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Smoothness in Banach spaces. Selected problems

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Abstract. This is a short survey on some recent as well as classical results and open problems in smoothness and renormings of Banach spaces. Applications in general topology and nonlinear analysis are considered. A few new results and new proofs are included. An effort has been made that a young researcher may enjoy going through it without any special pre-requisites and get a feeling about this area of Banach space theory. Many open problems of different level of difficulty are discussed. For the reader convenience, an ample list of literature quotations is included.

Diferenciabilidad en espacios de Banach. Problemas escogidos

Resumen. Este artículo dibuja un breve panorama sobre algunos problemas recientes y otros clásicos en la teoría de diferenciabilidad y renormamiento en espacios de Banach. Se consideran aplicaciones a la topología general y al análisis no lineal. Se incluyen algunos resultados nuevos así como nuevas pruebas de resultados anteriores. Se ha intentado realizar un esfuerzo para que un investigador joven pueda apreciar el artículo sin prerrequisitos especiales y perciba la situación de esta área de la teoría de los espacios de Banach. Se discuten muchos problemas abiertos de variada dificultad. Para conveniencia del lector, se incluye una amplia lista de referencias.

1. Introduction

We will use the standard notation in the Banach space theory. In particular, S_X or B_X will denote the unit sphere or the unit ball of a Banach space X respectively. Often, we will say a norm and mean an equivalent norm and say space and mean a Banach space. If we say C^1 function we mean Fréchet C^1 smooth or differentiable function.

In geometry of Banach spaces as well as in smooth approximation and optimization, one of the most important questions is how big is the supply of differentiable functions/norms on a given Banach space.

A basic notion in this area is the notion of a smooth bump function or smooth bump on the space. This is a real valued function with bounded nonempty support that is smooth on the space.

Smooth bump functions are most often constructed by using smooth norms on spaces: If $\|\cdot\|$ is a smooth norm on X (away from the origin), and if τ is a smooth real valued function on the real line such that $\tau(t) = 1$ for $|t| < \delta$ for some $0 < \delta < 1$ and $\tau(t) = 0$ for $|t| > 1$, then $\tau(\|\cdot\|)$ is a smooth bump on X.

A special care is needed around the point 0 (for example a flattening) since if $\|\cdot\|^2$ is twice Fréchet differentiable at the origin, then the second derivative at the origin produces an inner product on X that

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shows that X is then isomorphic to a Hilbert space. On the other hand, if $\|\cdot\|$ is a Hilbertian norm, then $\|\cdot\|^2$ is a polynomial (quadratic form) and thus C^∞ smooth on the whole space.

Therefore a Banach space X admits an equivalent norm $\|\cdot\|$ such that $\|\cdot\|^2$ is twice Fréchet differentiable on X if and only if X is isomorphic to a Hilbert space.

However, there are many other spaces that can be renormed by equivalent norms that are C^1 or even C^∞ away from the origin. We call such norms C^1 or C^∞ smooth (or differentiable) norms.

For example, we will see below that any reflexive space admits a Fréchet smooth norm and $C(K)$ admits a C^∞ smooth norm if K is a countable compact.

On the other hand, Kurzweil showed [79] that $C[0,1]$ admits no Fréchet smooth bump (cf. e.g. [17, Ch.I]).

Haydon constructed [69] a nonseparable Banach space X that admits a Lipschitz C^∞ smooth bump and yet X does not admit any Gâteaux differentiable norm (cf. e.g.[17, Ch. VII]).

There is a deep connection of smooth norms and bumps with the differentiability of convex and Lipschitz functions on Banach spaces (see Section 2, 3 below), with the shrinking properties of bases (see Section 5 below), with the weak compactness in spaces (see Sections 3 below), type and cotype on spaces (see Section 6 below), with topology of Corson compacta (see Section 3 below), with approximation and optimization (Sections 7), containment of ℓ_1 (Section 4 below) etc.

Generally speaking, the notion of smoothness is closely connected to the weak compactness, while the dual notion of rotundity is closely connected to the covering properties of Banach spaces like the Nagata-Smirnov theorems etc (see e.g. [88]).

Many basic problems in this area remain open. For example, it is not known if a separable space that admits a twice Fréchet differentiable bump admits a Lipschitz C^2 smooth bump or if the existence of a C^2 smooth bump on a separable space X implies the existence of C^2 smooth norm on X. Neither is known if $C(K)$ admits a C^∞ smooth bump if K is scattered. It is not known if, in full generality, any continuous function on X can be approximated by C^k smooth functions if X admits a C^k smooth norm ($k \geq 1$).

This survey article is intended to show that this area is a fertile field in the garden of the Banach space theory, with many interesting open problems and applications, ranging from nonlinear analysis, topology, or convexity.

Only a very small part of the renorming theory will be discussed in the present survey. In fact, besides a short review on the basic classical results needed in this survey, we will focus only on the relationship of smoothness to the weak compactness, to topology of special compacta and to nonlinear analysis.

In particular, we will not discuss the recent development in the renorming theory that deals with the relation of rotund norms with covering properties of spaces. Neither will we discuss nonlinear surjective smooth operators between Banach spaces. Nor will we study the rôle of smoothness in questions on uniform homeomorphisms between Banach spaces. All of these areas are recently a subject of an intensive research.

We refer to [17] and [26] for an account of the renorming theory as it was 10 years ago. We also refer to [50] and [110] for a survey on a part of this area as it was 5 years ago. The text [88] contains the study of the relationship of the renorming theory with covering properties of Banach spaces. The book [65], that is to appear in 2006, deals with the connection of the smoothness in Banach spaces with biorthogonal systems.

In our opinion, there is a need for a comprehensive, reader friendly, not encyclopedic book that would describe the present state of affair in the renorming theory overall.

2. Review on some classical basic results

Around the year of 1940, Šmulyan proved his fundamental dual characterization of smoothness of norms (cf.e.g. [17, Ch.I]).

A version of it for the case of Fréchet smoothness reads that the norm $\|\cdot\|$ on X is Fréchet differentiable at $x \in S_X$ if and only if $\{f_n\}$ is norm convergent whenever $f_n \in S_{X^*}$ are such that $\lim f_n(x) = 1$.

For the case of uniform Gâteaux differentiability, it reads that the norm $\|\cdot\|$ on X is uniformly Gâteaux differentiable if and only if $f_n - g_n \to 0$ in the weak star topology of X^* whenever $f_n, g_n \in S_{X^*}$ are such that $\|f_n + g_n\| \to 2$.

Recall that the norm $\|\cdot\|$ is Gâteaux respectively Fréchet respectively uniformly Gâteaux (UG) respectively uniformly Fréchet (UF) differentiable if

$\lim_{t\to 0^+} \frac{1}{t}(\|x+th\| + \|x-th\| - 2) = 0$ for every $x, h \in S_X$

respectively

$\lim_{t\to 0^+} \sup_{x\in S_X} \frac{1}{t}(\|x+th\| + \|x-th\| - 2) = 0$ for every $h \in S_X$.

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At the same time Šmulyan proved that every separable Banach space admits an equivalent UG norm (cf. e.g. [17, Ch. II]).

His norm was the predual norm to the norm defined on X^* by $\|f\|^2 = \|f\|_\infty^2 + \sum \frac{1}{2^i} f^2(x_i)$, where $\{x_i\}$ is dense in S_X and $\|\cdot\|_\infty$ is the canonical supremum norm of X^*.

The proof of his result is now standard:

If $2\|f_n\|^2 + 2\|g_n\|^2 - \|f_n + g_n\|^2 \to 0$, then by convexity, similar relation holds for each coordinate and then $(f_n - g_n)x_i \to 0$ for each i from the elementary property of real numbers. Therefore the predual norm is UG by the Šmulyan duality lemma.

Kadets later gave a beautiful, seminal proof to his result that X admits a Fréchet smooth norm if X^* is separable (cf. e.g.[17, Ch. II]).

Here is a sketch of the proof of Kadets' result: The required norm is a predual to the norm defined on X^* by

$$\|f\|^2 = \|f\|_\infty^2 + \sum_n \frac{1}{2^n} f^2(x_n) + \sum_n \frac{1}{2^n} \mathrm{dist}^2(f, L_n))^2,$$

where L_n is the line through $f_n \in S_{X^*}$, $\{f_n\}$ is norm dense in S_{X^*}, and $\{x_n\}$ is dense in S_X.

The proof of Kadets result goes as follows:

We use Šmulyan's duality lemma: If, in the new norm, given $x \in S_X$, $f \in S_{X^*}$ such that $f(x) = 1$ and $f_n \in S_{X^*}$ such that $f_n(x) \to 1$, then we have $\lim(2\|f\|^2 + 2\|f\|^2 - \|f_n + f\|^2) = 0$. Then the same holds by convexity for each coordinate and thus $\lim_n (f_n - f)(x_i)) =$ for each i. Similarly, $\lim_n \mathrm{dist}(f_n, L_i) = \mathrm{dist}(f, L_i)$ for each i and from that we get by using the compactness in finite dimensional spaces that $\lim \|f_n - f\| = 0$. Then by the Šmulyan's duality lemma, we get the Fréchet smoothness of the predual norm at x.

We showed in fact that the dual norm was locally uniformly rotund (LUR), i.e.

$\lim \|f_n - f\| = 0$ whenever $f_n, f \in S_{X^*}$ are such that $\lim \|f_n + f\| = 2$.

Day showed (cf. e.g. [17, Ch. II]) that the norm $\|\cdot\|$ defined on $c_0(\Gamma)$ by

$$\|x\|^2 = \sup\Big\{\sum_{i=1}^n \frac{1}{2^i} x^2(j_i),\ \{j_i\}_1^n \text{ a sequence of distinct indexes}, n \in I\!N\Big\}$$

is a strictly convex norm (in fact a locally uniformly rotund norm) on $c_0(\Gamma)$. This is due to the fact that the supremum is uniquely attained at the decreasing order of $\{x(j_i)\}$. and hence a Hilbertian behaviour of the norm can be used.

This was an idea that produced and still is producing many important results (Haydon calls such conditions rigidity conditions). It is well seen in spaces of continuous functions on tree spaces. We refer to [69], [102] and [103] for a nice recent treatment of these topics.

Troyanski (cf.e.g.[17, Ch. VII]) used a modification of the Day norm where he attached (added) to finite dimensional blocks in the Day norm the distances to the corresponding finite dimensional spaces. Then from the rigidity condition in the Day norm, we get that if $\{x_n\} \subset S_X$ satisfies $2\|x_n\|^2 + 2\|x\|^2 - \|x + x_n\|^2 \to 0$, then $\{x_n\}$ is relatively norm compact by the norm compactness of balls in finite dimensional spaces. This was a pioneering idea that produced a breakthrough in this area and still produces new results. Troyanski used it to construct an LUR norm on WCG spaces (cf. e.g. [17, Ch. VII]).

A Banach space X is weakly compactly generated (WCG) if there is a weakly compact set $K \subset X$ such that the norm closed linear hull of K is X.

Troyanski's result had among other consequences a profound impact on the study of Radon-Nikodym properties and their topological counterparts.

The Day norm was used by Amir and Lindenstrauss who showed in [1] that every WCG space admits a Gâteaux differentiable norm. In their proof they used their technique of projectional resolutions of identity on WCG spaces (cf. e.g. [17, Ch. VI]).

A projectional resolution of the identity (PRI) on a space X is a transfinite sequence of norm one projections P_α, $\alpha \leq \mu$ where μ is the smallest ordinal of cardinality dens$X$ such that dens$P_\alpha X <$ dens$X$ if $\alpha < \mu$, $P_\alpha P_\beta = P_\beta P_\alpha = P_\alpha$ if $\alpha < \beta$, the map $\alpha \to P_\alpha(x)$ is continuous from the ordinals into the norm topology of X for all $x \in X$ and the last projection is the identity.

Later on Mercourakis proved that any Vašák space admits a Gâteaux differentiable norm (cf e.g. [17, Ch. VII]).

A Banach space X is a Vašák space if there is a sequence $\{B_n\}$ of weak star compact sets in X^{**} such that given $x \in X$ and $u \in (X^{**} \setminus X)$, there is n such that $x \in B_n$ and $u \notin B_n$.

Note that if X is WCG, then for such sets we may take a reindexed family of $nB_X + \frac{1}{m}K$, $n, m \in I\!N$, where K is a weakly compact set that generates X.

As an application of smoothness in Banach spaces, let us first mention the following Šmulyan's theorem (cf.e.g. [17, Ch. II]).

Theorem 1. *If X is a Banach space and the dual canonical norm of X^* is Fréchet smooth, then X is reflexive.*

PROOF. Let $f \in S_{X^*}$ be given. Let $x_n \in S_X$ be such that $\lim f(x_n) = 1$. By Šmulyan's dual characterization of Fréchet smoothness of norms (cf. e.g. [17, Ch. I]), we have that x_n converges to some $x_0 \in S_X$. Then $f(x_0) = 1$ and thus every element of X^* attains it norm. By James' theorem, X is reflexive (cf. e.g. [17, Ch. I] or [33, Ch. 3]).

∎

If φ is a Lipschitz Gâteaux differentiable bump on a Banach space X and $\psi(x) = \varphi^{-2}(x)$ or $\psi(x) = +\infty$ if $\varphi(x) = 0$ and $f \in X^*$, then by using the smooth variational principle (cf. e.g. [17, Ch. I]) we get that there is $x \in X$ for which $\psi(x) \neq +\infty$ and such that the graph of $\psi - f$ is supported from below at x by the graph of a Lipschitz Gâteaux differentiable bump the supnorm on X of which together with the supnorm on X of its derivative is arbitrarily small. This gives that $\psi'(x) - f$ is arbitrarily small in norm and thus we get that $\overline{\text{span}}\,\varphi'(X) = X^*$, where $\varphi'(X) = \{\varphi'(x)\,; x \in X\}$.

Thus then dens$X^* \leq$ card$X$ and hence, for instance, $\ell_\infty(I\!N)$ does not admit any Lipschitz Gâteaux differentiable bump (cf. e.g. [33, p. 424]).

Alternatively, we can use [74], where a notion of the so called strong roughness was studied.

If the derivative φ' is moreover continuous, then dens$X^* \leq$ dens$X$.

Thus X is an Asplund space if X admits a Lipschitz C^1 smooth bump.

A Banach space X is Asplund if Y^* is separable for every separable subspace $Y \subset X$.

Note that the result does not hold for non Lipschitz situation:

Indeed, it is not true that the function $\|\cdot\|^2$ can be supported below by a C^2 smooth function on c_0 if $\|\cdot\|$ is a LUR norm on c_0. This is because c_0 would then be superreflexive (see e.g.[17, Ch. V]).

The space is superreflexive if it admits a norm that is uniformly convex, i.e. $\lim \|x_n - y_n\| = 0$ whenever $x_n, y_n \in S_X$ are such that $\lim \|x_n + y_n\| = 2$.

The proof of the smooth variational principle uses the fact that the set of differentiable functions that support below a given function out of a small neighborhood of a point, at which they are above the graph of the given function is an open set in the space of differentiable functions in the sup norm for them as well as for their derivative. Then the Baire category theorem gives the result (cf. e.g.[17, Ch. I]). This principle was a new version with a new proof of the Borwein-Preiss smooth variational principle where norms were used instead of bumps (cf. e.g. [17, Ch. I]).

If X admits a Lipschitz Gâteaux differentiable bump, then B_{X^*} in its weak star topology is fragmentable and thus X is a weak Asplund space ([44], [26, Ch. V]). Therefore $\ell_1(\Gamma)$ does not admit any Lipschitz Gâteaux differentiable bump if Γ is uncountable as the canonical norm in this space is nowhere Gâteaux differentiable (cf. e.g. [17, Ch. I]).

Recall that a topological space T is fragmentable if there is a metric ρ on T such that for every nonempty set $M \subset T$ and for every $\varepsilon > 0$ there is an open set Ω in T such that $\Omega \cap M \neq \emptyset$ and the ρ diameter of $\Omega \cap M$ is less than ε.

A Banach space X is called a weak Asplund space if every continuous convex function on X is Gâteaux differentiable on a G_δ dense set in X.

Before the smooth variational principle was proved, Kurzweil showed in [79] that X does not admit any Fréchet differentiable bump if it admits a so called rough norm i.e. a norm such that for some $\varepsilon > 0$,

$$\limsup_{h \to 0} \frac{1}{\|h\|}(\|x+h\| + \|x-h\| - 2) \geq \varepsilon$$

for every $x \in S_X$. By using the canonical norm of $C[0,1]$ as a rough norm he obtained that $C[0,1]$ does not admit any Fréchet differentiable bump.

A rough norm was then constructed by Leach and Whitfield on any separable space with nonseparable dual (cf.e.g. [17, Ch. III]).

Hence they obtained by the Kurzweil method that no separable space with nonseparable dual can admit a Fréchet smooth bump (note that the Lipschitz property was not used in the proof).

Kurzweil's method of proof consisted of comparing the growth of the smooth bump with the one of a rough norm in the following way: Assume that φ is a differentiable "upside down" bump that is equal to 0 at zero and equals to 4 outside the unit ball of the rough norm. Put $x_1 = 0$ and construct inductively step points x_n such that φ is below the graph of the rough norm. Take such steps of almost maximal possible distance from the preceding point, but smaller then 1. Then, either the norm of the step points prevail 2 for some first time which leads to contradiction as the bump φ is equal to 4 at such point and the rough norm is less or equal than 3 so the bump is above the norm. Or, if the step points remain bounded, then by the roughness property they are a Cauchy sequence converging to some point x_0 where the possible step point can be used to see that the length of steps cannot tend to zero. This is a contradiction with the convergence of the points. So, X cannot admit a Fréchet smooth bump.

This method has since been a source of many important results in this area, including the results on tree spaces, spaces that do not contain copies of c_0 etc.

This together with Kadets' result gave that a separable X admits a Fréchet smooth bump if and only if X admits a Fréchet smooth norm if and only if X^* is separable.

The following theorem came from an effort of many mathematicians (cf. e.g. [17], [26]), the final version is due to M. Valdivia (cf. e.g. [26, p. 154]).

Theorem 2. *A weakly Lindelöf determined Banach space X admits a Fréchet differentiable bump if and only if X admits a Fréchet smooth norm if and only if X is an Asplund space.*

A Banach space X is a weakly Lindelöf determined (WLD) space if B_{X^*} is Corson in its weak star topology.

A compact space K is called a Corson compact, if K is homeomorphic to a set S in $[-1,1]^\Gamma$ for some Γ in its pointwise topology, where S is formed by countably supported functions on Γ.

For example, any metrizable compact is a Corson compact or any weakly compact set in a Banach space is a Corson compact or the dual ball for a Vašák space in its weak star topology is a Corson compact (cf. e.g. [17, Ch. VI], [33, Ch. 12]).

There are WLD spaces that do not admit any Lipschitz Gâteaux differentiable bump. This is seen by using a result of Argyros and Mercourakis, who constructed in [2] a WLD space that is not weak Asplund.

The space $X = C[0, \omega_1]$ of all continuous functions on the ordinal space $[0, \omega_1]$, where ω_1 is the first uncountable ordinal, admits a C^∞ smooth norm ([69]) and X is not WLD (as the ordinal space is not angelic, i.e. cluster points of subsets are not in general reachable by limits of sequences in subsets, the property shared by Corson' compacts (cf. e.g. [33, p. 427])).

Using Kurzweil's idea mentioned above and a special "kind of roughness" of the canonical supremum norm on $C[0, \omega_1]$, Haydon showed in [69] (cf. e.g. [17, Ch. VII]) the following result.

Theorem 3. ([69]) *Let $C_0[0, \omega_1]$ be a subspace of $C[0, \omega_1]$ formed by functions that vanish at ω_1. Then $C_0[0, \omega_1]$ does not admit any Gâteaux differentiable norm $\|\cdot\|$ that would have the following property:*

$$\|x + \lambda\chi_{(\beta,\gamma]}\| = \|x\| \quad \textit{whenever} \quad \textit{supp}\, x \subset [0, \beta], \beta < \gamma, \quad \textit{and} \quad 0 \le \lambda \le x(\beta)$$

where supp denotes the support and χ denotes the characteristic function.

Theorem 3 was crucial for Haydon's result in[69] that there is a compact space K (an Alexandroff compactification of a tree) that admits no Gâteaux differentiable norm.

On the other hand, for any such tree space K, $C(K)$ admits a Lipschitz C^∞ smooth bump ([69], cf. e.g. [17, Ch. VII]).

The situation is different for the case of first degree uniform differentiability. This is because convex hulls produce uniformly smooth norms from uniformly smooth bumps ([39], cf. e.g. [17, Ch. V]).

Thus we get ([39], [104], [27], [17, Ch. IV]).

Theorem 4. *(i) A Banach space X admits a bump with uniformly continuous derivative if and only if X admits an UF norm if and only if X is super-reflexive*

(ii) X admits a uniformly Gâteaux differentiable bump if and only if X admits a UG norm if and only if B_{X^*} is a uniform Eberlein compact in its weak star topology. Thus, in particular, any space with UG norm is a subspace of a WCG space

A Banach space X is super-reflexive if it admits a uniformly rotund norm, i.e. such a norm $\|\cdot\|$ that $\|x_n - y_n\| \to 0$ whenever $x_n, y_n \in S_X$ are such that $\|x_n + y_n\| \to 2$.

Recall that a compact is (uniform) Eberlein if it is homeomorphic to a subset of $c_0(\Gamma)$ for some Γ (a Hilbert space) in its weak topology.

Recall also that $C(K)$ is WCG if K is Eberlein ([1], cf. e.g. [17, Ch. VI]).

To prove (i) we use the fact that if X admits a uniformly Fréchet differentiable bump, then X admits such a norm. Then, we use that X is superreflexive if and only if X admits a UF norm (cf. e.g. [17, Ch IV]).

To prove (ii), if B_{X^*} is uniform Eberlein, then Benyamini and Starbird proved in [8] that there is a bounded linear operator from a Hilbert space onto a dense set in $C(B_{X^*})$. Then it is standard that $C(B_{X^*})$ admits a UG norm (cf. e.g. [17, Ch. II]).

If X admits a UG bump, it admits a UG norm ([104]). Then we will show that X is Vašák. Then by using a projectional resolution of the identity constructed by Vašák (cf. e.g. [17, Ch. VI]) and following an adapted Troyanski's technique [106], we will use UG again to show that B_{X^*} is uniform Eberlein ([31], [27]).

In order to show that a space with UG norm is a Vašák space, we proceed as follows.

By using the Šmulyan duality lemma (cf. e.g. [17, Ch. II]), we have that then $f_n - g_n \to 0$ in the weak star topology whenever $f_n, g_n \in S_{X^*}$ are such that $\|f_n + g_n\| \to 2$.

For $\varepsilon > 0$ and $n \in I\!N$, put

$$B_n^\varepsilon = \{x \in B_X\,; |(f-g)(x)| < \varepsilon \quad \text{if} \quad f, g \in B_{X^*} \quad \text{satisfy} \quad \|f+g\| > 2 - \frac{1}{n}\}$$

We have for every $\varepsilon > 0$ that $\cup_n B_n^\varepsilon = B_X$.

We claim that for each $\varepsilon > 0$ and each n,

$$\overline{B_n^\varepsilon}^{w^*} \subset X + 4\varepsilon B_{X^{**}}$$

Indeed, if not, take $x_0 \in \overline{B_n^\varepsilon}^{w^*} \subset X^{**}$ with the distance greater then 2ε from X. Then take $F \in S_{X^{***}}$ such that F equals to 0 on X and $F(x_0) = 2\varepsilon$.

Let $f_\alpha \in S_{X^*}$ be such that $f_\alpha \to F$ in the weak star topology of X^{***}.

Then $\|f_\alpha + f_\beta\| \to 2$ and thus $|(f_\alpha - f_\beta)(x)| < \varepsilon$ for all $x \in B_n^\varepsilon$ for large α, β. As f_α weak star converge to F, we have $|(f_\alpha - F)(x)| \leq \varepsilon$ for all $x \in B_n^\varepsilon$ for large α. Since $F = 0$ on X in particular on B_n^ε, we have $|f_\alpha(x)| \leq \varepsilon$ for every $x \in B_n^\varepsilon$ and thus $|f_\alpha(x_0)| \leq \varepsilon$ for large α from the continuity of f_α in the weak star topology of X^{**}. Since $f_\alpha \to F$ in the weak star topology of X^{***} we get $|F(x_0)| \leq \varepsilon$ which is a contradiction.

Note that a space with UG smooth norm need not in general be a WCG space: For example, let W be a non WCG subspace of WCG space $L_1(\mu)$ constructed by Rosenthal ([98]). As $L_2(\mu)$ is dense in $L_1(\mu)$, we have that $L_1(\mu)$ and thus also W admits a UG norm ([17, Ch. II]).

As an application of these results let us present a proof of the following known result as it is in [27].

Corollary 1. ([7]) *Any continuous image of a uniform Eberlein compact is a uniform Eberlein compact.*

Indeed, if φ is a continuous map from K onto $\varphi(K)$, then $C(\varphi(K))$ is a subspace of a space $C(K)$ that admits a UG norm. So $\varphi(K)$ is uniform Eberlein by Theorem 4.

Let us mention that a new proof of the known result ([7]) that a continuous image of a Eberlein compact is Eberlein can also be given along these lines ([30]).

We also present a new proof of the following result.

Corollary 2. ([81], cf. e.g. [17, Ch.IV] [110]) *There is a reflexive Banach space X that does not admit any UG norm.*

Indeed, if K is Eberlein that is not uniform Eberlein (cf. e.g. [17, Ch. IV], [33, p. 419]), then $C(K)$ is WCG ([1]) and does not admit a uniformly Gâteaux differentiable norm (Theorem 4). By a standard method (cf. e.g. [17, Ch. II]) neither does a reflexive space that factorizes to $C(K)$ by Davis, Johnson et al factorization ([14]). This factorization result says that for every WCG space X there is a reflexive space Z and a bounded linear operator from Z onto a dense set in X.

Recall that X is called weakly uniformly rotund (WUR) if the dual norm if UG.

Hájek proved in [55] that any WUR space is Asplund (for another proof of it see e.g. [29]).

Now we can state the following stronger result.

Corollary 3. ([27]) *If X is WUR, then X admits a Fréchet smooth norm, in particular X is an Asplund space.*

Indeed, if X is WUR, then by Šmulyan's lemma, X^* is UG and we showed above that X^* is Vašák. Therefore it is enough to use the following Fabian's result (see e.g. [17, Ch. VII])

Theorem 5. *If X^* is Vašák, then X admits an equivalent Fréchet differentiable norm.*

PROOF. The proof of this Fabian's result uses Godefroy's transfer technique the main idea of which is the following ([51]).

If the dual norm of X^* is LUR and T is a weak star-weak star continuous operator of X^* onto a norm dense set in Y^* then $(T(B_{X^*}) + \varepsilon B_{Y^*})$ produces a norm that is LUR "up to ε"). Then it suffices to take a countable product of these for $\varepsilon = \frac{1}{n}$. The method uses heavily the weak star compactness of dual balls and produced a breakthrough in this area of Banach space theory. For a Fabian's proof of it see e.g. [17, Ch. VII]).

In the introduction we mentioned flattening to reach the smoothness of bumps. In this direction, we can state the following.

Theorem 6. ([39]) *If the norm of a Banach space is LUR and Fréchet differentiable with the locally uniformly continuous derivative, then X is superreflexive.*

This is because such norm easily produces a bump with uniformly continuous derivative and we can use Theorem 4.

This in particular shows that the Asplund averaging procedure (cf. e.g. [17, Ch. II]) does not work for locally uniformly continuous derivatives.

Motivated by this was the following procedure:

If the space does not contain copies of c_0 (this means does not contain isomorphic copies of c_0), then there is available a uniformization argument for local uniform differentiability that is based on the Bessaga-Pełczynski theorem on characterization of such spaces (cf. e.g. [33]). This is a concept of a compact variational principle that is the following (cf. e.g. [17, Ch. V]).

If X does not contain a copy of c_0 and if f is a continuous symmetric real function on X, $f(0) \le 0$ and $\inf_{S_X} f > 0$, then for every $\delta > 0$ there is a finite set K_δ such that

$$\inf\{f_{K_\delta}(x) - f_{K_\delta}(0); \|x\| \ge \delta\} > 0$$

where $f_{K_\delta}(x) = \sup\{f(x+k); k \in K_\delta\}$.

The proof of this result goes in the direction of Kurzweil's proof mentioned above, only instead of a point we get a compact set by considering partial sums $\sum \varepsilon_i x_i$ and Bessaga-Pełczynski result (cf. e.g. [33]) that in the space that does not contain a copy of c_0, such sums form a norm relatively compact set if they all stay bounded (for details see [17, Ch. V]).

We get the following result.

Theorem 7. ([39], cf.e.g.[17, Ch. V]) *If X does not contain copies of c_0 and admits a bump the k - th derivative of which is locally uniformly continuous, then X admits a bump with uniformly continuous k -th derivative.*

Note that this means that Kadets' result mentioned above in case of, say, separable reflexive not superreflexive space gives a Fréchet differentiable norm whose derivative is (automatically) continuous but cannot be locally uniformly continuous.

Based on Kurzweil's method is also the following Nemirovskii-Meškov- like result.

Theorem 8. (cf.e.g.[17, Ch. V], [109]) *If both X and X^* admit a bump with locally Lipschitz derivative, then X is isomorphic to a Hilbert space. If both X and X^* admit continuous twice Gâteaux differentiable bumps, then X is isomorphic to a Hilbert space.*

An elementary proof of this uses, besides the compact variational principle, integral convolutions for lines to show the following fact.

Theorem 9. ([39]) *If a separable space X admits a bump whose derivative is locally Lipschitz, then X admits a bump that is twice Gâteaux differentiable.*

In this direction let us mention Troyanski's result that $\ell_3(\Gamma)$ admits a bump that is 3 times Gâteaux differentiable if and only if Γ is countable. It seems to be unknown if $\ell_3(I\!N)$ admits four times Gâteaux differentiable norm. Further results of smoothness of bumps on ℓ_p spaces are in [86].

Let us also mention Vanderwerff's result that X is an Asplund space if X admits a continuous twice Gâteaux differentiable bump [109].

We will say that a function f on X locally depends on a finite number of coordinates if for every $x \in X$ there is a neighborhood U of x, elements $f_1, \cdots f_n \in X^*$ and a function φ on $I\!R^n$ such that $f(z) = \varphi(f_1(z), f_2(z), \cdots f_n(z))$ for every $z \in U$.

Bumps that locally depend on finitely many coordinates are usually as good as smooth bumps (and sometimes even better). Moreover, they are easier to construct.

The article [68] in this volume is devoted to bumps that locally depend on finitely many coordinates.

The C^∞ norm on $c_0(\Gamma)$ that locally depend on finitely many coordinates is constructed as follows:

Let τ be a positive C^∞ even convex function $\tau(t)$ on reals that is zero for all $|t| \leq \frac{1}{2}$ and tends to ∞ at ∞. Then use the function $\varphi(x) = \sum \tau(x_\alpha)$ and consider the Minkowski functional of $\{x; \varphi(x) \leq 1\}$ (see e.g.[17, Ch. V]).

Preiss showed in [38] that the function $\varphi(x) = \sum x_n^{2n}$ gives rise (via the Minkowski functional of the set $\{x; \varphi(x) \leq 1\}$ to a real analytic norm on $c_0(I\!N)$.

A function is real analytic if it can be locally represented by Taylor's series.

The final solution to some problems in this direction is the following result.

Theorem 10. ([60]) *$C(K)$ admits a real analytic norm if and only if K is countable.*

This covers the earlier result of Haydon that $C(K)$ admits a C^∞ smooth norm if K is countable and Pelczyński result that $c_0(\Gamma)$ does not admit any real analytic norm if Γ is uncountable [93]. Since $c_0(\Gamma)$ for any Γ admits a C^∞ norm this provides for uncountable Γ an example of a Banach space that admits an C^∞ norm but admits no real analytic norm. To get a separable example was much more difficult.

Theorem 11. ([67]) *There is a separable Banach space that admits C^∞ smooth norm but admits no real analytic norm.*

Recall that X is said to have a countable James boundary if there is a countable set $S \subset S_{X^*}$ such that for any $x \in S_X$ there is $s \in S$ such that $s(x) = 1$.

Note that then by a nice argument due to Fonf, X admits a norm that locally depends on finitely many coordinates. This argument consists of the following: If $\{f_n\}$ is a James boundary for X, then the norm $\|x\| = \sup\{(1 + \frac{1}{n}) f_n(x)\}$ locally depends on a finite number of coordinates. We have

Theorem 12. (([21],[22]) *If X admits a countable James boundary, then X admits a real analytic norm.*

Indeed, if $\{f_n\}$ is a countable James boundary for X, then the function $\varphi(x) = \sum((1 + \frac{1}{n}) f_n(x))^{2n}$ is used to produce a real analytic norm on X.

By using the Kurzweil method, we get

Theorem 13. ([17, Ch. V], [41]) *If X admits a bump that locally depends on finitely many coordinates, then X is Asplund and contains an isomorphic copy of c_0.*

In this direction let us mention that this method also yields that a normed space that has $\aleph_0$ linear dimension, admits a C^∞ norm ([21], [22])

Summing up,

Theorem 14. *For a metrizable compact K, the following are equivalent:*

- *(a) K is scattered, i.e. K is countable, i.e. $C(K)$ is Asplund*
- *(b) $C(K)$ admits a C^1 smooth norm*
- *(c) $C(K)$ admits a C^∞ smooth norm*
- *(d) $C(K)$ admits a real analytic norm*
- *(e) $C(K)$ admits a C^1 smooth bump*
- *(f) $C(K)$ admits a C^∞ smooth bump*
- *(g) $C(K)$ admits a norm that locally depend of a finite number of coordinates*
- *(h) $C(K)$ admits a bump function that locally depend on a finite number of coordinates*

For general compacts, it is not known if (a) implies (e) or if (a) implies (f), or if (b) implies (c). Haydon showed in [69] that if $\tilde{T}$ is an Alexandrov compactification of a tree T, then $C(\tilde{T})$ satisfies (f) but does not need to satisfy (b) in a stronger sense that it may not even admit Gâteaux differentiable norm.

The following result is proved in [63]:

Theorem 15. ([63]). *If $C(K)^*$ admits a dual LUR norm, then $C(K)$ admits a C^∞ smooth norm.*

We will discuss the following generalization of the property of local finite dependence, that is close to the notion of Gâteaux differentiability, namely the notion of bumps that locally depend on countably many coordinates.

We will say ([40]) that a function φ on X locally well depends on countably many coordinates if for every $x \in X$ there is a neighborhood U of X and countably many functionals $\{f_i\} \subset B_{X^*}$ and a function ψ on ℓ_∞ such that $\varphi(y) = \psi(f_1(y), \dots f_n(y), \dots)$ for each $y \in U$ and such that $X/\{f_i\}_\perp$ is separable. An example of such function is the canonical supremum norm on the space $C_0[0, \omega_1]$ of continuous functions on the ordinal segment $[0, \omega_1]$ that vanish at ω_1.

If the norm of X locally well depends on countably many coordinates and $M \subset B_{X^*}$ is such that $\overline{M}^{w^*} = B_{X^*}$, then given $f \in B_{X^*}$, there is a countable set $C \subset M$ such that $f \in \overline{C}^{w^*}$. This implies the result of Kalenda ([76], [77]) that the unit dual ball of the space $C_0[0, \omega_1]$ is not a Valdivia compact, though the unit dual ball of $C[0, \omega_1]$ is Valdivia [40].

A compact K is Valdivia if it is homeomorphic to a subset S of $[-1, 1]^\Gamma$ in its pointwise topology such that the countable supported elements of S are dense in S.

It can be shown that if B_{X^*} is weak star separable and the norm of X locally well depends on countably many coordinates, then B_{X^*} is weak star sequentially separable.

If we assume the Continuum Hypothesis, the Čech-Pospíšil result on sequentially compact spaces implies that B_{X^*} is weak star sequentially compact if the norm of X locally well depends on countably many coordinates [40].

Also, if the norm of X locally well depends on countably many coordinates, then $\text{dens}X^* \leq \text{card}X$. Thus ℓ_∞ does not admit such a norm.

Theorem 16. ([28]) *Assume that the norm of X is Gâteaux differentiable and that B_{X^*} is Valdivia in the weak star topology. Then X is WLD.*

This motivated the following result of Kalenda:

Theorem 17. ([78]) *If in every equivalent norm on X, B_{X^*} is Valdivia in its weak star topology, then X is WLD. In particular, if dens $X \leq \omega_1$ and X admits a PRI in every equivalent norm, then X is WLD.*

This was the final step in the effort of many mathematicians to produce the following result.

Theorem 18. ([78]) *Assume that the density of X is ω_1. Then X is WLD if and only if X has a PRI in every equivalent norm on X.*

3. Smoothness and weak compactness

If M is a bounded total set in X (i.e. a bounded set M in X such that $\overline{\text{span}}\ M = X$), we will say that a norm $\|\cdot\|$ is $M-$ Fréchet smooth if

$\lim_{t\to 0^+} \frac{1}{t} \sup_{h\in M}(\|x + th\| + \|x - th\| - 2) = 0$

for every $x \in S_X$.

A norm $\|\cdot\|$ of a Banach space X is called 2-rotund (2R) if $\{x_n\}$ is norm convergent whenever $x_n \in S_X$ are such that $\lim_{m,n\to\infty} \|x_m + x_n\| = 2$.

If M is a bounded total set in X, we will say that the norm $\|\cdot\|$ on X is dually $M-2$- rotund ($M-2R$) if $\{f_n\}$ is convergent to some $f \in B_{X^*}$ uniformly on M whenever $f_n \in S_{X^*}$ are such that $\lim_{n,m\to\infty} \|f_n + f_m\| = 2$.

If M is a bounded total set in X, we will say that the norm $\|\cdot\|$ is $\sigma-$ Fréchet differentiable if for every $\varepsilon > 0$ there is a decomposition $B_X = \cup_n B_n^\varepsilon$ such that $\limsup_{t\to 0^+} \sup_{h\in B_n^\varepsilon} \frac{1}{t}(\|x+th\|+\|x-th\|-2) \leq \varepsilon$ for every $x \in S_X$ and every n.

Note that any σ- Fréchet differentiable norm is Gâteaux differentiable, that a dually $M-2R$ norm is $M-$ Fréchet differentiable, that any Gâteaux differentiable norm on a separable space is $M-$ Fréchet differentiable for any compact total set M and thus then σ -Fréchet differentiable. Moreover any UG norm on a separable Banach space is dually $M - 2R$ for any compact total M in X (cf. e.g. [37]).

Theorem 19. ([92]) *A separable Banach space X is reflexive if and only if X admits an equivalent $2R$ norm.*

We will prove only that X is reflexive if it satisfies the condition in Theorem 19. Let $F \in S_{X^{**}}$ be given. Choose $f_n \in S_{X^*}$ such that $\lim F(f_n) = 1$. Then $\lim \|f_n + f_m\| = 2$ and thus f_n converges in norm to some $f \in S_{X^*}$ and thus $F(f) = 1$. Therefore each element of $S_{X^{**}}$ attains its norm and the space X is reflexive by James' theorem (cf. e.g. [17] or [33, Ch. III]).

Note that it is not known if the separability of X has to be assumed in Theorem 19.

Lemma 1. *Assume that M is a bounded weakly closed set in a Banach space X. Assume that the norm $\|\cdot\|$ of X is dually $M - 2R$. Then M is weakly compact.*

Proof. Let $S \subset M$ be a countable subset of M and assume that $x \in \overline{S}^{w^*} \subset X^{**}$ be such that $x \notin X$. Let $F \in S_{X^{***}}$ be such that $F \in X^{\perp}$ and $F(x) = \text{dist(x ,X)} > 0$. Let $\{y_i\} \subset B_{X^{**}}$ be such that $\sup_i F(y_i) = 1$. Let $f_n \in S_{X^*}$ be such that $\lim_n (f_n - F)(s) = 0$ for all $s \in S$, that $\lim(f_n - F)(y_i) = 0$ for all i and that $\lim_n (f_n - F)(x) = 0$. The existence of $\{f_n\}$ follows from a "metrizable version" of the Goldstine theorem ([33, p. 73]). Then $\lim_{n,m} \|f_n + f_m\| = 2$ and thus by the rotundity assumed, $\lim_n (f_n - F)(s) = 0$ and $\lim_n (f_n - F)(x) = 0$ uniformly on $(S \cup x)$. As all f_n are continuous on $(S \cup x) \subset X^{**}$ in its relative pointwise topology, so is their uniform limit on this set, which is not the case as F is zero on S and $F(x) > 0$. Therefore $x \in X$ and thus M is countably weakly compact which means that M is weakly compact by the Eberlein- Šmulyan theorem (cf. e.g. [33, Ch. IV].

∎

Theorem 20. *Assume that X is a separable Banach space and M is a bounded set in X. Then M is weakly relatively compact if and only if X admits an equivalent dually $M - 2R$ norm.*

Proof. Assume that M is relatively weakly compact. Let T be a bounded linear operator from a separable reflexive space Z such that $T(B_Z) \supset M$ ([14], cf. e.g. [33, p. 366]). Let $\|\cdot\|_0$ be a 2- rotund norm on Z constructed by Odell and Schlumprecht in [92]. Let $\|\cdot\|_1$ denotes the dual norm defined on X^* by $\|f\|_1^2 = \|f\|_\infty^2 + \|f\|_0^2$, where $\|\cdot\|_\infty$ is the canonical norm of X^* and $\|\cdot\|_0$ denotes the dual norm to the Odell- Schlumprecht norm on Z. Then it is standard to check that $\|\cdot\|_1$ is M- 2- rotund on X^*.

If the condition holds true, then M is relatively weakly compact by Lemma 1.

Theorem 21. *A Banach space X is WCG if and only if X admits a dually $M - 2R$ norm for some bounded total set in X.*

Proof. Assume that X is WCG. Then there is a bounded linear one-to-one operator T from X^* into some $c_0(\Gamma)$. Let $\|\cdot\|_D$ be the Day norm on $c_0(\Gamma)$. By the result of Hájek and Johanis ([64]), the norm defined on X^* by $\|f\|_1^2 + \|Tf\|_D^2$ is dually $M - 2R$, where $M = T^*(e_\alpha)$ where $\{e_\alpha\}$ are the unit vectors in $c_0(\Gamma)$. By Lemma 1, the set M is relatively weakly compact in X and the closed linear hull of it equals to X.

The other implication is contained in Lemma 1.

Theorem 22. ([36]) *A Banach space X is WCG if and only if X is WLD and there is a bounded total set M in X and a norm $\|\cdot\|$ on X that is $M-$ Fréchet smooth.*

Summing up with the result in [66] used in (iii) and the result [55], we have the following corollary.

Corollary 4. *Let X be a Banach space. Then*

(i) X^* is WCG if and only if X^* contains a bounded norm total set $M \subset X^*$ and X admits a norm $\|\cdot\|$ such that if $x_n \in S_X$ are such that $\lim_{n,m\to\infty} \|x_n + x_m\| = 2$, then $\{f(x_n)\}$ is uniformly Cauchy on $f \in M$.

(ii) X^* is a subspace of WCG if X admits a WUR norm.

(iii) The James tree space JT admits a norm $\|\cdot\|$ such that $f(x_n)$ is convergent for each $f \in X^*$ whenever $x_n \in S_X$ are such that $\lim_{n,m\to\infty} \|x_n + x_m\| = 2$.

(iv) The Hagler tree space JH does not admit a norm whose second dual on JH^* is strictly convex.

We refer to [70], [83], [54] cf. e.g. [33, p. 199] for James' tree space JT and for Hagler's space JH.

Theorem 23. ([29]) *A Banach space X is a subspace of a Hilbert generated space if and only if X admits a UG norm.*

A space X is Hilbert generated space, if there is a Hilbert space H, and a bounded linear operator T from H into X such that $T(H)$ is dense in X.

Theorem 24. ([29]) *A Banach space X is Hilbert generated if and only if there is a bounded total set M in X and a constant $C > 0$ such that*

$$\sup_{x\in S_X, h\in M} (\|x + th\| + \|x - th\| - 2) \le Ct^2$$

for every t.

The compact space is descriptive if it has a sigma discrete network. A network is a family of (not necessarily open) sets such that every open set is a union of some of them and a system of sets is sigma discrete if it decomposes into countably many subsystems each one is discrete, i.e. every set in it is disjoint from the closure of the union of others in it.

It is known that B_{X^*} in its weak star topology is descriptive if X is a Vašák space ([96]).

The dual norm $\|\cdot\|$ of X^* is weak star LUR if f_n weak star converge to f whenever $f_n, f \in S_{X^*}$ are such that $\|f_n + f\| \to 2$.

Theorem 25. ([96]) *A compact set K is descriptive if and only if $C(K)$ admits a norm whose dual is weak star LUR.*

We will say that a norm $\|\cdot\|$ on X is P- uniformly rotund (P stands for pointwise) if there is a weak star dense bounded set $M \subset X^*$ such that $f(x_n - y_n) \to 0$ whenever $x_n, y_n \in S_X$ are such that $\|x_n + y_n\| \to 2$ and $f \in M$.

Theorem 26. ([99]) *Let K be a compact space. Then*

(i) If K is descriptive, then $C(K)^*$ admits a dual norm that is p- uniformly rotund.

(i) K carries a strictly positive Radon measure if and only if $C(K)$ admits a P- uniformly rotund norm.

An interesting thing here is that Theorem 25 as well as Theorem 26 are not proved by using the Day norm but rather by the Godefroy transfer norm.

4. Smoothness in higher duals and containment of ℓ_1

Related to Theorem 1 in this survey is the following stronger version of a classical result of Dixmier.

If the third dual norm on X^{***} is Gâteaux smooth, then X is reflexive (cf. e.g. [33, p. 276])

A separable space X is not reflexive if and only if X admits a norm $\|\cdot\|$ such that there is a point in X that is a point of Gâteaux smoothness of $\|\cdot\|$ but not a point of smoothness of the second dual norm to $\|\cdot\|$ ([53]).

If X is separable and the second dual norm on X^{**} is Gâteaux smooth, then X^* is separable (cf. e.g. [33, p. 275]).

Separable spaces whose dual spaces are separable are characterized as separable spaces not admitting a rough norm (Section 2) and also as separable spaces X where every convex continuous function on X is

Fréchet differentiable at some points (cf. e.g. [17, Ch. I]) and also as separable Banach spaces, in which every Lipschitz function is Fréchet differentiable at some points ([95]).

Separable Banach space in which every Gâteaux differentiable convex continuous function is Fréchet differentiable at some points are characterized as separable spaces for which every convex weak star compact subset $K \subset X^*$ and for every $\varepsilon > 0$ there is a nonempty weak star open subset $O \subset K$ such that the diameter of O is less than ε (cf. e.g. [17, Ch.III]).

Separable spaces that do not contain a copy of ℓ_1 are characterized as separable spaces not admitting an octahedral norm (cf. e.g. [17, Ch. III]).

A norm $\|\cdot\|$ on X is octahedral if for every finite dimensional $F \subset X$ and for every $\eta > 0$ there exists $y \in S_X$ such that for every $x \in F$ we have $\|x + y\| \geq (1 - \eta)(\|x\| + 1)$.

If X contain a copy of ℓ_1, then X^* contains copy of $\ell_1(c)$ by a result of Pełczynski (cf. e.g. [33, p. 155]). Thus (see Section 2), X^* does not admit any Gâteaux smooth norm.

Hájek showed the following theorem.

Theorem 27. ([55])

(i) The predual of the James space J admits a norm whose second dual is UG

(ii) The James tree space JT admits a norm whose second dual is strictly convex

(iii) The Hagler space JH does not admit a norm whose second dual is strictly convex.

We refer to [84, p. 25] for the James' space J. Here we mention only that J is a separable Banach space that is isomorphic to J^{**} and the codimension of J in J^{**} is 1.

M. Smith proved in [101] that the space J admits a norm whose third dual on J^{***} is strictly convex.

In the end of this section we mention an applications of the rotundity in nonlinear analysis.

Let us call a Banach space in the norm $\|\cdot\|$ Lipschitz separated if for every closed convex set $C \subset X$ and every bounded $1-$ Lipschitz function f on C and every $x \notin C$ there exist 1- Lipschitz extensions f_1 and f_2 on X such that $f_1(x) \neq f_2(x)$ ([10]).

In [10] it is proved in particular

Theorem 28. ([10]) *If the norm $\|\cdot\|$ on X is WUR, then X is Lipschitz separated in $\|\cdot\|$. On the other hand if X is Lipschitz separated in $\|\cdot\|$, then the second dual norm to $\|\cdot\|$ is strictly convex.*

In [66], it is proved the following.

Theorem 29. *The space JT admits a norm under which it is Lipschitz separated.*

From Theorem 28 and Corollary 4 it follows that the space JH does not admit a norm in which it is Lipschitz separated.

5. Special norms

The norm $\|\cdot\|$ of a Banach space X is called strongly subdifferentiable (SSD) if for every $x \in S_X$, $\lim_{t\to 0^+} \frac{1}{t}(\|x + th\| - \|x\|)$ exists uniformly on $h \in S_X$.

The Šmulyan duality lemma reads that the norm $\|\cdot\|$ is SSD at $x \in S_X$ if and only if $\mathrm{dist}(x_n^*, J(x)) \to 0$ whenever $x_n^* \in B_{X^*}$ are such that $x_n^*(x) \to 1$, where $J(x) = \{X^* \in S_{X^*}\,; x^*(x) = 1\}$.

Note that the norm $\|\cdot\|$ is Fréchet differentiable if and only if it is Gâteaux differentiable and at the same time SSD. From the monotonicity of the differential quotient for convex functions and from the Dini theorem on monotone uniform convergence we get that any norm that locally depends on a finite number of coordinates is SSD.

Theorem 30. ([52]) *If X is separable and X^* is not separable, then X admits a norm that is nowhere strongly subdifferentiable except at the origin.*

The following is the result of G. Godefroy.

Theorem 31. ([52]) *If X admits an SSD norm, then X is Asplund.*

Theorem 32. *(cf. e.g.* ([52]) *Assume that X has a PRI. Assume that the norm of X is SSD. Then the PRI is shrinking.*

Recall that PRI is called shrinking if P_α^* form a PRI for X^*.

An interesting thing here is that in the results on SSD norms, a usual use of the Bishop-Phelps theorem is replaced by the use of the Simons' lemma (cf. e.g.[17, Ch. I], [33, Ch. III]).

Theorem 33. ([35]) *Let X be a subspace of WCG and S be a weak star dense subset of B_{X^*}. Then X admits an equivalent Gâteaux differentiable norm that is $S-$ lower semi-continuous.*

The space $C[0,\omega_1]$ admits a C^∞ smooth norm ([69]). However, we have:

Theorem 34. ([35]) *The space $C[0,\omega_1]$ does not admit any equivalent Gâteaux differentiable norm that is $[0,\omega_1)$ lower semi-continuous.*

6. Smoothness, type and cotype

We recall that a Banach space X has type 2 if there is a constant $K > 0$ such that

$$\sum_{\varepsilon_i=\pm 1} \|\sum \varepsilon_i x_i\|^2 \leq 2^n K \sum \|x_i\|^2$$

for all $x_1, \ldots, x_n \in X$.

If the inequality turns to the opposite inequality, we speak about the cotype 2 of the space.

The proof of Theorem 7 gives

Theorem 35. ([39], cf.e.g.[17, Ch. V]) *If X admits a C^2 smooth bump and does not contain a copy of c_0 then X is of type 2.*

Theorem 36. ([15], cf. e.g. [17, Ch. V]) *Let X admit a separating polynomial of degree p. Then X is of cotype equal to the even part of p.*

A polynomial P on a Banach space X is a separating polynomial if there is $\delta > 0$ such that $|P(x)| > \delta$ for all $x \in S_X$.

This leads to the following result.

Theorem 37. ([15], [17, Ch. V]) *Assume that X admits a C^∞ smooth bump and that X does not contain a copy of c_0. Then X is of cotype equal to (*$\inf q; X$ *is of cotype q). This infimum is an even integer and X contains a copy of some ℓ_p where p is an even integer.*

This is in contrast with the result of Tsirelson (cf. e.g. [84, p. 95]) that there is a Banach space that contains no isomorphic copy of c_0 or ℓ_p, $1 \leq p < \infty$.

The following is Makarov's result.

Theorem 38. (cf. e.g. [17, Ch. V]) *Assume that X admits a C^2 bump and that X is saturated with Hilbertian spaces. Then X is isomorphic to a Hilbert space.*

Recall that X is saturated with Hilbertian spaces if every infinite dimensional subspace of X contains an infinite dimensional subspace that is isomorphic to a Hilbert space.

Note that a nonseparable Banach space may admit a C^∞ smooth norm, be saturated with copies of c_0 and yet, not contain any $c_0(\Gamma)$ or $\ell_p(\Gamma)$ for Γ uncountable. Such space is for instance the Johnson-Lindenstrauss space JL_0 ([75], cf. e.g.[110, p. 1757]).

Theorem 39. ([19], [39]) *The space $(\sum \ell_4^n)_2$ admits no twice Fréchet differentiable norm, though it admits a twice Gâteaux differentiable norm.*

7. Smooth approximation and optimization

Theorem 40. *(a) Assume that X is WLD or that $X = C(K)$. Assume that X admits a C^k smooth bump. Let f be a continuous function on X and $\varepsilon > 0$. Then there is a C^k-smooth function on X such that $|f(x) - g(x)| < \varepsilon$ for all $x \in X$.* (cf. e.g. [17, Ch. VII], [63]).

b) Assume that X admits a LUR norm whose dual norm is LUR. *Let f be a continuous function on X and $\varepsilon > 0$. Then there is a C^1 smooth function g on X such that $|f(x) - g(x)| < \varepsilon$* ([108], cf. e.g. [17, Ch. VII]).

(c) Assume that a separable X admits a separating polynomial, that f is a continuous function on X and $\varepsilon > 0$. Then there is a real analytic function g on X such that $|f(x) - g(x)| < \varepsilon$ for every $x \in X$ [80].

(d) Assume that f is a uniformly continuous function on $c_0(\mathbb{N})$ and $\varepsilon > 0$. Then there is a real analytic function on c_0 such that $|f(x) - g(x)| < \varepsilon$ for all $x \in c_0(\mathbb{N})$ [13], [47].

(e) Assume that X is a super-reflexive space and that f is a uniformly continuous function on X. Then there is a sequence $\{f_n\}$ of differentiable functions on X such that f'_n is uniformly continuous on every bounded set for each n and $\lim\ f_n = f$ uniformly on bounded sets [11],[12].

(f) Assume that X is super-reflexive. Then X admits partitions of unity formed by uniformly Fréchet differentiable functions [73].

(g) Assume that X is separable space such that the norm of X is k times Fréchet differentiable so that the $k-$ th derivative is bounded on the sphere. Assume that X has a Schauder basis. Then any norm on X can be approximated on bounded sets by C^k smooth norms [21], [22].

(h) Any equivalent norm on a separable Hilbert space is approximated by real analytic norms [21], [22].

(i) Any Lipschitz mapping from a separable Banach space into a Banach space can be uniformly approximated by Lipschitz uniformly Gâteaux differentiable mappings [72].

An important rôle here is played by smooth partitions of unity (locally finite and subordinated to any open cover). Of crucial importance in nonseparable spaces has been the result of Torunczyk that $c_0(\Gamma)$ admits such partitions formed by C^∞ smooth functions that locally depend on a finite number of coordinates. This allowed for the use of smooth homeomorphisms into by maps that are smooth coordinate-wise only (cf. e.g. [17, Ch. VII]).

The result in (h) should be compared with Vanderwerff's result ([108]) that there is a norm on ℓ_2 that cannot be approximated by norms with uniformly continuous second derivative on bounded sets.

We will now discuss ranges $f'(X) = \{f'(x);\, x \in X\} \subset X^*$ for bump functions on X.

By James' theorem ([17, Ch. I]), [33, Ch. III]), if $\{\|x\|', x \in S_X\} = S_{X^*}$, then X is reflexive.

However we have

Theorem 41. *(a) If X admits a C^1 smooth bump then X admits such a bump f with $f'(X) = X^*$*([4]).

(b) If X admits a C^1 smooth bump, then X admits such a bump f with $f'(X) = B_{X^*}$ ([9]).

(c) If X admits a Lipschitz bump with uniform continuous derivative, then X admits such a bump f with $f'(X) = B_{X^*}$ ([49]).

(d) If X is separable, then X admits a Gâteaux differentiable bump f with $f'(X) = X^*$ ([5]).

(e) If M is an analytic set in separable dual space X^* (in norm topology) such that for every $f \in M$ there is a continuous map φ from $[0, 1]$ into X^* with $\varphi(0) = 0, \varphi(1) = f$ and $\varphi([0, 1)) \subset IntM$. Then there is a C^1 smooth bump b on X such that $b'(X) = M$ ([34]).

Note that the set is analytic if it is a continuous image of $\mathbb{N}^{\mathbb{N}}$ in its pointwise topology.

(f) If f is a real valued function on a Banach space X that is Fréchet differentiable, then $f'(X)$ is connected ([87]).

(g) There is a Fréchet smooth map f from $\mathbb{R}^2$ into itself such that $f'(\mathbb{R}^2)$ is not connected ([97]).

(h) If X is infinite dimensional Banach space then there is a Gâteaux differentiable bump f on X so that f' is norm to weak star continuous and $\|f'(0) - f'(x)\| \geq 1$ for every $x \in X$, $x \neq 0$. If X^* is moreover separable, we can get that f is C^1 on $X \setminus \{0\}$ ([20]). *However, if X is a Banach space and f is a real*

valued Lipschitzian Gâteaux differentiable function, then for every $x \in X$ and every $\varepsilon > 0$ there exist y, z within the distance $\varepsilon > 0$ from x such that $y \neq z$ and $\|f'(y) - f'(z)\| \leq \varepsilon$.

(i) If X is separable, then the function $f(x) = \sum 2^{-i} f_i^2(x)$, where $\{f_i\}$ is weak star dense in S_{X^*} is a C^∞ function on X such that $f(0) = 0$ and $f(x) > 0$ for $x \neq 0$. This has the following generalization [3], cf. e.g. [65]*:*

If X^* is hereditarily weak star separable and C is a closed convex set in X, then there is a C^∞ smooth convex function f on X such that $f(c) = 0$ for all $c \in C$ and $f(x) > 0$ for all $x \notin C$.

(j) If f is a real valued function on c_0 with locally uniformly continuous derivative, then $f'(c_0)$ is included in a countable union of norm compact sets in ℓ_1 [57]. *This implies that if Γ in uncountable, then there is no real valued function f on $c_0(\Gamma)$ with f' locally uniformly continuous that would attain its minimum exactly at one point* [57]. *This answered a question of J.A. Jaramillo.*

8. Selected open problems

8.1. Problems on separable spaces

(a) Assume that a separable Banach space X admits a C^k smooth norm for all $k \in I\!N$. Does X admit a C^∞ smooth norm? For the case of bumps, this problem has a positive solution if X does not contain c_0 ([15], cf. e.g. ([17, Ch. V]).

(b) Assume that a separable space admits a C^k smooth bump ($k > 1$). Does X admit a C^k smooth norm?

(c) Assume that X/Z is separable and that Z admits an equivalent Gâteaux differentiable norm. Does X admit an equivalent Gâteaux differentiable norm?

(d) Does the space of compact operators on ℓ_2 admit a real analytic norm?

(e) Does the convexified Tsirelson space T_2 admit a C^2 smooth norm?

For more on this question see [24].

(f) Assume that a separable X admits a C^k smooth bump, $k > 1$. Does there exist an infinite dimensional subspace Y of X and a C^k smooth norm on Y?

(g) Assume that X^* is separable, Y is a subspace of X and $\|\cdot\|$ is Fréchet smooth norm on Y. Does there exist a norm on X that is an extension of the norm $\|\cdot\|$ from Y and that is Fréchet differentiable on $X \setminus 0$?

For Gâteaux smooth norms solved negatively in [111].

8.2. Problems on $C(K)$ spaces

(a) Assume that K is a scattered compact. Does $C(K)$ admit a C^1 smooth or even C^∞-smooth bump ? Does $C(K)$ for Kunen's space ([90] or [65, Ch. III]) admit a C^∞ (C^1) smooth bump?

(b) Assume $C(K)$ admits a Fréchet smooth norm. Does $C(K)$ admit a C^∞ smooth norm?

(c) Does there exist a LUR norm that is C^1 on $C[0, \omega_1]$?

(d) Are Fréchet smooth norms on $C[0, \omega_1]$ dense (residual) in all equivalent norms on this space?

(e) Does the renorming by Fréchet smooth norm have the three space property? The same question for WUR.

Note that the three space property means that X has the property if there is $Y \subset X$ so that both Y and X/Y have the property.

(f) If Γ is uncountable does $c_0(\Gamma)$ admit a C^2 smooth norm which is uniformly Gâteaux differentiable?

(For Γ countable, the answer is yes [42]).

(g) Does $c_0(\Gamma)$, if Γ uncountable admits partitions of unity formed by uniformly Gâteaux differentiable functions?

(h) Classify trees T so that the space of continuous functions on Alexandroff's compactification of T admits an M-Fréchet differentiable norm for some M (or σ- Fréchet differentiable norm).

(i) Classify trees T so that $C(T)$ admits an SSD norm. For some recent results on tree spaces we refer to [102] and [103].

8.3. Problems on WLD spaces

(a) Assume that X is WLD and that X admits a Lipschitz Gâteaux smooth bump. Does X admit a Gâteaux differentiable norm?

(b) If X is WLD and X admits a Gâteaux differentiable norm. Does X admit a norm whose dual norm is strictly convex?

(c) Let X be a WLD space. Is every convex continuous function on X Gâteaux differentiable at some points? (for more on this problem we refer to [2]).

8.4. Problem on polynomials

Assume that a Banach space admits a separating polynomial. Does X admit a C^∞ smooth norm?

8.5. Problems on separable spaces that do not contain copies of ℓ_1

(a) Assume that X is a separable Banach space that does not contain a copy of ℓ_1. Does X^* necessarily admit a Gâteaux differentiable norm?

Compare with the remark preceding Theorem 27.

(b) Assume that X is a separable Banach space that does not contain a copy of ℓ_1. Let $\|\cdot\|$ be the norm of X. Does the second dual norm on X^{**} have a point of Fréchet smoothness?

For James' tree space solved positively in [100].

(c) Assume that X is a separable space not containing a copy of ℓ_1. Does X^* admit a LUR norm?

8.6. Problem on special norms

Assume that X is a superreflexive Banach space with Schauder basis. Does there exist a uniformly Fréchet smooth norm on X such that the given basis is monotone in it? (The same question for uniformly rotund norms)

8.7. Problems on approximation and optimization

(a) Is any continuous function on c_0 approximable by a real analytic function?

(b) If f is a continuous convex function on c_0, does there exist an $x_0 \in c_0$ so that there is $C > 0$ such that $f(x_0 + h) + f(x_0 - h) - 2f(x_0) \le C\|h\|^2$?

(c) Let Γ be uncountable. Are all equivalent norms on $\ell_2(\Gamma)$ approximated by C^∞ smooth norms? For lattice norms on $c_0(\Gamma)$ C^∞ smooth approximation was done in [32].

(d) Let X be a separable Banach space and $\|\cdot\|$ be the norm of X. Assume that the restriction of $\|\cdot\|$ to any subspace of X has a point of Fréchet differentiability. Is X^* necessarily separable?

(e) Assume that every Gâteaux differentiable convex continuous function on X has a point of Fréchet smoothness. Does every Lipschitz Gâteaux differentiable function on X have a point of Fréchet smoothness? For more information in this direction see [17, Ch. III].

8.8. Problems on general spaces

(a) Assume that X is an Asplund space. Does X admit a Fréchet C^1 smooth bump? In more generality: Let M be a bounded total set in a Banach space X. Assume that every convex continuous function on X is M-differentiable on a dense set in X. Does X admit an M-differentiable bump?

(b) Assume that X is weak Asplund. Does X admit a Lipschitz Gâteaux differentiable bump?

Note that Moors and Somasundaram [89] found an example of a space X where every convex function is differentiable on dense set and yet X is not weak Asplund. This space thus does not admit any Lipschitz Gâteaux differentiable bump (see Section 2).

(c) Assume that X admits a C^k smooth bump (C^k- smooth norm), $k \geq 1$ or $k = \infty$. Does X admit C^k smooth partitions of unity?

(d) Assume that X admits a Fréchet smooth bump. Does X admit a Lipschitz C^1 smooth bump?

(e) Assume that a Banach space X admits a twice Fréchet differentiable norm. Does X possess the weak Banach Saks property (i.e. If $x_n \to 0$ weakly does there exist a subsequence x_{n_k} such that $\|\frac{1}{k}(x_{n_1} + \cdots + x_{n_k})\| \to 0$)?

(f) Assume that a Banach space X is Lipschitz homeomorphic to a Banach space Y. Let X admit Fréchet differentiable bump (norm). Does Y admit a Fréchet differentiable bump (norm)?

(g) Does James' long space $J(\eta)$ admit a smooth norm (bump?) ([25]). This space admits a norm with the Mazur intersection property (i.e. each convex closed bounded set is an intersection of a family of balls)([16]).

(h) Assume that X admits a $\sigma-$ Fréchet differentiable norm. Is X a subspace of a space that admits an $M-$ differentiable norm for some total M? This problem is related to the known problem if a continuous image of Radon- Nikodym compact space is Radon- Nikodym, i.e. homeomorphic to a weak star compact in the dual to an Asplund space.

(i) Assume that X admits a norm that well depends on a countable number of coordinates (see Section 3). Does X admit a Gâteaux smooth norm?

(j) Let K be a Kunen compact ([90], or [65]). Does $C(K)$ admit an equivalent norm that well depends on countably many coordinates?

(k) Assume that X is a Vašák space. Does X admit a norm that has the following property: $\{f_n\}$ is weak star convergent to some $f \in B_{X^*}$ whenever $f_n \in S_{X^*}$ are such that $\lim_{m,n\to\infty} \|f_m + f_n\| = 2$?

(l) Does the Johnson-Lindenstrauss space JL_0 admit a norm with the same property as in (k)?

(m) Assume that X has an unconditional basis and admits a Gâteaux differentiable norm. Does X admit a norm the dual of which is rotund?

(n) Assume that X admits a Lipschitz Fréchet smooth bump. Does X admit an equivalent norm with the Mazur intersection property? This holds true if X has the Radon-Nikodym property, i.e. if bounded sets have arbitrarily small cuts by halfspaces (slices) [18].

(o) Assume that B_{X^*} in its weak star topology is fragmented. Does X admit a Lipschitz Gâteaux differentiable bump?

The converse implication is true [44].

(p) Assume that for any weak star dense subset S of B_{X^*} there is an equivalent Gâteaux differentiable norm on X that is $S-$ lower semi-continuous. Is X a subspace of WCG? (Compare with Theorem 33).

(q) Assume that X is an Asplund space. Does X admit a continuous (not necessarily equivalent) Fréchet differentiable norm?

(r) Assume that X admits a norm whose derivative is p -Hölder and a norm the dual of which has the q -Hölder derivative. Does there exist one norm on X that has these two properties at the same time?

Compare with Theorem 8.

(s) Assume that X admits a C^∞ smooth norm. Does X admit an LUR norm?

(s) Assume that X admits both Gâteaux differentiable norm and an SSD norm. Does X admit a Fréchet smooth norm? This is a counterpart to the result of Troyanski that X admits an equivalent LUR norm if it admits both rotund norm and also a norm with the property that on the unit sphere the norm and weak topology coincide (cf. e.g. [17, Ch IV], for a short Raja's proof of it see e.g. [33, p. 281]).

(t) Assume that a super-reflexive space X admits a C^2 smooth norm. Does X admit a UF norm that is at the same time C^2 smooth?

The same question on uniformly rotund norms. Compare with [17, p. 192].

(s) Assume that X is a WLD space that admits a Fréchet C^2-smooth norm. Does X admit a norm that is LUR and is a limit (uniform on bounded sets) of C^2 smooth norms?

For $X = c_0(\Gamma)$ the answer is yes, see [17, Ch. II].

(t) Does every Asplund space admit an equivalent SSD norm? In particular, does the space $C(K)$ admit an equivalent SSD norm if K is a tree space?

(u) Assume that X is a nonseparable non Asplund space. Does X admit an equivalent norm that is nowhere SSD except at the origin? For separable non Asplund space the answer is yes [52].

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