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Universal spaces for strictly convex Banach Spaces

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Abstract. We show that if a separable Banach space X contains an isometric copy of every strictly convex separable Banach space, then X contains an isometric copy of ℓ_1 equipped with its natural norm. In particular, the class of strictly convex separable Banach spaces has no universal element. This provides a negative answer to a question asked by J. Lindenstrauss.

Espacios universales para los espacios de Banach estrictamente convexos

Resumen. Probamos que si un espacio separable X contiene una copia isométrica de todo espacio estrictamente convexo separable, entonces X contiene una copia isométrica de ℓ_1 con su norma natural. En particular, la clase de los espacios de Banach estrictamente convexos no tiene elemento universal. Esto responde negativamente a una pregunta de J. Lindenstrauss.

1. Introduction

A classical result, which goes back to Mazur, asserts that every separable Banach space is isometric to a linear subspace of the space of continuous functions on the Cantor set. Let us consider more generally some isometric property (P) of separable Banach spaces, and assume that (P) is hereditary, that is, any subspace of a space with (P) shares this property. It is natural to wonder whether there is a universal space for (P). In other words, is there a separable space U enjoying (P) such that every space with (P) is isometric to a subspace of U?

When (P) yields to some modulus, such as uniform convexity or uniform smoothness, is is usually easy to show that the above question has a negative answer, by considering spaces with (P) but an "arbitrarily bad" modulus. Hence quantitative properties are quite easy to deal with in this context. Qualitative properties such as strict convexity are more difficult to handle. And in fact, a question which goes back to J. Lindenstrauss ([11], p. 241) asks if there is a strictly convex separable space which isometrically contains every strictly convex separable Banach space.

Quite naturally, the answer to this problem is also negative, and the solution relies again on a somewhat quantitative argument, but this time of a transfinite nature. Indeed the gist of our approach is that the collection of separable strictly convex Banach spaces is a coanalytic non Borel family (which means that the "modulus of strict convexity" is a countable ordinal) while the collection of subspaces of a given space is analytic.

Proving the non Borel character of the property of strict convexity requests the construction of strictly convex spaces whose modulus is an arbitrarily large countable ordinal. These spaces will be supported by well-founded trees of arbitrarily large height, and we will rely heavily on the method of [3].

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Let us outline the contents of this note. The first section contains the construction of a special norm on ℓ_1. The corresponding space is what will be obtained on every infinite branch. Section 2 displays the space $E(\omega^{<\omega})$ where our work is done, and it is proved that a subspace $E(T)$ supported by a tree T is strictly convex if and only if T is well-founded. The main results follow quite easily in section 3, through some simple topological lemmas. Finally, section 4 presents a topological frame in which these results can easily be expressed, and which is an isometric version of a notion introduced in the recent work of S. Argyros and P. Dodos [1]. Our hope is to stimulate research along these lines and some open questions conclude the article.

The notation we use is classical or it will be explained before use. We denote by $\omega = \{0, 1, 2, ...\}$ the set of integers.

2. A special norm on ℓ_1

This section is devoted to the construction of a peculiar norm on $\ell_1(\omega) = \ell_1$. The technical Proposition 1 gathers the properties of this norm which will be useful later. We equip ℓ_1 with its natural basis $(e_k)_{k\geq 0}$. We denote by $(e_k^*)_{k\geq 0}$ the coordinate functionals, and by $(\pi_n)_{n\geq 0}$ the projections such that

$$\pi_n(x) = \sum_{k=0}^{n} e_k^*(x)e_k.$$

For $x \in \ell_1$, we denote

$$\text{supp }(x) = \{n \in \omega; e_n^*(x) \neq 0\}$$

Proposition 1 *There exists an equivalent norm $|||\cdot|||$ on ℓ_1 such that:*

i For every $x \in \ell_1$ and every $n \geq 1$, $|||\pi_n(x)|||^2 \geq |||\pi_{n-1}(x)|||^2 + \frac{1}{2}|e_n^*(x)|^2$

ii For every $v \in \ell_1$ with $v \neq 0$ and $\text{supp }(v)$ *finite, $|||\cdot|||$ is uniformly convex in the direction v.*

iii There exists a subspace X of ℓ_1 such that $|||y||| = \|y\|_1$ for all $y \in X$ and such that X is isometric to $(\ell_1, \|\cdot\|_1)$.

PROOF.

We write $\omega = \bigcup_{j\geq 0} I_j$, where the I_j's are disjoint infinite subsets of ω. For all $j \geq 0$, let $x_j \in \ell_1$ be such that $\|x_j\|_1 = 1$ and $\text{supp }(x_j) = I_j$. Let

$$X = \overline{\text{span}}\,((x_j)_{j\geq 0})$$

and let $Q = \ell_1 \to \ell_1/X$ be the canonical quotient map. Let

$$T = \ell_1/X \to \ell_2$$

be a linear continuous one-to-one map such that $\|T\| < \frac{1}{2}$. We let $S = TQ$ and we define our equivalent norm on ℓ_1 by

$$|||x|||^2 = \|x\|_1^2 + \|S(x)\|_2^2.$$

Condition (iii) is clearly satisfied.

For all $x \in \ell_1$, we have

$$S\pi_n(x) - S\pi_{n-1}(x) = e_n^*(x)Se_n$$

hence

$$\begin{aligned} \Big|\|S\pi_n(x)\|_2^2 - \|S\pi_{n-1}(x)\|_2^2\Big| &\leq \Big(\|S\pi_n(x)\|_2 + \|S\pi_{n-1}(x)\|_2\Big)\Big|e_n^*(x)\Big|\|Se_n\|_2 \\ &\leq \|S\|^2\Big(\|\pi_n(x)\|_1 + \|\pi_{n-1}(x)\|_1\Big)|e_n^*(x)|. \end{aligned}$$

On the other hand,

$$\|\pi_n(x)\|_1^2 - \|\pi_{n-1}(x)\|_1^2 = \Big(\|\pi_n(x)\|_1 + \|\pi_{n-1}(x)\|_1\Big)|e_n^*(x)|$$

and thus

$$\begin{aligned} |||\pi_n(x)|||^2 - |||\pi_{n-1}(x)|||^2 &\geq \Big(1-\|S\|^2\Big)\Big(\|\pi_n(x)\|_1 + \|\pi_{n-1}(x)\|_1\Big)|e_n^*(x)| \\ &\geq \Big(1-\|S\|^2\Big)|e_n^*(x)|^2 \\ &> \frac{1}{2}|e_n^*(x)|^2 \end{aligned}$$

and this shows (i).

Finally, let $v \neq 0$ with finite support, and let (x_n) and (y_n) be two sequences in ℓ_1 such that

$$x_n - y_n = \lambda_n v$$

and

$$\lim \Big[2\Big(|||x_n|||^2 + |||y_n|||^2\Big) - |||x_n + y_n|||^2\Big] = 0$$

The usual convexity argument ([6, Fact II. 2. 3]) shows that

$$\lim \Big[2\Big(|||S(x_n)|||^2 + |||S(y_n)|||^2\Big) - |||S(x_n) + S(y_n)|||_2^2\Big] = 0$$

hence by the parallelogram identity

$$\lim |||S(x_n - y_n)|||_2^2 = \lim \lambda_n^2 |||S(v)|||_2^2 = 0$$

Since $v \notin X$ we have $S(v) \neq 0$ and thus $\lim(\lambda_n) = 0$, which shows (ii). ■

Remark 2 *The space X is contractively complemented in $(\ell_1, \|\cdot\|_1)$ and we have $\|x\|_1 = |||x|||$ for all $x \in X$, and $\|y\|_1 \leq |||y|||$ for all $y \in \ell_1$. It follows that X is contractively complemented in $(\ell_1, |||.|||)$.*

3. The spaces $E(T)$

Let $\omega^{<\omega}$ be the set of all finite sequences of elements of ω. If $s \in \omega^{<\omega}$, we denote by $|s|$ the length of s. A subset T of $\omega^{<\omega}$ is called a *tree* if for all $s = (s_0, s_1, \dots, s_{|s|-1}) \in T$ and all $k < |s|, s_{\restriction k} = (s_0, s_1, \dots, s_k) \in T$. Of course, $\omega^{<\omega}$ itself is a tree. If there is $\sigma \in \omega^\omega$ such that $\sigma_{\restriction k} \in T$ for all $k \in \omega$, we say that T is *not well-founded*, and T is *well-founded* otherwise. We denote by $\mathcal{T}$ the set of all trees, and by $WF \subseteq \mathcal{T}$ the set of all well-founded trees. The set $\mathcal{T}$ is a closed subset of $2^{(\omega^{<\omega})}$ and therefore it is a compact metric space for the induced topology.

Let $c_{00}(\omega^{<\omega})$ be the vector space of all functions from $\omega^{<\omega}$ to $\mathbb{R}$ with finite support.

If $\sigma \in \omega^\omega$ and $s \in \omega^{<\omega}$, we write $s \prec \sigma$ if there is $k \in \omega$ such that $s = \sigma_{\restriction k}$. We denote

$$\sigma^* = \{s \in \omega^{<\omega}; s \not\prec \sigma\}$$

We define a norm $\|\cdot\|$ on $c_{00}(\omega^{<\omega})$ by the formula

$$\|y\|^2 = \sup_{\sigma \in \omega^\omega} |||\sum_{s \prec \sigma} y(s) e_{|s|}|||^2 + \frac{1}{2}\sum_{s \in \sigma^*} y(s)^2, \tag{1}$$

and we denote by $E(\omega^{<\omega})$ the completion of $c_{00}(\omega^{<\omega})$ with respect to this norm. It is easily seen that the sequence $\{\chi_s; s \in \omega^{<\omega}\}$, where

$$\left.\begin{array}{ll} \chi_s(s') = 1, & \text{if } s' = s \\ \chi_s(s') = 0, & \text{if } s' \neq s \end{array}\right\}$$

is an unconditional basis of the space $E(\omega^{<\omega})$. For every tree $T \subseteq \omega^{<\omega}$, we define the subspace $E(T)$ of $E(\omega^{<\omega})$ by

$$E(T) = \overline{\text{span}}\{\chi_s; s \in T\}$$

With this notation, we have:

Proposition 3 *If T is not well-founded then $E(T)$ contains a subspace isometric to $(\ell_1, |||\cdot|||)$.*

PROOF. Let $\sigma \in \omega^\omega$ be such that

$$(\sigma) = \{s; s \prec \sigma\} \subseteq T.$$

We claim that

$$E((\sigma)) = \overline{\text{span}}\{\chi_s; s \prec \sigma\} \equiv (\ell_1, |||\cdot|||),$$

where $\equiv$ means "is isometric to". For $y \in E(\omega^{<\omega})$ and $\sigma' \in \omega^\omega$, we denote

$$\sigma'(y) = \sum_{s \prec \sigma'} y(s) e_{|s|}. \tag{2}$$

If $y \in E((\sigma))$ and $\sigma' \neq \sigma$, there is $n \in \omega$ such that

$$\sigma'(y) = \sum_{|s| \leq n} y(s) e_{|s|},$$

hence if we let

$$\gamma_n(y) = ||| \sum_{|s| \leqslant n} y(s) e_{|s|} |||^2 + \frac{1}{2} \sum_{|s| > n} y(s)^2$$

we have by (1) that

$$\Big\| \sum_{s < \sigma} y(s) \chi_s \Big\|^2 = \sup_{n \in \omega} (\gamma_n)$$

But condition (i) of Proposition 1 shows that (γ_n) is increasing, and thus

$$\Big\| \sum_{s \prec \sigma} y(s) \chi_s \Big\| = ||| \sum_{s \prec \sigma} y(s) e_{|s|} |||.$$

■

Remark 4 *It follows easily from the monotonicity of the natural basis of ℓ_1 for the norm $|||\cdot|||$ (a consequence of condition (i)) that the space $E((\sigma))$ is contractively complemented in $E(\omega^{<\omega})$, the projection being the restriction to (σ). Thus, by Remark 2, $E(T)$ contains a contractively complemented subspace isometric to $(\ell_1, \|\cdot\|_1)$ if $T \notin WF$.*

We now check what happens when T is well-founded.

Proposition 5 *If $T \in WF$, then $E(T)$ is strictly convex.*

PROOF. Let $y, z \in E(T)$ be such that

$$\|y\| = \|z\| = \left\|\frac{y+z}{2}\right\| = 1. \tag{3}$$

For $\sigma \in \omega^\omega$ and $u \in E(\omega^{<\omega})$, we denote

$$\|u\|_\sigma^2 = |||\sigma(u)|||^2 + \frac{1}{2}\sum_{s\in\sigma^*} u(s)^2,$$

where $\sigma(u)$ is defined by (2), and then (1) reads

$$\|u\| = \sup_{\sigma\in\omega^\omega} \|u\|_\sigma.$$

Let $(\sigma_n) \subseteq \omega^\omega$ be such that

$$\sup_n \left\|\frac{y+z}{2}\right\|_{\sigma_n} = 1. \tag{4}$$

By (3), we have

$$\sup_n \|y\|_{\sigma_n} = \sup_n \|z\|_{\sigma_n} = 1. \tag{5}$$

Define $v = y - z \in E(T)$, and assume by contradiction that $v \neq 0$. ■

Fact 6 *The support* $\mathrm{supp}\,(v)$ *is finite, and there exists* $N \in \omega$ *such that* $\mathrm{supp}\,(v) \subseteq (\sigma_n)$ *if* $n \geq N$.

PROOF. We have to show that if $v(t) \neq 0$, then $t \in (\sigma_n)$ for n large enough. We have

$$\left\|\frac{y+z}{2}\right\|_{\sigma_n}^2 = |||\sigma_n\Big(\frac{y+z}{2}\Big)|||^2 + \frac{1}{2}\sum_{s\in\sigma_n^*}\left(\frac{y+z}{2}\right)(s)^2.$$

If $t \in \sigma_n^*$ for some n, we use the identity

$$\left(\frac{y+z}{2}\right)(t)^2 + \left(\frac{y-z}{2}\right)(t)^2 = \frac{1}{2}\left[y(t)^2 + z(t)^2\right]$$

to obtain

$$\left\|\frac{y+z}{2}\right\|_{\sigma_n}^2 \leq \frac{1}{2}\left(\|y\|_{\sigma_n}^2 + \|z\|_{\sigma_n}^2\right) - \frac{1}{2}\left(\frac{y(t)-z(t)}{2}\right)^2 \leq 1 - \frac{1}{2}\left(\frac{v(t)}{2}\right)^2,$$

and (4) shows that this fails for n large enough.

It follows that for every $t \in \mathrm{supp}\,(v)$, there is $N(t) \in \omega$ such that $t \in (\sigma_n)$ if $n \geq N(t)$. This implies in particular that if $t, t' \in \mathrm{supp}\,(v)$ then t and t' are comparable (for $\prec$) in $\omega^{<\omega}$, and since $v \in E(T)$ and T is well-founded, it follows that $\mathrm{supp}\,(v)$ is finite.

Now, taking $N = \max\{N(t); t \in \mathrm{supp}\,(v)\}$ concludes the proof of the Fact. ■

Fact 6 implies that the vector

$$\sigma_n(y) - \sigma_n(z) = \sigma_n(v) \in \ell_1$$

is independent of $n \geq N$. On the other hand, it follows from (4), (5) and the standard convexity argument that

$$\lim_n 2\Big[|||\sigma_n(y)|||^2 + |||\sigma_n(z)|||^2\Big] - |||\sigma_n(y+z)|||^2 = 0$$

but since $|||\cdot|||$ is uniformly convex in the direction $\sigma_n(v)$ by condition (ii) of Proposition1, this implies that $\sigma_n(v) = 0$ and thus $v = 0$. ■

4. Main results

We refer to [2] for a display —and use— of a proper parametrization of the collection of separable Banach spaces, which turns it into a standard Borel space. In short, we consider it as the set $\mathcal{SE}(\mathcal{C}(2^\omega))$ of closed vector subspaces of the universal space $\mathcal{C}(2^\omega)$, and $\mathcal{SE}(\mathcal{C}(2^\omega))$ is a Borel subset of the set $\mathcal{F}(\mathcal{C}(2^\omega))$ equipped with the Effros-Borel structure, and thus it is a standard Borel space –in which the Lusin-Suslin theory of analytic sets applies.

We denote by $\underline{x} = (x_i)_{i\in\omega}$ sequences in $\mathcal{C}(2^\omega)^\omega = \mathcal{S}$.

Lemma 7 *The relations:*

(i) $\equiv$*: "is isometric to"*

(ii) $\overset{=}{\hookrightarrow}$*: "embeds isometrically into"*

(iii) Q*: "is a quotient of"*

are analytic subsets of $\mathcal{SE}(\mathcal{C}(2^\omega))^2$.

PROOF. We first prove (i) and (ii). If $\underline{x}$ and $\underline{y}$ belong to $\mathcal{C}(2^\omega)^\omega = \mathcal{S}$, we denote $\underline{x} \equiv \underline{y}$ if

$$\left\| \sum \lambda_i x_i \right\| = \left\| \sum \lambda_i y_i \right\|$$

for all $(\lambda_i) \in \mathbb{R}^{<\omega}$. The relation $\equiv$ is closed in $\mathcal{S}^2$. The set

$$\{(X, \underline{x}) \in \mathcal{SE}(\mathcal{C}(2^\omega)) \times \mathcal{S}; \overline{\operatorname{span}}\,(\underline{x}) = X\}$$

is Borel in $\mathcal{SE}(\mathcal{C}(2^\omega)) \times \mathcal{S}$ ([2, Lemma 2.6]). It follows that the set of all

$$(\underline{x}, \underline{y}, X, Y) \in \mathcal{S}^2 \times \mathcal{SE}(\mathcal{C}(2^\omega))^2$$

such that $\overline{\operatorname{span}}\,(\underline{x}) = X$, $\overline{\operatorname{span}}\,(\underline{y}) = Y$ and $\underline{x} \equiv \underline{y}$ is Borel, and (i) follows by projection on $\mathcal{SE}(\mathcal{C}(2^\omega))^2$. The proof of (ii) follows the same lines, except that the condition $\overline{\operatorname{span}}\,(\underline{y}) = Y$ is replaced by the condition $\underline{y} \subseteq Y$, which is also Borel.

For showing (iii), we observe that $Y \equiv Q(X)$ for some quotient map Q if and only if there exist $\underline{y}$, $\underline{x}$, and $\underline{x'}$ such that $\overline{\operatorname{span}}\,(\underline{y}) = Y$, $\overline{\operatorname{span}}\,(\underline{x}) = X$, $\underline{x'} \subseteq X$, and moreover for every $(\lambda_i) \in \mathbb{Q}^{<\omega}$

$$\left\| \sum \lambda_i y_i \right\| = \inf_{(\alpha_j)\in\mathbb{Q}^{<\omega}} \left\| \sum \lambda_i x_i + \sum \alpha_j x'_j \right\|,$$

and this last equation defines a Borel subset of $\mathcal{S}^3$. Then (iii) follows by projection. ■

We call J an *isometric linear embedding* of $E(\omega^{<\omega})$ into $\mathcal{C}(2^\omega)$.

Lemma 8 *The map* $\Psi = \mathcal{T} \to \mathcal{SE}(\mathcal{C}(2^\omega))$ *defined by* $\Psi(T) = J(E(T))$ *is Borel.*

PROOF.
The Effros-Borel structure is generated by the sets

$$B_V = \{F; F \cap V \neq \emptyset\},$$

where V is an open subset of $\mathcal{C}(2^\omega)$. It is easy to check that the set $\{T \in \mathcal{T}; \Psi(T) \cap V \neq \emptyset\}$ is open in $\mathcal{T}$: this follows from the definition of $E(T)$ and the fact that for all $s \in \omega^{<\omega}$, $\{T \in \mathcal{T}; s \in T\}$ is open in $\mathcal{T}$. This shows the lemma. ■

Our first main result provides a strong negative answer to J. Lindenstrauss' question.

Theorem 9 *Let X be a separable Banach space which contains and isometric copy of every strictly convex separable Banach space. Then X contains an isometric copy of $(\ell_1, \|\cdot\|_1)$.*

PROOF.
By Lemma 7 (ii), the set

$$\mathcal{A} = \{Y \in \mathcal{SE}(\mathcal{C}(2^\omega)); Y \overset{=}{\hookrightarrow} X\}$$

is analytic. By Lemma 8, the set $\Psi^{-1}(\mathcal{A})$ is analytic as well. Since X contains an isometric copy of every strictly convex space, Proposition 5 shows that $WF \subseteq \Psi^{-1}(\mathcal{A})$. But WF is *not* analytic (see [9]) and thus there exists $T \notin WF$ such that $\Psi(T) \in \mathcal{A}$. The corresponding space $E(T)$ embeds isometrically into X and contains an isometric copy of $(\ell_1, \|\cdot\|_1)$. ∎

We note that by ([7, Cor. 3.3]), a separable Banach space Y embeds isometrically (as a metric space) into a Banach space X if and only if there exists a linear isometric embedding from Y into X. Hence, "isometric copy" can be understood in the metric sense in Theorem 9.

Since the canonical norm $\|\cdot\|_1$ of ℓ_1 is not strictly convex, Theorem 9 implies that there is no universal strictly convex space.

It is natural to investigate alternative notions of universality, where quotient maps are involved. In this direction, we obtain the following satisfactory result.

Theorem 10 *Let X be a separable Banach space. If every strictly convex separable Banach space is isometric to a quotient (resp. a subspace of a quotient) of X, then every separable Banach space is isometric to a quotient (resp. a subspace of a quotient) of X.*

PROOF.
If $\mathcal{Z}(X)$ be the subset of $\mathcal{SE}(\mathcal{C}(2^\omega))$, consisting of spaces which are isometric to a quotient of X. By Lemma 7 the set $\mathcal{Z}(X)$ is analytic. Since $WF \subseteq \Psi^{-1}(\mathcal{Z}(X))$, it follows as above that there is $T \notin WF$ such that $E(T) \in \mathcal{Z}(X)$. By Remark 4, the space $(\ell_1, \|\cdot\|_1)$ is contractively complemented in $E(T)$ and thus $(\ell_1, \|\cdot\|_1) \in \mathcal{Z}(X)$. Since every separable Banach space is isometric to a quotient of $(\ell_1, \|\cdot\|_1)$, the result follows.

A simpler argument provides the "subspace of quotient" assertion. Again by Lemma 7, the set $\mathcal{SZ}(X)$ of subspaces of quotients of X is analytic and thus it contains some $E(T)$ with $T \notin WF$. By Proposition 3 and 1, the space $(\ell_1, \|\cdot\|_1)$ belongs to $\mathcal{SZ}(X)$ and thus every separable Banach space does. ∎

Remark 11 *Known results on the topological complexity of families of norms already provide (through Lemma 7 and statements similar to Lemma 8) negative answers to Lindenstrauss' question and other universality problems. For instance [3] implies that no strictly convex separable Banach space contains isometric copies of all renormings of $c_0(\omega) = c_0$ which are uniformly rotund in every direction.*

Similarly, it follows from [4] that we may replace "uniformly rotund in every direction" by "locally uniformly rotund" in the above, and (again by [4]) that no Gâteaux-smooth separable Banach space contains isometrically every Fréchet-smooth renorming of c_0, or of ℓ_2. Quotients maps can be involved as well, as in Theorem 10 and this provides more negative results.

However, Theorem 9 and its proof provide a somewhat better information, which is relevant to a weak notion of "closure" on the collection of separable Banach spaces. We will display this notion in our last section 5.

5. The Bossard Topology

The concept we now display in the isometric version of a notion which is implicitely contained in the recent article [1]. Our terminology follows [1] and refers to [2]. We recall that $\equiv$ denotes isometry on the set $\mathcal{SE}(\mathcal{C}(2^\omega))$. A subset H of $\mathcal{U}$ is $\equiv$-*saturated* if $X \in H$ and $X \equiv Y$ implies that $Y \in H$. It is *hereditary* if $X \in H$ and $Y \subseteq X$ implies that $Y \in H$. The word "Borel" refers of course to the Effros-Borel structure on $\mathcal{U}$.

Definition 12 *A subset F of $\mathcal{U}$ is Bossard closed if F is an intersection of $\equiv$-saturated hereditary Borel sets. The Bossard topology β is the topology on $\mathcal{U}$ whose closed sets are the Bossard closed sets.*

Note that Definition 12 is valid since the collection of Bossard-closed sets is stable under intersection and finite (actually, countable) union. Since this topology (denoted below by β) deals with $\equiv$-saturated sets, it can as well be defined on U/$\equiv$. However, it is handy to consider it on $\mathcal{U}$. Note that it is not Hausdorff, even when considered on U/$\equiv$.

Here is an important example of β-closed set.

Lemma 13 *Every analytic $\equiv$-saturated hereditary subset A of $\mathcal{U}$ is β-closed.*

PROOF.
Assume that $X \notin A$. Let

$$\tilde{X} = \{Y \in \mathcal{U}; X \overset{\equiv}{\hookrightarrow} Y\}.$$

By Lemma 7, $\tilde{X}$ is analytic and $A \cap \tilde{X} = \emptyset$. By Suslin's separation theorem, there exists a Borel set $B_0 \supseteq A$ such that $B_0 \cap \tilde{X} = \emptyset$. Let A_1 be the smallest hereditary $\equiv$-saturated set containing B_0.

By Lemma 7, A_1 is analytic and we still have $A_1 \cap \tilde{X} = \emptyset$. Hence there is $B_1 \supseteq A_1$ a Borel set with $B_1 \cap \tilde{X} = \emptyset$. Continuing in this way, we construct a sequence (B_n) of $\equiv$-saturated hereditary Borel sets. If

$$B = \cup_{n \geq 0} B_n$$

we have $A \subseteq B$, $X \notin B$ and B is Borel $\equiv$-saturated and hereditary. This shows that A is β-closed. ■

Remark 14

1. Lemma 13 shows that we may replace "Borel" by "analytic" in Definition 12.
2. If we replace the relation $\equiv$,"being isometric", by the isomorphism relation $\simeq$, we can define a topology β' as in Definition 12. Except for the notation, this is done in [1]: in their terminology, $\mathcal{C}$ is Bossard X-generic if and only if X belongs to the β'-closure of $\mathcal{C}$.
3. For any $X \in \mathcal{U}$, the set
$$\mathcal{SE}(X) = \{Y \in \mathcal{U}; Y \overset{\equiv}{\hookrightarrow} X\}$$
is analytic by Lemma 7, hereditary and $\equiv$-saturated, hence it is β-closed. Therefore, every class $\mathcal{C}$ which admits a universal space is β-closed. However, the converse is false. For instance, the set $\mathcal{SE}(\ell_1) \cap \mathcal{SE}(\ell_2)$ is β-closed but contains no universal space.

We now revisit Lemma 13 and show:

Proposition 15 *Let $X \in \mathcal{U}$, and $\mathcal{C}$ a subset of $\mathcal{U}$. If there exists a Borel map $\Psi = \mathcal{T} \to \mathcal{U}$ such that*

1. *If $T \in WF$, then $\Psi(T) \in \mathcal{C}$*

2. *If $T \notin WF$, then $X \overset{=}{\hookrightarrow} \psi(T)$. Then $X \in \overline{\mathcal{C}}^{\beta}$*

PROOF. Let $B \supseteq \mathcal{C}$ be a Borel hereditary $\equiv$-saturated set. Since $\Psi^{-1}(B) \supseteq WF$ and is Borel, there exists $T_0 \notin WF$ such that $\Psi(T_0) \in B$. Therefore $X \overset{=}{\hookrightarrow} \Psi(T_0)$ and thus $X \in B$. This concludes the proof. ■

Examples: Sections III and IV above show that $(\ell_1, \|\cdot\|_1)$ belongs to the β-closure of the set $\mathcal{SC}$ of strictly convex spaces. Actually, if a space $(X, |||\cdot|||)$ with a basis and the corresponding sequence of projections (π_n) satisfies conditions (i) and (ii) of Proposition 1, then $(X, |||\cdot|||)$ belongs to the β-closure of $\mathcal{SC}$.

Along similar lines, it follows from Proposition 15 and [3] that the non-strictly convex space $(c_0, |||\cdot|||)$ (in the notation of [3]) belongs to the β-closure of the set of URED renormings of c_0.

Also, it is shown in [10] through James tree spaces (see [5]) that the quasi-reflexive James space J [8] belongs to the β-closure of the set of reflexive spaces with a uniformly Kadec-Klee norm.

This work leaves open a fair number of natural questions. Let us formulate some of them.

1. Problem. Which classical non-smooth spaces belong to the β-closure of the set of Gâteaux-smooth spaces? Of the set of Fréchet-smooth spaces?

   The next problem is really to know whether the optimal version of Theorem 9 holds true. Along these lines, note that an open problem which goes back to S. Rolewicz ([12], Pb. IX. 9. 4) asks whether a separable Banach space which contains an isometric copy of every finite dimensional Banach space, contains also isometric copies of every separable Banach space.

2. Problem. Let X be a separable Banach space which contains isometric copies of every strictly convex separable Banach space. Does X contain an isometric copy of every separable Banach space?

   In view of Theorem 14, our last problem is a strong version of the previous one.

3. Problem. Does the space $(\mathcal{C}(2^{\omega}), \|\cdot\|_{\infty})$ belong to the β-closure of the set of strictly convex separable Banach spaces?

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