathcal{A}_{\overline{\eta}}}, (S_X^\omega)_{5\overline{\delta}})$ holds. Using Theorem 1 we can block (E_i) into (G_i) so that every skipped block of (G_i) in $(S_X^\omega)_{5\overline{\delta}}$ (as a subset of S_Z) is in $\overline{\mathcal{A}_{2\overline{\eta}}}$.

Assume $(x_i) \subset S_X$ is a $\overline{\delta}$-skipped block sequence of (G_i) and let $1 \leq k_1 < \ell_1 < k_2 < \ell_2 < \dots$ in $\mathbb{N}$ so that

$$\|x_n - P^E_{(k_n,\ell_n]}(x_n)\| < \delta_n, \text{ for all } n \in \mathbb{N}.$$

The sequence (z_n) with $z_n = P^E_{(k_n,\ell_n]}(x_n)/\|P^E_{(k_n,\ell_n]}(x_n)\|$, for $n \in \mathbb{N}$, is a skipped block sequence of S_Z and we deduce that

$$\begin{aligned}\|x_n - z_n\| &\leq \|x_n - P^E_{(k_n,\ell_n]}(x_n)\| + \|P^E_{(k_n,\ell_n]}(x_n)\| \Big|1 - \frac{1}{\|P^E_{(k_n,\ell_n]}(x_n)\|}\Big| \\ &\leq \delta_n + (1+\delta_n)\frac{\delta_n}{1-\delta_n} \leq 5\delta_n.\end{aligned}$$

This implies that $(z_n) \in \overline{\mathcal{A}_{2\overline{\eta}}}$ and thus ($\delta_i < \eta_i/5$ for $i \in \mathbb{N}$),

$$(x_i) \in \overline{(\overline{\mathcal{A}_{2\overline{\eta}}})_{\overline{\eta}}} \subset \overline{\mathcal{A}_{\overline{\varepsilon}}},$$

which finishes the verification of (b).

(b)⇒(a) is clear since for any blocking (G_i) of (E_i) and any null sequence $\overline{\delta} = (\delta_i) \subset (0,1]$ every tree $(x_\alpha)_{\alpha \in T_\infty}$ in S_X with the property that $x_{(\alpha,n)} \in X_n$, whenever $n > n_\ell$ and $\alpha = (n_1, \dots n_\ell) \in T_\infty \cup \emptyset$ has a full subtree all of whose branches are $\overline{\delta}$-skipped block sequences of (G_i).

Now assume that X has a separable dual, or (E_i) is a boundedly complete FDD of Z and B_X in Z w^*-closed.

It is clear that (c) or (d), respectively, imply (a). Secondly, since for any null sequence $\overline{\delta} = (\delta_i) \subset (0,1]$ and any blocking (G_i) every weakly null tree (in the case that X, has a separable dual) or every w* null tree (in the boundedly complete case) has a full subtree which is a $\overline{\delta}$-skipped block of G_i we deduce that (b) implies (c) or (d) respectively.

In view of asymptotic structures we introduce the following "coordinate-free" variant of our games. Again let X be a separable Banach space and for $\mathcal{A} \subset S_X^\omega$ we consider the following *coordinate-free $\mathcal{A}$-game*.

$$\begin{aligned}
&\text{Player I chooses } X_1 \in \operatorname{cof}(X)\\
&\text{Player II chooses } x_1 \in X_1\ , \|x_1\| = 1,\\
&\text{Player I chooses } X_2 \in \operatorname{cof}(X)\\
&\text{Player II chooses } x_2 \in X_2,\ \|x_2\| = 1,\\
&\vdots
\end{aligned}$$

As before, Player I wins if $(x_i) \in \mathcal{A}$. We will show that X can be embedded into a space Z with FDD so that for all $\overline{\varepsilon} = (\varepsilon_i) \subset (0,1]$ Player I has a winning strategy in the coordinate-free $\overline{\mathcal{A}_{\overline{\varepsilon}}}$-game, which we will denote by $WI(\mathcal{A}, \operatorname{cof}(X))$, if and only if for $\overline{\varepsilon} = (\varepsilon_i) \subset (0,1]$ he has a winning strategy for the $(\overline{\mathcal{A}_\varepsilon}, Z)$-game.

First note that since we only considering fattened sets and their closures, Player II has a winning strategy if and only he has a winning strategy choosing his vectors out of a dense and countable subset of S_X determined before the game starts. But this implies that there is countable set of cofinite dimensional subspaces, say $\{Y_n : n \in \mathbb{N}\}$ from which player I can choose if he has a winning strategy. Moreover if we consider a countable set $\mathcal{B}$ of coordinate free games, there is a countable set $\{Y_n : n \in \mathbb{N}\}$ so that for all $\mathcal{A} \in \mathcal{B}$

$$\forall \overline{\varepsilon} \subset (0,1] \quad WI(\overline{\mathcal{A}_{\overline{\varepsilon}}}, \operatorname{cof}(X)) \iff \forall \overline{\varepsilon} \subset (0,1) \quad WI(\overline{\mathcal{A}_{\overline{\varepsilon}}}, \{Y_n : n \in \mathbb{N}\}), \tag{2..6}$$

where we write $WI(\overline{\mathcal{A}_{\overline{\varepsilon}}}, \{Y_n : n \in \mathbb{N}\})$, if player I has a winning strategy for the coordinate-free $\mathcal{A}$-game, even if he can only choose his spaces out of the set $\{Y_n : n \in \mathbb{N}\}$. Note that by passing to $(\bigcap_{i=1}^n Y_i)$ we can always assume that the Y_n's are decreasing in $n \in \mathbb{N}$. In case that X has a separable dual and we let (x_n^*) be a dense subset of X^*, we can put for $n \in \mathbb{N}$

$$Y_n = \{x_1^*, x_2^*, \dots, x_n^*\}^\perp = \{x \in X : \forall i \le n\ x_i^*(x) = 0\},$$

and observe that (2..6) holds for all $\mathcal{A} \subset S_Z^\omega$.

The following result was shown in [24, Lemma 3.1]and its proof was based on techniques and results of W. B. Johnson, H. Rosenthal and M. Zippin [JRZ] we derive the following Lemma.

Lemma 2. *Let (Y_n) be a decreasing sequence of closed subspaces of X, each having finite codimension. Then X is isometrically embeddable into a space Z having an FDD (E_i) so that (we identify X with its isometric image in Z)*

a) $c_{00}(\oplus_{i=1}^\infty E_i) \cap X$ is dense in X.

b) For every $n \in \mathbb{N}$ the finite codimensional subspace $X_n = \oplus_{i=n+1}^\infty E_i \cap X$ is contained in Y_n.

c) There is a $c > 0$, so that for every $n \in \mathbb{N}$ there is a finite set $D_n \subset S_{\oplus_{i=1}^n E_i^*}$ such that whenever $x \in X$

$$\|x\|_{X/Y_n} = \inf_{y \in Y_n} \|x - y\| \le c \max_{w^* \in D_n} w^*(x)\ . \tag{2..7}$$

From (a) it follows that $c_{00}(\oplus_{i=n+1}^\infty E_i) \cap X$ is a dense linear subspace of X_n.

Moreover if X has a separable dual (E_i) can be chosen to be shrinking (every normalized block sequence in Z with respect to (E_i) converges weakly to 0, or, equivalently, $Z^* = \oplus_{i=1}^\infty E_i^*$), and if X is reflexive Z can also be chosen to be reflexive.

So assume that for a countable set $\mathcal{B}$ of games that (Y_n) is a sequence of decreasing finite codimensional closed spaces satisfying the equivalences of (2..6). We then use Lemma 2 to embed X into a space Z with an FDD (E_i).

Note that b) of Lemma 2 implies that for all $\mathcal{A} \in \mathcal{B}$

$$\forall \overline{\varepsilon} \subset (0,1) \quad WI(\overline{\mathcal{A}_{\overline{\varepsilon}}}, \mathrm{cof}(X)) \iff \forall \overline{\varepsilon} \subset (0,1) \quad WI(\overline{\mathcal{A}_{\overline{\varepsilon}}}, Z)$$

Using the embedding of X given by Lemma 2 a result similar to Proposition 2 can be shown. The proof is very similar, therefore we will only present a sketch.

Proposition 4. *Assume that X is a Banach space and $\{Y_n : n \in \mathbb{N}\}$ a decreasing sequence of cofinite dimensional subspaces. Let Z be a space with an FDD (E_i) which satisfies the conclusion of Lemma 2.*

Assume that $\mathcal{A} \subset S_X^\omega$ such that we have $WI(\overline{\mathcal{A}_{\overline{\varepsilon}}^X}, \{Y_n : n \in \mathbb{N}\})$ for all null sequences $\overline{\varepsilon} \subset (0,1]$.

Then for all null sequences $\overline{\varepsilon} = (\varepsilon_i) \subset (0,1]$, $WI(\overline{\mathcal{A}_{\overline{\varepsilon}}}, \overline{(S_X^\omega)_{\overline{\delta}}})$ holds, where $\overline{\delta} = (\delta_i) = (\varepsilon_i/28cK)$, with c as in Lemma 2, K is the projection constant of (E_i) in Z, and where the fattenings $\overline{\mathcal{A}_{\overline{\varepsilon}}}$ and $\overline{(S_X^\omega)_{\overline{\delta}}}$ are taken in Z.

PROOF. [Sketch of proof] Note that instead of condition (2..4) the following condition is satisfied.

$$\begin{gathered}\text{There is a } C > 0 \text{ so that for all } m \in \mathbb{N} \\ \|x\|_{X/Y_m} \leq C \|P^E_{[1,m]}(x)\| \text{ whenever } x \in S_X.\end{gathered} \tag{2..8}$$

Also note that $WI(\overline{\mathcal{A}_{\overline{\varepsilon}}}, \{Y_n : n \in \mathbb{N}\})$ means that every tree $(x_\alpha) \subset S_X$, with the property that for $\alpha = (m_1, m_2, \dots m_\ell) \in T_\infty$ we have that $x_\alpha \in Y_{m_\ell}$, has a branch in $\overline{\mathcal{A}_{\overline{\varepsilon}}}$.

We follow the proof of Proposition 2 until choosing the x_α's which we will not choose in X_{m_ℓ} but in Y_{m_ℓ} instead. Then the proof continues as the proof of Proposition 2.

Using Proposition 4 and Theorem 1 we deduce the following.

Corollary 2. *Let $\mathcal{B}$ be a countable set of $\mathcal{A} \subset S_X^\omega$ and assume that Z is a space with an FDD which contains X and satisfies the conclusion of Lemma 2*

For $\mathcal{A} \in \mathcal{B}$ the following conditions are equivalent

a) For all null sequences $\overline{\varepsilon} = (\varepsilon_n) \subset (0,1]$, $WI(\overline{\mathcal{A}_{\overline{\varepsilon}}^X}, \mathrm{cof})$ holds.

b) For all null sequences $\overline{\varepsilon} = (\varepsilon_n) \subset (0,1]$, $WI(\overline{\mathcal{A}_{\overline{\varepsilon}}^X}, Z)$ holds.

c) For all null sequences $\overline{\varepsilon} = (\varepsilon_n) \subset (0,1]$, every $\overline{\varepsilon}/420CK$-skipped block sequence (z_n) is in $\mathcal{A}_{\overline{\varepsilon}}$.

In the case that X has a separable dual (a),(b) and (c) are equivalent to the following condition (which is independent of the choice of Z)

d) For all null sequences $\overline{\varepsilon} = (\varepsilon_n) \subset (0,1]$ every weakly null tree in S_X has a branch in $\overline{\mathcal{A}_{\overline{\varepsilon}}}$

Moreover, in the case that X has a separable dual we deduce from the remarks after the equivalence (2..6), Corollary 1 and Proposition 3 that above equivalences hold for any embedding *of X into a space Z having a shrinking FDD.*

3. Banach spaces of Szlenk index ω

In this section we will present (Theorem 3) a long list of equivalent conditions for a space X to have Szlenk index ω. We also show how Kalton's c_0 theorem (Theorem 4) can be proved with our techniques. We begin with some definitions that will be used in later sections as well as this one.

Definition 3. *Let $1 \leq q \leq p \leq \infty$ and $C < \infty$. A (finite or infinite)* FDD (E_i) *for a Banach space Z is said to satisfy C-(p,q)*-estimates *if for all $n \in \mathbb{N}$ and block sequences $(x_i)_{i=1}^n$ w.r.t. (E_i),*

$$C^{-1}\Big(\sum_1^n \|x_i\|^p\Big)^{1/p} \leq \Big\|\sum_1^n x_i\Big\| \leq C\Big(\sum_1^n \|x_i\|^q\Big)^{1/q}.$$

*A space X satisfies C-(p,q)-*tree estimates *if for all weakly null trees in S_X there exist branches $(x_i)_{i=1}^\infty$ and $(y_i)_{i=1}^\infty$ satisfying for all $(a_i)\in c_{00}$,*

$$C^{-1}\Big(\sum|a_i|^p\Big)^{1/p}\le\Big\|\sum a_ix_i\Big\| \text{ and } \Big\|\sum a_iy_i\Big\|\le C\Big(\sum|a_i|^q\Big)^{1/q}. \tag{3..1}$$

*If $X\subseteq Y^*$, a separable dual space, we say that X satisfies C-(p,q)-w^*-*tree estimates *if each w^* null tree in S_X admits branches (x_i) and (y_i) satisfying* (3..1).

*We will say that X satisfies (p,q)-*tree estimates *if it satisfies C-(p,q)-tree estimates for some $C<\infty$ and similarly for (p,q)-w^* tree estimates.*

It is perhaps worth noting that if every weakly null tree in X admits a branch dominating the unit vector basis of ℓ_p (not assuming that the constant of domination can be chosen independently of the tree) then X satisfies $(p,1)$-tree estimates (and similar remarks hold for (∞,q)-tree estimates or (p,q)-w^*-tree estimates). Indeed, if no uniform constant existed one could assemble a tree with no branch dominating the unit vector basis of ℓ_p [26][Proposition 1.2].

Definition 4. *[28] Let $1\le p\le\infty$ and let Z be a Banach space with an* FDD *$E=(E_i)$. Then $Z_p(E)$ is the completion of $c_{00}(\oplus_{i=1}^\infty E_i)$ under*

$$\|z\|_p=\sup\Big\{\Big(\sum_j\|P^E_{I_j}z\|^p)^{1/p}: I_1<I_2<\cdots \text{ are intervals in } \mathbb{N}\Big\}.$$

$(E_i)_{i=1}^\infty$ is then a bimonotone FDD *for $Z_p(E)$ which satisfies 1-$(p,1)$-estimates. Moreover, if Z is isomorphic to $\widetilde Z$ then $Z_p(E)$ is naturally isomorphic to $\widetilde Z_p(E)$.*

Our main tool for proving Theorem 3 is the following result which is a non-reflexive version of Theorem 2.1 a in [25]

Theorem 2. *Let Z be a Banach space with a boundedly complete* FDD *$E=(E_i)$ and let X be a subspace of Z with B_X being a w^*-closed subspace of Z $(=(Z^{(*)})^*)$. Let $1\le p\le\infty$. If X satisfies $(p,1)$-w^*-tree estimates in Z then there exists a blocking $F=(F_i)$ of (E_i) so that X naturally embeds into $Z_p(F)$.*

To prove this we need a blocking lemma which appears in various forms in [19], [24], [25], [26] and ultimately results from a blocking trick of Johnson [9]. We will use this lemma as well in section 4..

Lemma 3. *Let X be a subspace of a space Z with B_X being a w^*-closed subset OF z having a boundedly complete* FDD *(E_i) with projection constant K. Let $\delta_i\downarrow 0$. Then there exists a blocking (F_i) of (E_i) given by $F_i=\oplus_{j=N_{i-1}+1}^{N_i}E_j$ for some $0=N_0<N_1<\cdots$ with the following properties. For all $x\in S_X$ there exists $(x_i)_{i=1}^\infty\subseteq X$ and for all $i\in\mathbb{N}$ there exists $t_i\in(N_{i-1},N_i)$ satisfying*

a) $x=\sum_{j=1}^\infty x_j$.

b) $\|x_i\|<\delta_i$ *or* $\|P^E_{(t_{i-1},t_i)}x_i-x_i\|<\delta_i\|x_i\|$.

c) $\|P^E_{(t_{i-1},t_i)}x-x_i\|<\delta_i$.

d) $\|x_i\|<K+1$.

e) $\|P^E_{t_i}x\|<\delta_i$

Moreover, the above hold for any blocking of (F_i) (which would redefine the N_i's).

PROOF. We observe that for all $\varepsilon > 0$ and $N \in \mathbb{N}$ there exists $n > N$ such that if $x \in B_X$, $x = \sum y_i$ with $y_i \in E_i$ for all i, then there exists $t \in (N, n)$ with

$$\|y_t\| < \varepsilon \text{ and } \operatorname{dist}\left(\sum_{i=1}^{t-1} y_i, X\right) < \varepsilon .$$

Indeed, if this was not true for any $n > N$ we can find $y^{(n)} \in B_X$ failing the conclusion for $t \in (N, n)$. Choose a subsequence of $(y^{(n)})$ converging w^* to $y \in X$ and choose t so that $\|P^E_{[t,\infty)} y\| < \varepsilon/2K$. Then choose $y^{(n)}$ from the subsequence so that $t < n$ and $\|P^E_{[1,t]}(y - y^{(n)})\| < \varepsilon/2K$. Thus

$$\|P^E_{[1,t)} y^{(n)} - y\| \le \|P^E_{[1,t)}(y^{(n)} - y)\| + \|P^E_{[t,\infty)} y\| < \frac{\varepsilon}{2K} + \frac{\varepsilon}{2K} < \varepsilon .$$

Also

$$\|P^E_t y^{(n)}\| \le \|P^E_t (y^{(n)} - y)\| + \|P^E_t y\| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon .$$

This contradicts our choice of $y^{(n)}$.

Let $\varepsilon_i \downarrow 0$ and by the observation choose $0 = N_0 < N_1 < \cdots$ so that for all $x \in S_X$ there exists $t_i \in (N_{i-1}, N_i)$ and $z_i \in X$ with $\|P^E_{t_i} x\| < \varepsilon_i$ and $\|P^E_{[1,t_i-1)} x - z_i\| < \varepsilon_i$ for all $i \in \mathbb{N}$. Set $x_1 = z_1$ and $x_i = z_i - z_{i-1}$ for $i > 1$. Thus $\sum_{i=1}^n x_i = z_n \to x$ so a) holds. Also

$$\|P^E_{(t_{i-1},t_i)} x - x_i\| \le \|P^E_{[1,t_i)} x - z_i\| + \|P^E_{[1,t_{i-1}]} x - z_{i-1}\| < \varepsilon_i + 2\varepsilon_{i-1} ,$$

and

$$\|P^E_{(t_{i-1},t_i)} x_i - x_i\| = \|(I - P^E_{(t_{i-1},t_i)})(x_i - P^E_{(t_{i-1},t_i)} x)\| < (K+1)(\varepsilon_i + 2\varepsilon_{i-1}) .$$

From these inequalities b), c) and d) follow if we take (ε_i) so that $(K+1)(\varepsilon_i + 2\varepsilon_{i-1}) < \delta_i^2$.

PROOF. [Proof of Theorem 2] We may assume that E is a bimonotone FDD for Z and that X satisfies C-$(p,1)$-w^*-tree estimates in Z.

Let

$$\mathcal{A} = \left\{ (x_i)_{i=1}^\infty \in S_X^\omega : \left\| \sum a_i x_i \right\| \ge C^{-1} \Big(\sum |a_i|^p \Big)^{1/p} \text{ for all } (a_i) \in c_{00} \right\} .$$

Choose a null sequence $\overline{\varepsilon} = (\varepsilon_i) \subset (0,1)$ so that

$$\overline{\mathcal{A}_{\overline{\varepsilon}}} \subset \left\{ (x_i)_{i=1}^\infty \in S_X^\omega : \left\| \sum a_i x_i \right\| \ge (2C)^{-1} \Big(\sum |a_i|^p \Big)^{1/p} \text{ for all } (a_i) \in c_{00} \right\} .$$

By Corollary 1 there exist $\bar\delta = (\delta_i)$ with $\delta_i \downarrow 0$ and a blocking of (E_i), which we still denote by (E_i) so that if $(x_i)_{i=1}^\infty \in S_X^\omega$ is a $\bar\delta$-skipped block sequence w.r.t. (E_i) then $(x_i) \in \overline{\mathcal{A}_{\overline{\varepsilon}}}$. Wlog $\sum_1^\infty \delta_i < \frac{1}{2}$. We will produce a blocking (F_i) of (E_i) and $A < \infty$ so that for all $0 = n_0 < n_1 < \cdots$ and $x \in S_X$,

$$\left(\sum_{j=1}^\infty \|P^F_{(n_{j-1},n_j]} x\|^p \right)^{1/p} \le A \tag{3..2}$$

and this will finish the proof.

(F_i) will be the blocking given by Lemma 3 for (δ_i). We will show, using Lemma 3, that

$$\left(\sum_j \|P^F_j x\|^p \right)^{1/p} \le A \tag{3..3}$$

and by the "moreover" part of the lemma the same proof will yield (3..2).

Let $F_j = \oplus_{i=N_{j-1}+1}^{N_j} E_i$ be as in Lemma 3, $x \in S_X$ and let (x_i), (t_i) be as in the Lemma. Set $B = \{i \geq 2;\, x_i \neq 0$ and $\|P^E_{(t_{i-1},t_i)} x_i - x_i\| < \delta_i \|x_i\|\}$ and for $i \in B$ let $\bar{x}_i = x_i/\|x_i\|$. Note that if $i \geq 2$, $i \notin B$, then $\|x_i\| < \delta_i$.

Now $(\bar{x}_i)_{i\in B}$ is a $\bar{\delta}$-skipped block sequence w.r.t. (E_i) and so

$$2C \Big\| \sum_{i\in B} x_i \Big\| \geq \Big(\sum_{i\in B} \|x_i\|^p \Big)^{1/p} .$$

Also

$$\Big\| \sum_{i\notin B} x_i \Big\| \leq \|x_1\| + \sum \delta_i < 2 + 1 = 3 \qquad (\text{since } K = 1) .$$

Thus

$$\Big\| \sum_{i\in B} x_i \Big\| \leq \|x\| + \Big\| \sum_{i\notin B} x_i \Big\| < 4 .$$

It follows that

$$\begin{aligned} \sum \|x_i\|^p &\leq \|x_1\|^p + \sum_{i\in B} \|x_i\|^p + \sum_{\substack{i\notin B\\ i\geq 2}} \|x_i\|^p \\ &< 2^p + (8C)^p + 1 \equiv D^p . \end{aligned}$$

For $i \in \mathbb{N}$ set $y_i = P^E_{(t_{i-1},t_i]} x$. Then

$$\|y_i - x_i\| \leq \|P^E_{(t_{i-1},t_i)} x - x_i\| + \|P^E_{t_i} x\| < 2\delta_i .$$

Hence

$$\Big(\sum \|y_i\|^p \Big)^{1/p} \leq \Big(\sum \|x_i\|^p \Big)^{1/p} + \Big(\sum (2\delta_i)^p \Big)^{1/p} < D + 1 .$$

Finally write $x = \sum z_i$ where $z_i \in F_i$ for all i. Then $z_i = P^F_i (y_i + y_{i+1})$, so $\|z_i\| \leq \|y_i\| + \|y_{i+1}\|$. Hence

$$\Big(\sum \|z_i\|^p \Big)^{1/p} \leq 2(D+1) \equiv A .$$

Before stating Theorem 3 we need some definitions and preliminaries.

Definition 5. *X has the* w^*-UKK *if for all $\varepsilon > 0$ there exists $\delta^*(\varepsilon) > 0$ so that if $(x_n^*) \subseteq B_{X^*}$ converges w^* to x^* and $\liminf_{n\to\infty} \|x_n^* - x^*\| \geq \varepsilon$ then $\|x^*\| \leq 1 - \delta^*(\varepsilon)$.*

We have defined $S_z(X)$ but there is another view of this index which shall prove useful. In [2, Theorem 4.2] an index $I_{\ell_1^+,w}(X)$ is defined and shown to equal $S_z(X)$ if X does not contain an isomorph of ℓ_1. The precise definition need not concern us here. However we note that one consequence is $S_z(X) = \omega$ iff $\forall\, K > 1\, \exists\, n(K)$ so that if $(e_i)_{i=1}^n \in \{X\}_n$ is an $\ell_1^+ - K$ sequence, i.e., $\| \sum_1^n a_i e_i \| \geq K^{-1} \sum_{i=1}^n a_i$ for $(a_i)_{i=1}^n \subseteq [0,\infty)$, then $n \leq n(K)$.

Definition 6. *For a Banach space X the* modulus of asymptotic uniform smoothness *$\bar{\rho}_X(t)$ is given for $t > 0$ by*

$$\bar{\rho}_X(t) = \sup_{\|x\|=1} \inf_{Y\in \mathrm{cof}(X)} \sup_{y\in tB_Y} \|x+y\| - 1 .$$

The modulus of asymptotic uniformly convexity $\bar{\delta}_X(t)$ is given for $t > 0$ by

$$\bar{\delta}_X(t) = \inf_{\|x\|=1} \sup_{Y\in \mathrm{cof}(X)} \inf_{\substack{y\in Y\\ \|y\|\geq t}} \|x+y\| - 1 .$$

X is asymptotically uniformly smooth *(a.u.s.) if* $\lim_{t\to 0^+} \bar\rho_X(t)/t = 0$.
X is asymptotically uniformly convex *(a.u.c.) if for* $t > 0$, $\bar\delta_X(t) > 0$.
X is a.u.s. of power type p *if for some* $K < \infty$, $\bar\rho_X(t) \le K\, t^p$ *for* $t > 0$.
X is a.u.c. of power type p *if for some* $K > 0$, $\bar\delta_X(t) \ge K\, t^p$ *for* $t > 0$.
If X is a dual space we can define similar modulii $\bar\delta^*_X(t)$ and $\bar\rho^*_X(t)$ using

$$\mathrm{cof}^*(X) = \Big\{ Y \subseteq X : Y \textit{ is a } w^*\textit{-closed finite co-dimensional subspace of } X \Big\}$$

More about these modulii can be found in [11] but we shall extract a few things we need in proving Theorem 3. In the case where X^* is separable a.u.s. and a.u.c. say something about weakly null trees and $\{X\}_n$. Let $(e_i)_{i=1}^n \in \{X\}_n$ and let $(a_i)_{i=1}^n \subseteq (0,1]$. Assume that $\bar\delta_X(t) \ge K\, t^p$ for some K and all $t > 0$. Using that there exists $c > 0$ with $K\, t^p \ge (1 + c\, t^p)^{1/p} - 1$ for $t > 0$ we obtain

$$\begin{aligned} \Big\| \sum_1^n a_i e_i \Big\|^p &\ge \Big\| \sum_1^{n-1} a_i e_i \Big\|^p \Big(1 + \bar\delta_X\Big(\frac{|a_n|}{\| \sum_1^{n-1} a_i e_i \|} \Big) \Big)^p \\ &\ge \Big\| \sum_1^{n-1} a_i e_i \Big\|^p + c|a_n|^p \\ &\ge \cdots \ge c \sum_1^n |a_i|^p \,. \end{aligned}$$

Similarly if we begin with a weakly null tree in S_X we can extract a branch (x_i) satisfying

$$\Big\| \sum a_i x_i \Big\| \ge \frac{c}{2} \Big(\sum |a_i|^p \Big)^{1/p}$$

for all $(a_i) \in c_{00}$.

With a similar argument for $\bar\rho_X(t)$ we obtain

Proposition 5. *[11]*

a) Let X^* be separable. If X is a.u.c. of power type p then X satisfies $(p,1)$-tree estimates.

b) Let X^* be separable. If X is a.u.s. of power type q then X satisfies (∞,q)-tree estimates.

c) Let $X = Y^*$ be a separable dual. If X is w^*-a.u.c. of power type p (i.e., $\bar\rho^*_X(t) \ge K\, t^p$) then X satisfies $(p,1)$-w^*-tree estimates.

Theorem 3. *Let X^* be separable. The following are equivalent.*

(1) $S_z(X) = \omega$.

(2) $\exists\, q > 1\ \exists\, K < \infty\ \forall\, n\ \forall\, (e_i)_{i=1}^n \in \{X\}_n\ \forall\, (a_i)_{i=1}^n \subseteq \mathbb{R}$,

$$\Big\| \sum_1^n a_i e_i \Big\| \le K \Big(\sum_1^n |a_i|^q \Big)^{1/q} .$$

(3) $\exists\, q > 1$ so that X satisfies (∞,q)-tree estimates.

(4) $\exists\, p < \infty$ so that X^* satisfies $(p,1)$-w^*-tree estimates.

(5) $\exists\, p < \infty\ \exists$ a Banach space Z with a boundedly complete FDD *E so that X^* embeds into $Z_p(E)$ as a w^*-closed subspace.*

(6) *X can be renormed to be a.u.s. of power type q for some $q > 1$.*

(7) *X can be renormed to be a.u.s.*

(8) *X can be renormed so that $\bar\rho_X(t) < t$ for some $t > 0$.*

(9) *X can be renormed to be w^*-UKK with modulus $\delta^*(\varepsilon) \ge c\,\varepsilon^p$ for some $p < \infty$.*

(10) *X can be renormed to be w^*-UKK.*

(11) *$\exists\, p < \infty$ so that X can be renormed so that $\bar\delta^*_{X^*}(t)$ is of power type p.*

(12) *X can be renormed to be w^*-a.u.c.*

PROOF.
$(2) \Rightarrow (1)$. (2) implies that $I_{\ell_1^+,\omega}(X) = \omega$.
$(1) \Rightarrow (2)$. This follows from the fact that for $n \in \mathbb{N}$ there exists $q > 1$ so that every normalized monotone basis which does not admit a normalized block basis of length n which is ℓ_1^+ with constant 2 is 6-dominated by the unit vector basis of ℓ_q^n (proved in [8], [9]). Since $I_{\ell_1^+,w}(X) = \omega$ (2) follows by our earlier remarks and the fact that if $(x_i)_{i=1}^m$ is a block of some sequence $(e_i)_{i=1}^n \in \{X\}_n$ then $(x_i)_{i=1}^m \in \{X\}_m$.
$(2) \Rightarrow (3)$. Let $X \subseteq Y$, a space with a shrinking FDD (E_i) (by 1.2). Using our discussion of asymptotic structure, applying Corollary 2 to $\mathcal{B} = \{\mathcal{A}^{(n)} : n \in \mathbb{N}\}$, with

$$\mathcal{A}^{(n)} = \left\{ (x_i) \in S_X^\omega : \begin{array}{c} \exists (e_i)_{i=1}^n \in \{X\}_n \text{ so that} \\ (x_i)_{i=1}^n \text{ is 2-equivalent to} (e_i)_{i=1}^n \end{array} \right\} \tag{3..4}$$

and a diagonal argument we can find $\bar\delta = (\delta_i)$, $\delta_i \downarrow 0$, and a blocking (F_i) of (E_i) with the following property. For all $n \in \mathbb{N}$ if $(x_i)_{i=1}^n \subseteq S_X$ is a $(\delta_i)_{n+1}^{2n}$-skipped block sequence w.r.t. $(\oplus_{i=1}^n F_i, F_{n+1}, F_{n+2}, \ldots)$ then $d_b((x_i)_{i=1}^n, \{X\}_n) < 2$. Let $(x_\alpha)_{\alpha \in T_\infty}$ to be a weakly null tree in X. Then the exists a branch $(x_i)_{i=1}^\infty$ so that for all n if $(y_i)_{i=1}^n$ is a normalized block basis of $(x_i)_n^\infty$ then $(y_i)_{i=1}^n$ is a $(\delta_i)_{n+1}^{2n}$-skipped block sequence w.r.t. $(\oplus_1^n F_i, F_{n+1}, \ldots)$ and so satisfies $2K$-upper ℓ_q^n estimates. Now it follows [19, Proposition 3.5] that for any $q > \bar q > 1$, (x_i) satisfies $(\infty, \bar q)$-estimates. Thus (3) holds.
$(3) \Rightarrow (4)$ follows from the following

Lemma 4. *Let X^* be separable. If X satisfies (∞, q)-tree estimates for $q > 1$ then X^* satisfies $(q', 1)$-w^*-tree estimates ($\frac{1}{q} + \frac{1}{q'} = 1$).*

PROOF. Let X satisfy K-(∞, q)-tree estimates. Note the following. If (x_i^*) is normalized w^*-null in X^* then there exists $(x_i) \subseteq S_X$, (x_i) is weakly null, and a subsequence $(x_{n_i}^*)$ of (x_i^*) with $\lim_i x_{n_i}^*(x_i) \ge \frac{1}{2}$. Indeed we choose $(y_i) \subseteq S_X$ with $\lim x_i^*(y_i) = 1$ and pass to a weak Cauchy subsequence (y_{k_i}) so that $\lim_i x_{k_i}^*(y_{k_{i-1}}) = 0$. Let $x_{n_i}^* = x_{k_{2i}}^*$ and $x_i = (y_{k_{2i}} - y_{k_{2i-1}})/\|y_{k_{2i}} - y_{k_{2i-1}}\|$.

Let $(x_\alpha^*)_{\alpha \in T_\infty}$ be a w^*-null tree in X^*. Using the above remark we can pass to a full subtree and find a weakly null tree $(x_\alpha)_{\alpha \in T_\infty} \subseteq S_X$ so that $x_\alpha^*(x_\alpha) > 1/3$ for all α. By further pruning we can also assume that, given $\eta > 0$, $|x_\alpha^*(x_\beta)| < 2^{-m-n}\eta$ and $|x_\beta^*(x_\alpha)| < 2^{-m-n}\eta$ if $\alpha < \beta$ and $|\alpha| = m$, $|\beta| = n$. This pruning uses only that each node in (x_α^*) is w^*-null and each node in (x_α) is weakly null. An easy calculation shows that if $(x_i)_{i=1}^\infty$ is a branch in (x_α) which is K-dominated by the unit vector basis of ℓ_q, then the corresponding branch (x_i^*) in (x_α^*) satisfies, for small η,

$$\Big\| \sum a_i x_i^* \Big\| \ge \frac{1}{3K} - \frac{1}{K}\eta > \frac{1}{4K}$$

if $(\sum |a_i|^{q'})^{1/q'} = 1$. □

$(4) \Rightarrow (5)$. By 1.1 X is a quotient of a space with a shrinking basis and hence X^* embeds as a w^*-closed subspace into a space Z with a boundedly complete FDD $E = (E_i)$. Since any w^*-null tree is S_{X^*} is a w^*-null tree w.r.t. Z, (5) follows from (4) by Theorem 2.

$(5) \Rightarrow (6)$. Let X^* be embedded into $Z_p(E)$ as in (5). We renorm X via

$$|x| = \sup\{|x^*(x)| : x* \in X^*,\ \|x^*\|_p \leq 1\} .$$

It follows easily that $\bar{\rho}_X(t) \leq (1+t^q)^{1/q} - 1$ where $\frac{1}{p} + \frac{1}{q} = 1$ which proves (6).

$(6) \Rightarrow (7) \Rightarrow (8)$ is trivial.

$(8) \Rightarrow (1)$. Assume (1) fails so $S_z(X) = I_{\ell_1^+,w}(X) > \omega$. Then there exists $K \geq 1$ so that for all n there exists an $\ell_1^+ - K$ sequence in $\{X\}_n$. By James' argument that ℓ_1 is not distortable (which also works in the ℓ_1^+ case) we obtain that there exists $(e_1, e_2) \in \{X\}_2$ with

$$\|e_1 + te_2\| = 1 + t \text{ for all } t > 0 .$$

Since $S_z(X)$ is an isomorphic invariant, we have for all renormings of X, $\bar{\rho}_X(t) = t$ for all $t > 0$. Thus (8) fails.

$(5) \Rightarrow (9)$ by the renorming used in $(5) \Rightarrow (6)$. Indeed if $(x_n^*) \subseteq S_{X^*,\|\cdot\|_p}$ with $x_n^* \xrightarrow{w^*} x^*$ and $\lim_n \|x_n^* - x^*\|_p \geq \varepsilon$ then $\|x^*\|^p + \varepsilon^p \leq 1$.

$(9) \Rightarrow (10) \Rightarrow (1)$ is trivial.

$(5) \Rightarrow (11)$ holds again by the $(5) \Rightarrow (6)$ argument.
$(11) \Rightarrow (12)$ is trivial.
$(12) \Rightarrow (4)$ Assume (12) holds. By [7] (7) holds. Alternatively, it follows that there exists $n_0 \in \mathbb{N}$ so that if $(e_i)_{i=1}^n$ is in the w^*-asymptotic structure of the w^*-a.u.c. space X^* and there exist $(a_i)_{i=1}^n \subseteq [\frac{1}{2}, 1]$ with $\|\sum_1^n a_i e_i\| \leq 1$ then $n \leq n_0$. Indeed we obtain $\|\sum_1^n a_i e_i\| \geq \frac{1}{2}[1 + \overline{\delta}_X^*(\frac{1}{2})]^{n-1}$. This condition yields that there exists $p = p(n_0) < \infty$ so that the unit vector basis of ℓ_p^n 2-dominates $(e_i)_{i=1}^n$ for all $n \in \mathbb{N}$, ([8], [10], [19]). Arguing then as in $(2) \Rightarrow (3)$ we obtain (4).

We end this section with Kalton's c_0-theorem.

Definition 7. *X has the* bounded tree property *if there exists $C < \infty$ so that for all weakly null trees in S_X there exists a branch $(x_i)_{i=1}^\infty$ with*

$$\sup_n \Big\| \sum_{i=1}^n x_i \Big\| \leq C .$$

Note that if X has the bounded tree property and does not contain an isomorph of ℓ_1 then $S_z(X) = I_{\ell_1^+,\omega}(X) = \omega$.

Theorem 4. *[17] Let X have the bounded tree property. If X does not contain an isomorph of ℓ_1, then X embeds into c_0.*

PROOF. By (1.2) we may regard $X \subseteq Z$, a space with a bimonotone shrinking FDD $E = (E_i)$. Assume that X has the bounded tree property with constant C. Let

$$\mathcal{A} = \left\{ (x_i)_{i=1}^\infty \in S_X^\omega : \sup_n \Big\| \sum_1^n x_i \Big\| \leq C \right\} .$$

Choose $\varepsilon > 0$ so that with $\overline{\varepsilon} = (\varepsilon_i 2^{-i})$

$$\bar{\mathcal{A}}_{\overline{\varepsilon}} \subseteq \left\{ (x_i) \in S_X^\omega : \sup_n \Big\| \sum_1^n x_i \Big\| \leq 2C \right\} .$$

By Corollary 1 we may choose $\overline{\delta} = (\delta_i)$, $\delta_i \downarrow 0$, and a blocking of E which we still denote by $E = (E_i)$ so that any $\overline{\delta}$-skipped block sequence $(x_i) \subseteq S_X$ w.r.t. (E_i) is in $\bar{\mathcal{A}}_\varepsilon$. Since $(\pm x_i)$ is a $\overline{\delta}$-skipped block sequence when (x_i) is a $\bar{\delta}$-skipped block sequence it follows by a convexity argument that $\|\sum a_i x_i\| \leq 2C$ for $(a_i) \in c_{00}$, $(a_i) \subseteq [-1,1]$.

It follows that X satisfies (∞,∞)-tree estimates and hence by Lemma 4, X^* satisfies $(1,1)$-w^*-tree estimates. By Theorem 2, X^* embeds as a w^* closed subspace into some space $Z_1^*(F_i^*)$ which is $(\oplus_{i=1}^\infty F_i^*)_{\ell_1}$, where $F^* = (F_i^*)$ is some blocking of (E_i^*). From basic functional analysis we have that X is a quotient of $(\sum F_i)_{c_0}$. Hence X is isomorphically a subspace of a quotient of c_0 and hence embeds into c_0 since every quotient of c_0 embeds into c_0. [14].

4. Reflexive spaces

In this section we first discuss the problem of characterizing when a reflexive space X satisfies (p,q)-tree estimates for a given $1 \leq q \leq p \leq \infty$. The ultimate result is

Theorem 5. *Let X be a reflexive Banach space and let $1 \leq q \leq p \leq \infty$. The following are equivalent*

a) X satisfies (p,q)-tree estimates.

b) X embeds into a reflexive space Z having an FDD which satisfies (p,q)-estimates.

c) X is isomorphic to a quotient of a reflexive space Z having an FDD which satisfies (p,q)-estimates.

d) X^* satisfies (q',p')-tree estimates where $1/q' + 1/q = 1$ and $1/p' + 1/p = 1$.

e) X embeds into a reflexive space Z having an FDD which satisfies 1-(p,q)-estimates.

The duality between an FDD (E_i) satisfying (p,q)-estimates and (E_i^*) satisfying (q',p')-estimates is easy to establish [27]. Half of the tree estimate duality a) ⇔ d) follows from Lemma 4, which proves that if X satisfies (∞,q) estimates then X^* satisfies $(q',1)$-estimates, and if X^* satisfies (∞,p')-tree estimates X satisfies $(p,1)$-tree estimates. But we do not have a direct proof, i.e. without first showing (a) $\iff$ (b) and then using Pruss' result, which shows that if X satisfies $(p,1)$-tree estimates then X^* satisfies (∞,p') estimates.

Theorem 5 was proved in [25] and rather than just repeat that proof we shall give a sketch of the proof emphasizing the new ideas necessary to go beyond the proof of Theorem 3. But first let's see what is an easy consequence of our earlier arguments.

First consider the case where X satisfies $(p,1)$-tree estimates for some $1 < p < \infty$. Let $X \subseteq Z$, a reflexive space with a basis (by 1.3). From Theorem 2 there exists a blocking $E = (E_i)$ of the basis for Z so that X naturally embeds into $Z_p(E)$. E is a bimonotone FDD for $Z_p(E)$ which satisfies 1-$(p,1)$-estimates and thus is boundedly complete. Let $\mathcal{F} = \{\Sigma a_i f_i \colon (a_i) \in B_{\ell_{p'}}$ and (f_i) is a (finite or infinite) block sequence of (E_n^*) in $S_{Z^*}\}$. It is easy to check that $\mathcal{F}$ is a w^*-compact 1-norming (for $Z_p(E)$) subset of $B_{Z_p(E)^*}$ and thus $Z_p(E)$ embeds isometrically into $C(\mathcal{F})$. Furthermore it is again easy to check that each normalized block sequence of E in $Z_p(E)$ is point wise null on $\mathcal{F}$. Hence E is shrinking in $Z_p(E)$ and so $Z_p(E)$ is reflexive. Note for later that this argument only requires that E is a shrinking FDD.

So we have proved part of Theorem 5 in a special case. Assume now that X satisfies (p,p)-tree estimates. In this case things become simpler. We could follow the proof of Theorem 2 but after obtaining the FDD E for Z so that all $\bar{\delta}$-skipped block sequences of E is S_X $2C$-dominate the unit vector basis of ℓ_p we could repeat the argument for upper estimates and by blocking again obtain an FDD, still denoted by E, so that such $\bar{\delta}$-skipped block sequences are also $2C$-dominated by the unit vector basis of ℓ_p. Then by estimates as in the proof of Theorem 2 we could show that X naturally embeds into $(\Sigma F_n)_{\ell_p}$ for some blocking (F_n) of (E_n).

The more general cases of Theorem 5 present new difficulties. The norm defining $Z_p(E)$ yields $(p,1)$-estimates. There seems to be no natural way however to directly define a norm yielding (∞,q)-estimates. However if X satisfies (∞,q)-tree estimates then by Lemma 4 X^* satisfies $(q',1)$-tree estimates. We thus need to show that X^* is a quotient of a reflexive space $Y_{q'}(F)$ and obtain X embeds into $Z = Y_{q'}(F)^*$ which, as is easily seen, satisfies (∞, q)-estimates for the FDD (F_i^*). Then we use the "X embeds into $Z_p(G^*)$" argument above, for some blocking (G_i^*) of (F_i^*), to obtain X embeds into a space with an FDD satisfying (p, q)-estimates. Of course it needs to be checked that $Z_p(G_i^*)$ preserves the (∞, q)-estimates. This was proved by Pruss. In fact if $F = (F_i)$ is an FDD for Z satisfying C-(∞, q)-estimates then F satisfies C-(∞, q)-estimates for $Z_p(E)$. We will not give the proof but note that the same argument (due to Johnson and Schechtman) is used below in the proof of Theorem 11 (see Remark after proof of Theorem 11).

We thus require the following theorem of which part a) has been proved.

Theorem 6. *Let X be a reflexive space and let $1 < p < \infty$. If X satisfies $(p, 1)$-tree estimates then*

a) If X is a subspace of a reflexive space Z with an FDD E then there is a blocking $F = (F_i)$ of E so that X naturally embeds into the reflexive space $Z_p(F)$.

b) X is a quotient of a reflexive space with an FDD satisfying $(p, 1)$-estimates.

Theorem 5 follows readily from Theorem 6 (and Lemma 4). We are left with the
Sketch of the proof of Theorem 6 *b).*

By Lemma 2 we can regard $X^* \subseteq Z^*$ where Z^* is a reflexive space with a bimonotone FDD (E_i^*) such that $c_{00}\left(\oplus_{i=1}^{\infty} E_i^*\right) \cap X^*$ is dense in X^*. Thus we have a quotient map $Q\colon\ Z \to X$. By part a), $X \subseteq W$, a reflexive space with an FDD (F_i) satisfying C-$(p, 1)$-estimates.

By a fundamental blocking lemma of Johnson and Zippin [14] we may assume that for all $i \leq j$, $Q\left(\oplus_{n\in(i,j]} E_n\right)$ is essentially contained in $\oplus_{n\in[i,j]} F_n$.

We shall increase the norm on Z, obtaining a space $\widetilde{Z}$ for which (E_i), now designated $(\widetilde{E}_i)$, remains a shrinking FDD and so that Q, now called $\widetilde{Q}$, remains a quotient map. Then we shall find a blocking $\widetilde{H}$ of $\widetilde{E}$ so that $\widetilde{Q}\colon\ \widetilde{Z}_p(\widetilde{H}) \to X$ remains a quotient map. As noted above $\widetilde{Z}_p(\widetilde{H})$ is reflexive, since $(\widetilde{H})$ is shrinking.

For $z \in E_i$ we set $|||\tilde{z}||| = \|Q(z)\|$ and more generally for $\tilde{z} = \Sigma\tilde{z}_i \in c_{00}\left(\oplus_1^{\infty}\widetilde{E}_i\right)$ we set $|||\tilde{z}||| = \max_{m\leq n}\left\|\sum_{i=m}^{n} Q(z_i)\right\|$. Then one checks that $\widetilde{Q}$ remains a precise quotient map from $\widetilde{Z}$ = completion of $c_{00}\left(\oplus_1^{\infty}\widetilde{E}_i\right)$ under $|||\cdot|||$ onto X. In fact if $Qz = x$, $\|z\| = \|x\|$, then $|||\tilde{z}||| = \|z\|$, $\widetilde{Q}\tilde{z} = x$. Also $(\widetilde{E}_i)$ is a bimonotone FDD for $\tilde{z}$ (by blocking we may assume $\widetilde{E}_i \neq \{0\}$).

A key feature of $(\widetilde{Z}, |||\cdot|||)$ is following.

If $(\tilde{z}_i)$ is a block sequence of $(\widetilde{E}_i)$ in $B_{\widetilde{Z}}$ and $(Q\tilde{z}_i)$ is a basic sequence in X with projection constant $\overline{K}$ and $a \equiv \inf_i \|\widetilde{Q}\tilde{z}_i\| > 0$ then (4..1)

$$\left\|\sum a_i\widetilde{Q}(\tilde{z})_i)\right\| \leq \left|\left|\left|\sum a_i\tilde{z}_i\right|\right|\right| \leq \frac{3\overline{K}}{a}\left\|\sum a_i\widetilde{Q}(\tilde{z}_i)\right\|$$

for all scalars (a_i).

From (4..1) and the fact that $c_{00}\left(\oplus_{i=1}^{\infty} E_i^*\right) \cap X^*$ is dense in X^* one can deduce that $(\tilde{E}_i)$ is a shrinking FDD for $\widetilde{Z}$.

It remains only to prove that there exists $A < \infty$ and a blocking $\widetilde{H}$ of $\widetilde{E}$ satisfying the following. Let $x \in S_X$. There exists $\tilde{z} = \Sigma\tilde{z}_i$, $\tilde{z}_i \in \widetilde{H}_i$, so that if $(\tilde{w}_n)$ is any blocking of $(\tilde{z}_i)$ then $(\Sigma|||\tilde{w}_n|||^p)^{1/p} \leq A$ and $\|\widetilde{Q}\tilde{z} - x\| < 1/2$. Thus $\widetilde{Q}\colon\ \widetilde{Z}_p(\widetilde{H}) \to X$ remains a quotient map.

To accomplish this we first use the Johnson and Zippin [14] blocking lemma for our original $Q\colon\ Z \to X$ to produce a blocking (C_n) of (E_n), and corresponding blocking (D_n) of (F_n) so that if $x \in S_X$ is essentially contained in $\oplus_{s\in(i,j)}D_s$ then there exists $z\in B_Z$ with $Qz\approx x$ and $z\in C_{i,R}\oplus\big(\oplus_{s\in(i,j)}C_s\big)\oplus C_{j,L}$ where $C_{i,R}$ is the "right half" of the blocking of E_i's yielding C_i and $C_{j,L}$ is the "left half" of C_j.

Then we use Lemma 3.2 for suitable (δ_i) to obtain a blocking (G_n) of (D_n) and let (H_n) be the corresponding blocking of (C_n). If $x\in S_X$ we write $x=\Sigma x_i$, $(x_i)\subseteq X$, as in Lemma 3.2 and let

$$B=\{i\colon\ \|P^D_{(t_{i-1},t_i)}x_i-x_i\|<\delta_i\|x_i\|\},$$

$y=\sum\limits_{i\in B}x_i$. Then $\|y-x\|<1/4$ if $\Sigma\delta_i<1/4$. From our left half/right half construction above we can choose a block sequence $(z_i)_{i\in B}$ of (E_n) in B_Z with $\|Qz_i-\bar{x}_i\|\approx 0$ for $i\in B$ and $\bar{x}_i=x_i/\|x_i\|$. $(\bar{x}_i)_{i\in B}$ is a perturbation of a block sequence of (F_i) in W so admits $2C$-$(p,1)$-estimates. From 4..1 $(\tilde{z}_i)_{i\in B}$ is equivalent to $(\bar{x}_i)_{i\in B}$ and if we set $\tilde{z}=\sum\limits_{i\in B}\|x_i\|\tilde{z}_i$ we can show this has the desired property. □

Suppose that X is a reflexive space which can be renormed to be a.u.s. and can also be renormed to be a.u.c. From Theorem 3 it follows that there exists $1<q\le p<\infty$ so that X satisfies (p,q)-tree estimates. Thus we have from Theorem 5.

Theorem 7. *If X is a reflexive space with an equivalent a.u.s. norm and an equivalent a.u.c. norm then there exists $1<q\le p<\infty$ so that*

a) X embeds into a reflexive space with an FDD satisfying 1-(p,q)-estimates. Hence

b) X can be renormed to be simultaneously a.u.s. of power type q and a.u.c. of power type p.

Remark 1. *The hypothesis of Theorem 7 is equivalent to: X is reflexive and $S_z(X)=S_z(X^*)=w$.*

It is natural to ask if the results obtained above for (p,q)-estimates can be extended to more general estimates, say where ℓ_p is replaced by a space V with a normalized 1-unconditional basis (v_i) replacing the unit vector basis of ℓ_p and similarly for ℓ_q. This is done in [26]. The arguments have a similar flavor as do the ones above but the proofs are more technically difficult. The analog of Theorem 6 is the following result. The definitions are the analogs of the ones in the ℓ_p-case.

Theorem 8. *[26, Theorem 3.1] Let V be a Banach space with a normalized 1-unconditional basis (v_i) and let X be a reflexive space satisfying V-lower tree estimates (i.e. for some $C<\infty$ every weakly null tree in S_X admits a branch C-dominating (v_i)). Then*

a) For every reflexive space Z with an FDD $E=(E_i)$ containing X there is a blocking $H=(H_i)$ of E so that X naturally embeds into $Z_V(H)$.

b) There is a space Y with a shrinking FDD G so that X is a quotient of $Y_V(G)$.

The norm in $Z_V(H)$ is given by for $x\in c_{00}\left(\oplus_1^\infty H_i\right)$ by

$$\|x\|=\sup\left\{\left\|\sum_{j=1}^\infty\|P^H_{(n_{j-1},n_j]}x\|_Z\cdot v_i\right\|_v\ :\ 0=n_0<n_1<\cdots\right\}.$$

Unlike the ℓ_p case (H_i), which is an FDD for $Z_V(H)$, does not automatically admit a lower V-estimate on blocks. But this can be achieved with additional hypotheses on V.

Definition 8. *A normalized 1-unconditional basis (v_i) is* regular *iff*

i) (v_i) is dominated by every normalized block basis of (v_i).

ii) There exists $c>0$ so that for all $(a_i)\in c_{00}$ and $n\in\mathbb{N}$

$$\left\|\sum_{i=1}^\infty a_iv_{i+n}\right\|\ge c\left\|\sum_{i=1}^\infty a_iv_i\right\|.$$

iii) There exists $d > 0$ so that for all $m \in \mathbb{N}$ there exists $L = L(m) \geq m$ so that for all $k \leq m$

$$\left\| \sum_{i=L+1}^{\infty} a_i v_{i-k} \right\| \geq d \left\| \sum_{i=L+1}^{\infty} a_i v_i \right\| \quad \textit{whenever} \quad (a_i) \in c_{00}.$$

Theorem 9. *[26, Corollary 3.2] Let V be a reflexive space with a regular normalized 1-unconditional basis (v_i). Let X be a reflexive space with V-lower tree estimates. Then X is a subspace of a reflexive space Z with an FDD satisfying V-lower estimates and X is a quotient of a reflexive space Y with an FDD satisfying V-lower estimates.*

For an upper and lower estimate result we have

Theorem 10. *[26, Theorem 3.4] Let V and U^* be reflexive Banach spaces with regular normalized 1-unconditional bases (v_i) and (u_i^*), respectively. Assume that every subsequence of (u_i) dominates every normalized block basis of (v_i) and every normalized block basis of (u_i) dominates every subsequence of (v_i). If X is a reflexive space satisfying (V, U)-tree estimates then X embeds into a reflexive space Z with an FDD satisfying (V, U)-estimates.*

Examples of spaces (V, U) satisfying the hypothesis of Theorem 10 are the *convexified Tsirelson* spaces $(T_{p,\gamma}, T^*_{q',\gamma})$ where $1 \leq q \leq p \leq \infty$ and $0 < \gamma < 1/4$. If X is a reflexive asymptotic ℓ_p space (i.e. $\exists C \geq 1 \forall n\, \forall (e_i)_1^n \in \{X\}_n$,

$$C^{-1} \left(\sum_1^n |a_i|^p \right)^{1/p} \leq \left\| \sum_1^n a_i e_i \right\| \leq C \left(\sum_{i=1}^n |a_i|^p \right)^{1/p}$$

for all $(a_i)_1^n \subseteq \mathbb{R}$) then it can be easily seen that X satisfies $(T_{p,\gamma}, T^*_{p'\gamma})$-tree estimates for some $0 < \gamma < 1/4$. As an application we have

Corollary 3. *Let X be a reflexive asymptotic ℓ_p space. Then X embeds into a reflexive space with an asymptotic ℓ_p FDD. X is also a quotient of such a space.*

Similar results can be obtained analogous to those of Theorem 5.

5. Universal spaces

We begin with the solution to Bourgain's problem (see Section 1.). Note that (e.g., by Krivine's theorem [17]) if X contains an isomorph of ℓ_p for all $1 < p < \infty$, then c_0 and ℓ_1 are finitely represented in X so X cannot be superreflexive.

One step in the proof will be, given $1 < q \leq p < \infty$, to construct a space $Z_{(p,q)}$ with an FDD satisfying 1-(p, q)-estimates which is universal for all such spaces. We shall do this first before proceeding to the theorem. S. Pruss [28] has shown a similar result but we prefer to present a somewhat different argument which could prove useful elsewhere.

Lemma 5. *Let $1 \leq q \leq p \leq \infty$, and let F and G be two finite dimensional normed linear spaces. Denote the norm on F and G by $\|\cdot\|_E$ and $\|\cdot\|_G$ respectively. Let $|||\cdot|||$ be a norm on $F \oplus G$ and assume that (F, G) satisfies 1-(p, q)-estimates in $(F \oplus G, |||\cdot|||)$ and there are $1 < c < d < \infty$ so that*

$$\begin{aligned} &c\|f\|_F \leq |||f||| \leq d\|f\|_F \textit{ whenever } f \in F \textit{ and} \\ &c\|g\|_G \leq |||g||| \leq d\|g\|_G \textit{ whenever } g \in G. \end{aligned} \tag{5..1}$$

Then there is a norm $\|\cdot\|$ on $F \oplus G$ extending $\|\cdot\|_F$ and $\|\cdot\|_G$, so that (F, G) is an FDD satisfying 1-(p, q)-estimates in $(F \oplus G, \|\cdot\|)$ and

$$c\|f + g\| \leq |||f + g||| \leq d\|f + g\| \textit{ whenever } f \in F \textit{ and } g \in G. \tag{5..2}$$

PROOF. For $f \in F$ and $g \in G$ put:

$$\|f+g\| = \max\Big\{\big(\|f\|_F^p + \|g\|_G^p\big)^{1/p}, \frac{1}{d}|||f+g|||\Big\},$$

where we replace $\big(\|f\|_F^p + \|g\|_G^p\big)^{1/p}$ by $\max\{\|f\|_F, \|g\|_G\}$ if $p = \infty$. Clearly (F, G) satisfies 1-$(p,1)$-estimates, and since (F, G) satisfies 1-(∞, q)-estimates in $(F \oplus G, |||\cdot|||)$, this is also true for $(F \oplus G, \|\cdot\|)$. Moreover, for $f \in F$ and $g \in G$ we deduce

$$\begin{aligned}
c\|f+g\| &= \max\Big\{\big(\|cf\|_F^p + \|cg\|_G^p\big)^{1/p}, \frac{c}{d}|||f+g|||\Big\}\\
&\le \max\Big\{\big(|||f|||_F^p + |||g|||_G^p\big)^{1/p}, |||f+g|||\Big\}\\
&\le |||f+g||| \le d\|f+g\|.
\end{aligned}$$

We introduce the following Terminology.

Definition 9. *Let E_α be finite dimensional linear space for each $\alpha \in T_\infty$ and let $\|\cdot\|_\beta$ be a norm on $c_{00}(\oplus_{i=1}^\infty E_{\beta_i})$ for each branch $\beta = (\beta_i)_{i=1}^\infty$ of T_∞. We say that the family $(\|\cdot\|_\beta)$ indexed over all branches of T_∞ is* compatible *if*

1. *For every branch $\beta = (\beta_i)_{i=1}^\infty$ of T_∞, (E_{β_i}) is a bimonotone FDD for the completion X_β of $c_{00}(\oplus_{i=1}^\infty E_{\beta_i})$ under $\|\cdot\|_\beta$.*
2. *If $\alpha = (\alpha_i)$ and $\beta = (\beta_i)$ are two branches and if $\ell = \max\{i : \forall j \le i\ \alpha_j = \beta_j\}$ ($\ell = 0$ if $\alpha_1 \ne \beta_1$) then $\|\cdot\|_\alpha$ and $\|\cdot\|_\beta$ coincide on $\oplus_{i=1}^\ell E_{\alpha_i}$.*

Proposition 6. *Let $1 \le q \le p \le \infty$. Then there exists a tree $(E_\alpha)_{\alpha\in T_\infty}$ of finite dimensional linear spaces and a compatible family of norms $\|\cdot\|_\beta$ for each branch β of T_∞ satisfying the following*

1. *If $\beta = (\beta_i)$ is a branch in T_∞ then (E_{β_i}) satisfies 1-(p,q)-estimates for $\|\cdot\|_\beta$.*
2. *Let Y be any Banach space with norm $\|\cdot\|$ and with an FDD (F_i) satisfying 1-(p,q)-estimates in Y and let $d > 1$. Then there exists a branch $\beta = (\beta_i)$ of T_∞ and an isomorphism I from X_β (the completion of $c_{00}(\oplus E_{\beta_i})$) onto Y under $\|\cdot\|_\beta$) mapping E_{β_i} onto F_i, for $i \in \mathbb{N}$, satisfying*

$$\|x\|_\beta \le \|I(x)\| \le d\|x\|_\beta \text{ whenever } x \in X_\beta.$$

PROOF. For $n \in \mathbb{N}$ let T_n be the elements of T_∞ of length n. By induction on $n \in \mathbb{N}$ we will define the normed linear spaces E_α for all $\alpha \in T_n$ and norms $\|\cdot\|_\beta$ on $\oplus_{j=1}^n E_{(\alpha_j)}$ where $\beta = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ is a branch of length n in $\bigcup_{j=1}^n T_j$, i.e. for $i \le n$ $|\alpha_i| = i$ and α_i is a successor of α_{i-1} if $1 < i$.

The first level of $(E_\alpha)_{\alpha\in T_\infty}$ is any sequence of finite dimensional Banach spaces which is dense (with respect to the Banach-Mazur distance) in the set of all finite dimensional Banach spaces.

Assume we have defined for a branch $\beta = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ the space $E^\beta = \oplus_{i=1}^n E_{\alpha_i}$ along with a norm $\|\cdot\|_\beta$ on it. Let $\beta = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ be such a branch. The successors of α_n are chosen as follows. Let (G_i) be the spaces of level 1. For each G_i we consider the set of all extensions of $\|\cdot\|_\beta$ to $E^\beta \oplus G_i$ satisfying 1-(p,q)-estimates.

For any two such extensions $\|\cdot\|_1$ and $\|\cdot\|_2$ we define the distance between $\|\cdot\|_1$ and $\|\cdot\|_2$ by

$$d(\|\cdot\|_1, \|\cdot\|_2) = \ln\big(\|I\|\cdot\|I^{-1}\|\big),$$

where $I : \big(E_\alpha \oplus G_i, \|\cdot\|_1\big) \to \big(E_\alpha \oplus G_i, \|\cdot\|_2\big)$ is the identity. We then choose a countable dense subset of these extensions with respect to $d(\cdot,\cdot)$. The sequence of all successors will then be formed by the union over all i of these countable many extensions.

To see (2) we will use Lemma 5. Let $1 < d$ and let $1 < c_n < d_n < d$ with $c_n \searrow 1$ and $d_n \nearrow d$, if $n \nearrow \infty$. Let Y and (F_i) as in (2 and denote the norm on Y by $\|\cdot\|$. To start we find $\alpha_1 \in T_1$ and an isometry I_1 from F_1 onto $(E_{\alpha_1}, |||\cdot|||_1)$ where $|||\cdot|||_1$ is a norm on E_{α_1} with $c_1\|x\|_{\beta_1} \le |||x|||_1 \le d_1\|x\|_{\beta_1}$ ($\beta_1 = (\alpha_1)$).

Assume we constructed a branch $\beta = (\alpha_1, \dots \alpha_n)$ of length n along with a norm $|||\cdot|||_n$ on E^β and an isometry mapping, F_i into E_{α_i}, for $i = 1, 2 \dots, n$

$$I_n : \big(\oplus_{i=1}^n F_i, ||\cdot||\big) \to \big(E^\beta, |||\cdot|||_n\big)$$

satisfying

$$c_n\|x\|_\beta \le |||x|||_n \le d_n\|x\|_\beta \text{ for } x \in E^\beta.$$

Since (G_i) is dense in the set of all finite dimensional normed spaces we can find a $G = G_i$, whose norm we denote by $\|\cdot\|_G$, $\dim(G) = \dim(F_{n+1})$ and an isometry $J : F_{n+1} \to (G, |||\cdot|||)$ where $|||\cdot|||$ is a norm on G satisfying

$$c_n\|x\| \le |||x||| \le d_n\|x\|_G, \text{ whenever } x \in G.$$

Define

$$I_{n+1} : \oplus_{i+1}^n F_i \to E^\beta \oplus G, \quad \sum_{i=1}^{n+1} x_i \mapsto I_n\Big(\sum_{i=1}^n x_i\Big) + J(x_{n+1}),$$

and put

$$|||x + y|||_{n+1} = ||I_{n+1}^{-1}(x, y)|| \text{ whenever } x \in E^\beta \text{ and } y \in G.$$

By Lemma 5 we can find a norm $\|\cdot\|$ on $E^\beta \oplus G$ extending $\|\cdot\|_\beta$ on E^β and $||\cdot||_G$ on G for which (E^β, G) satisfies 1-(p, q)-estimates so that

$$c_n\|x + y\| \le |||x + y|||_{n+1} \le d_n\|x + y\| \text{ whenever } x \in E^\beta \text{ and } y \in G.$$

From our construction of $(E_\alpha)_{\alpha\in T_\infty}$ there exists a successor α_{n+1} of α so that for $\overline{\beta} = (\alpha_1, \alpha_2, \dots, \alpha_{n+1})$ and $x \in \oplus E^{\overline{\beta}}$

$$c_{n+1}\|x\|_{\overline{\beta}} \le |||(x)|||_{n+1} \le d_{n+1}\|x\|_{\overline{\beta}},$$

which finishes our recursive choice.

Taking now the infinite branch $\beta = (\alpha_i)_{i=1}^\infty$ yields our claim (2).

Theorem 11. *There exists a separable reflexive space X_u which is universal for $\{X : X$ is reflexive and $S_z(X) = S_z(X^*) = \omega\}$. In particular X_u contains an isomorph of all separable superreflexive spaces.*

Proof. We first note that if X is superreflexive then X satisfies (p, q)-tree estimates for some $1 < q \le p < \infty$ ([8], [6]). By Theorem 5 X then embeds into a reflexive space Z with an FDD satisfying 1-(p, q)-estimates. Moreover by Theorem 3 (applied to X and X^*) the same holds if X is reflexive with $S_z(X) = S_z(X^*) = \omega$. Thus it suffices to produce a space $Z_{(p,q)}$ with an FDD satisfying 1-(p, q)-estimates which is universal for all spaces with a 1-(p, q) FDD. We then take $X_u = (\sum Z_{(p_n,q_n)})_{\ell_2}$ where $p_n \uparrow \infty$ and $q_n \downarrow 1$.

To construct $Z_{(p,q)}$ we first let $(E_\alpha)_{\alpha\in T_\infty}$ along with compatible norms $\|\cdot\|_\beta$ for branches β in T_∞ be as constructed in Proposition 6 for p and q.

$Z_{(p,q)}$ is then the completion of $c_{00}(\oplus_{\alpha\in T_\infty} E_\alpha)$ under

$$\|z\| = \sup\left\{\Big(\sum_j \|P^E_{I_j} z\|^p\Big)^{1/p} : I_1, I_2, \dots \text{ are disjoint segments in } T_\infty\right\}.$$

For a segment I, $\|P^E_I z\| = \|P^E_I z\|_\beta$ where α is any branch containing I. $E = (E_\alpha)_{\alpha\in T_\infty}$ is thus an FDD for $Z_{(p,q)}$ when ordered linearly in any manner compatible with the tree order of T_α, e.g. $E_{(n_1,n_2,n_3)}$ comes after $E_{(n_1,n_2)}$. Moreover E, when thus ordered, satisfies 1-$(p,1)$-estimates and the norm on each branch of $(E_\alpha)_{\alpha\in T_\alpha}$ is preserved. Finally we check that $Z_{(p,q)}$ satisfies 1-(∞,q)-estimates.

Let $z = \sum z_i \in c_{00}(\oplus_{\alpha\in T_\infty} E_\alpha)$ where (z_i) is a block sequence of E. Let $\|z\| = (\sum \|P^E_{I_j} z\|^p)^{1/p}$ where $I_1, I_2, \dots$ are disjoint segments in T_∞. We decompose I_j into segments $I_{j,1}, I_{j,2}, \dots$ so that $P^E_{I_{j,i}} z_i = P^E_{I_{j,i}} z$ and $P^E_{I_{j,i}} z_s = 0$ if $s \neq i$. Then

$$\|z\| \le \Big(\sum_j \Big(\sum_i \|P_{I_{j,i}} z_i\|^q \Big)^{p/q} \Big)^{1/p}$$

by the 1-(∞,q)-estimates on each branch. Now

$$\Big(\sum_j \Big(\sum_i \|P_{I_{j,i}} z_i\|^q \Big)^{p/q} \Big)^{1/p} \le \Big[\sum_i \Big(\sum_j \|P_{I_{j,i}} z_i\|^p \Big)^{q/p} \Big]^{1/q}$$

by the reverse triangle inequality in $\ell_{p/q}$. Thus $\|z\| \le (\sum_i \|z_i\|^q)^{1/q}$.

Remark 2. *The clever argument for $1-(\infty,q)$ estimate is due to Johnson and Schechtman. It was used in [25] to show that if an FDD $E=(E_i)$ for a space Z satisfies $1-(\infty,q)$-estimates then it also satisfies $1-(\infty,q)$-estimates in $Z_p(E)$.*

We now turn to the universal problem for the classes (see Theorem 3)

$$\begin{aligned} \mathcal{C}_{\text{auc}} &= \{Y : Y \text{ is separable, reflexive and has an equivalent a.u.c. norm}\} \\ &= \{Y : Y \text{ is separable, reflexive and } S_z(Y^*) = \omega\} \end{aligned}$$

and

$$\begin{aligned} \mathcal{C}_{\text{aus}} &= \{Y : Y \text{ is separable, reflexive and has an equivalent a.u.s. norm}\} \\ &= \{Y ; Y \text{ is separable, reflexive and } S_z(Y) = \omega\} \end{aligned}$$

First note that the Tsirelson space $T(\frac{1}{2}, S_\alpha)$, $\alpha < \omega_1$, $S_\alpha = \alpha^{th}$ Schreier class [1] are all in $\mathcal{C}_{\text{auc}}$ since their unit vector basis has $(p,1)$-estimates for all $p>1$ and their duals are all in $\mathcal{C}_{\text{aus}}$. It follows by index arguments [4]) that any space universal for C_{auc} must contain ℓ_1 and any space universal for $\mathcal{C}_{\text{aus}}$ must contain c_0.

Proposition 7. *There exists a separable dual space X which is universal for $\mathcal{C}_{\text{auc}}$. X is the ℓ_2 sum of a.u.c. spaces.*

PROOF. The argument is much the same as that of Theorem 11. For $p<\infty$ we let $Z_{(p,1)}$ be the space constructed in the proof of Theorem 11. $Z_{(p,1)}$ has an FDD satisfying $1-(p,1)$-estimates and as such, having a boundedly complete FDD, is a separable dual space. By Theorem 5 $Z_{(p,1)}$ is universal for all spaces in $\mathcal{C}_{\text{auc}}$ satisfying $(p,1)$-estimates. Thus by Theorem 3, $X = \big(\oplus_{n=2}^\infty Z_{(n,1)}\big)_{\ell_2}$ is universal for $\mathcal{C}_{\text{auc}}$. X is a separable dual space.

Remark 3. *The spaces $Z_{(p,q))}$ constructed in Theorem 11 and Proposition 7 are actually complementably universal for the members of their respective classes which have (p,q) or $(p,1)$ FDD's. The space X of Proposition 7 is universal for the class*

$$\begin{aligned} &\{Y : Y \textit{ is a separable dual satisfying } w^* - (p,1) - \textit{estimates for some } p<\infty\} \\ &= \{Y : Y = W^* \textit{ with } S_z(W) = \omega\} \end{aligned}$$

Proposition 8. *There exists a space Y with separable dual which is universal for the class $\mathcal{C}_{\mathrm{aus}}$. Y is the ℓ_2 sum of a.u.s. spaces.*

PROOF. Let $q > 1$. Let $(E_\alpha)_{\alpha\in T_\infty}$ and a compatible set of norms $\|\cdot\|_\beta$ for each branch β of T_∞ be constructed as in Proposition 6 for (∞, q). We let Z_q be the completion of $c_{00}(\oplus_{\alpha\in T_\infty} E_\alpha)$ under the norm

$$\|z\| = \sup\left\{\|P^E_\beta z\| : \beta \text{ is a branch in } T_\infty\right\}.$$

If $(E_\alpha)_{\alpha\in T_\infty}$ is linearly ordered in a manner compatible with the order on T_∞ it becomes a bimonotone FDD for Z_q satisfying $1-(\infty, q)$-estimates.

Let $q_n \searrow 1$, if $n \nearrow \infty$ and set $Y = \left(\oplus_{n=1}^\infty Z_{q_n}\right)_{\ell_2}$. By Theorem 3 and Theorem 5 Y is universal for $\mathcal{C}_{\mathrm{aus}}$. Clearly Y^* is separable.

Remark 4. *For Y as constructed in Proposition* 8 *it follows that $S_z(Y) = \omega^2$.*

Finally we note the following result from [26].

Theorem 12. *Let $K < \infty$ and $1 < p < \infty$. There exists a reflexive asymptotic ℓ_p space which is universal for the class of all reflexive K-asymptotic ℓ_p spaces.*

We refer to [26] for the proof and for more general versions of this results.

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E. Odell
Department of Mathematics
The University of Texas at Austin
1 University Station C1200
Austin, TX 78712-0257
E-mail: odell@math.utexas.edu

Th. Schlumprecht
Department of Mathematics
Texas A&M University
College Station, TX 77843-3368
E-mail: schlump@math.tamu.edu